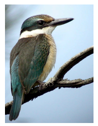
Fifth Southern Hemisphere Conference on Undergraduate Mathematics and Statistics Teaching and Learning

Proceedings of Kingfisher Delta'05

World Heritage Fraser Island, Queensland, Australia



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Editors

Michael Bulmer

Helen MacGillivray

Cristina Varsavsky

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Preface

Firstly, I acknowledge with thanks the support of people and organisations that have made it possible for Kingfisher Delta'05 to succeed. Delta'05 has been financially underwritten by the Faculty of Sciences at the University of Southern Queensland with further financial backing from the Australian Mathematical Society, the Australian Bureau of Statistics, CEANET, and a number of Australian university mathematics and statistics departments from the University of Southern Queensland, the University of Queensland, Central Queensland University, Queensland University of Technology, Monash University, and the University of Adelaide.

The organisation of Delta'05 is backed by an International Steering Committee made up of representatives from Southern Hemisphere countries, as well as two representatives from our Northern colleagues. This steering committee currently consists of; Chris Harman (Australia, Chair), Patricia Cretchley (Australia), Johann Engelbrecht (South Africa), Derek Holton (New Zealand), Ken Houston (UK), Deborah Hughes-Hallett (USA), Victor Martinez-Luaces (Uruguay), Jan Persens (South Africa), Ivan Reilly (New Zealand), and Cristina Varsavsky (Australia).

The Kingfisher Delta Conference is the fifth in a series of Southern Hemisphere biennial conferences on undergraduate teaching and leaming of Mathematics and Statistics. The first conference, Delta'97 (delta implying change) was held in Brisbane, Australia, in November 1997. Delta'99 followed in November 1999 at Laguna Quays, Queensland, Australia with the Warthog Delta'01 being held in Kruger National Park, South Africa, in July 2001, and the Remarkable Delta'03 held at Queenstown, New Zealand in November 2003.

The theme for Delta'97 was "What Can We Do to Improve Leaming?", for Delta'99 it was "The Challenge of Diversity", for Delta'01 it was "Gearing for Flexibility" and for Delta'03 it was "From all Angles". For this Delta'05 the theme is "Blending Beyond Boundaries".

Approximately 100 delegates are attending from 15 different countries and are responsible for about 60 contributed presentations, panel discussions and round table discussions. The conference has two publications: the first is a special edition of the International Journal of Mathematical Education in Science and Technology; the second is the Conference Proceedings, consisting of peer reviewed research papers and which are judged as Full or Short by peer referees.

This conference provides an important forum for dissemination and discussion of research into the crucial topic of undergraduate teaching and learning of mathematics and statistics. Excellence and proficiency have always been crucial in undergraduate work. More than ever, this is being better understood by society. In addition, more research and accountablility is now being asked of us by our students, as well as by our political and academic masters. Change accelerates at a greater rate day by day, and we must lead this change. I am certain that meetings such as this will have important influences on the use of new technologies, the development of sound methodologies, and the understandings of associated learning in undergraduate mathematics and statistics.

Thanks from all of us to the group who worked on editing, compiling and producing this issue of the Proceedings, especially to Michael Bulmer, Cristina Varsavsky, and Helen MacGillivray. Thanks also to Sang Mi Seo for her typesetting assistance.

And lastly, I must convey my special thanks to the hard-working and ever-patient Local Committee; Michael Bulmer, Patricia Cretchley, Marg Flanders, Ruth Hubbard, and Christine McDonald.

Chris Harman
Convenor
Delta'05

Delta'05 Reviewing

All papers appearing in the iJMEST Special Issue and the Conference Proceedings were blind reviewed by at least two peers from an international team of referees. Papers were accepted on the basis of being relevant to the interests of Delta and also had to

- contain a statement of the problem/issue and a discussion of its significance;
- contain some critical analysis of the research literature as it relates to the topic of the paper;
- contain conclusions and implications for mathematics education derived from the study.

Papers appearing in the iJMEST Special Issue and in the Full Papers section of this Proceedings were accepted by the reviewers as full papers of international standing.

We gratefully acknowledge the following reviewers who have made the excellent programme of Delta'05 possible.

Peter Adams
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Plenary Papers

Teaching Quantitative Literacy

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1. What is Quantitative Literacy?

When it comes to teaching mathematics to university-level science, mathematics, engineering, and technology (SMET) students, most educators agree that a stout regimen of algebra and trigonometry followed by the calculus sequence is essential. But a far more challenging and unresolved issue is teaching mathematics to non-SMET or liberal arts students. These students comprise a majority of the students that take mathematics courses at many universities, they are not calculus-bound, and yet they need additional quantitative skills for their careers and lives. What sort of mathematics should we teach to university students of fine arts, music, literature, political science, or history? One answer that has emerged in the last few years is *quantitative literacy* (QL), *quantitative reasoning*, or simply *numeracy*.

In the growing discussion of the need for vigorous QL courses and programs in our universities, many different concepts of QL have emerged. For example, the International Life Skills Survey [3] claims that QL is

> an aggregate of skills, knowledge, beliefs, dispositions, habits of mind, communication capabilities, and problem solving skills that people need in order to engage effectively in quantitative situations arising in life and work.

Lynn Steen, who has written extensively about QL [4, 7], describes it as involving

> confidence in mathematics, cultural appreciation, interpreting data, logical thinking, making decisions, mathematics in context, number sense, practical skills, prerequisite knowledge, symbol sense.

And New York Times science writer Gina Kolata [7] characterizes QL as follows:

> Beyond arithmetic and geometry, quantitative literacy also requires logic, data analysis, and probability.... It enables individuals to analyze evidence, to read graphs, to understand logical arguments, to detect logical fallacies, to understand evidence, and to evaluate risks. Quantitative literacy means knowing how to reason and how to think.

When distilled, the many characterizations of QL have some common features. First, QL is a set of diverse skills needed to survive in a world that requires decision-making based on quantitative information. Whether it is choosing the best life insurance policy, evaluating the results of clinical drug trials, investing personal savings wisely, interpreting surveys and polls, exchanging currency, making informed election decisions, or understanding how increases in the cost of living diminish the value of savings, QL skills are indispensable. They are necessary for successful careers and responsible citizenship.

Second, while QL appears be rooted in mathematics, it goes beyond mathematics. As one QL proponent said [4],

> unlike mathematics, numeracy does not so much lead upward in abstraction as it moves outward toward an ever richer engagement with life's diverse contexts and situations.

Some express concern that teaching QL may be done at the expense of traditional mathematics, notably algebra. For calculus-bound students, there is no question that algebra is a gatekeeper and its thorough mastery is essential. However for a more general audience of non-SMET students, selected algebraic skills are important; but there are other, equally vital topics and skills. More succinctly [4],

> quantitative literacy is not the same as statistics. Neither is it the same as mathematics, nor is it (as some fear) watered-down mathematics . As it turns out, it is not calculus, but numeracy that is the key to understanding our data-drenched society.

Finally, QL courses and programs must instill more than a passive appreciation of mathematics. It is not enough to tell students that mathematics is a critical ingredient of medical images, weather forecasting, DNA fingerprinting, and bar codes. QL means actively cultivating analytical skills, strengthening problem-solving proficiency, emphasizing communication of quantitative ideas, and developing confidence in dealing with quantitative issues.

2. Implementing QL

Individual courses aimed at strengthening QL skills have appeared at many institutions in the past few years. On the other hand, full-fledged QL programs designed to infuse QL throughout the undergraduate curriculum are still rare. In either case, several observations help explain why implementing QL initiatives is so challenging. First, most institutions that embrace QL courses and/or programs expect the mathematics department to take the lead. And yet, teaching non-SMET students is not a high priority in many mathematics departments and it is not a concern of many full-time faculty members. Ideally the development of QL initiatives should be a collaborative effort in which mathematics are leading partners.

Teaching and designing QL courses requires some special considerations. Students who take these courses are often victims of previous mathematics courses and instructors. Not surprisingly, they harbor genuine fears of mathematics, have lost confidence in their quantitative skills, and have little belief that mathematics might be of use in their future. Therefore providing QL students with a worthwhile experience in what may be their last mathematics course requires overcoming significant psychological obstacles. It cannot be done by subjecting students to more of the same experiences they have had in previous mathematics courses. It *can* be done by demonstrating the breadth and utility of mathematics with compelling examples of how it affects students' lives in immediate ways.

Another challenge is that most mathematics educators have a shared understanding of the content of an algebra course or a calculus course. By contrast, there is no common agreement, at the moment, about the content of a QL course. However some common topics and themes may be emerging. Here is one list of expectations that could define the core of a one-semester QL course.

- Students must possess strong critical and logical thinking skills, so they can navigate the media and be informed citizens;
- they should have a good number sense and be proficient at estimation, unit conversions, and the uses of percentages;

- they should be able to read a statistical study ? or at least a summary ? and evaluate it critically;
- they should possess the mathematical tools needed to make basic financial decisions;
- they should understand the role that probability plays in their lives and be able to compute or estimate probabilities and risks; and
- they should understand exponential growth and know that it governs everything from populations and prices to tumors and drugs in the blood.

Any remaining time (or a second semester) can be filled with a wealth of breadth topics, such as risk analysis, voting, political apportionment, mathematics and the arts, and graph theory, to name only a few.

All of the above topics should be motivated and supported by examples and problems taken from the media or from current contexts. Ideal problems involve the application of relatively elementary mathematics to practical situations. They are often open-ended problems that disabuse students of the belief that answers to mathematics problems are unique and always given in the back of the book. They may involve the use of library or Internet resources for background information. And they have the goal of strengthening students' problem solving confidence and communications skills.

Assessment in QL courses should differ from traditional mathematics courses. High-pressure in-class exams tend to reveal little about students' understanding of the course material. Instead, assignments and projects that involve problem solving and verbal or written comunication provide better measures of student performance.

3. Examples and Problems

For the sake of concreteness, the remainder of this paper is devoted to a selection of problems and examples that have been used by the author in QL courses. In all cases, the necessary mathematical background has already been presented in class, and the problems represent natural applications of familiar ideas. Most problems are amenable to follow-up questions, extensions, and discussion.

1. Marie and Alex just paid $250,000 for a house. They made a down payment of $50,000 and assumed a 30-year $200,000 mortgage with a fixed annual interest rate of 7.50%. The house will serve as a residence for several years, but Marie and Alex also view it as an investment, as property values in the neighborhood are projected to increase at a rate of 5% per year in the near future. Suppose the couple sells the house after eight years. Neglecting tax implications, discuss whether they come out ahead in their investment. Explain any assumptions you made in the analysis.
2. The following ballot initiative appeared before Colorado (USA) voters in 1992:

   > Shall there be an amendment to the Colorado constitution to prohibit the state of Colorado and any of its political subdivisions from adopting or enforcing any law or policy which provides that homosexual, lesbian, or bisexual orientation, conduct, or relationships constitutes or entitles a person to claim any minority or protected status, quota preferences, or discrimination?

   What does a yes vote mean?
3. The notes on a standard musical scale have the property that the frequency of the notes doubles every 12 steps (for example, middle C has a frequency of 260 cycles per second and the C that is 12 steps higher has a frequency of 520 cycles per second). Explain why the frequencies obey an exponential growth law and find the formula for the frequency of the note that is n steps above middle C, where $n = 0, 1, 2, 3, \ldots$
4. You've been charging your expenses to a credit card, and have accumulated a balance of $5000. Your credit card charges an annual interest rate of 18%. Assume you charge no additional expenses to your credit card.

(a) Suppose the credit card requires that you make minimum monthly payments of \$70. With minimum payments, how long will it take to pay off the balance?

(b) How long will it take to pay off the balance with payments of \$100 per month?

5. Suppose you pay a total of \$15.50 for a new DVD, after taxes. Assuming a local sales tax rate of 7.5%, what is the retail (before-tax) price of the DVD?

6. In Australia in 2004, there were approximately 210,240 births, approximately 136,660 deaths, and a net increase of 126,140 people due to immigration. What was the average rate of change of the population in units of people per hour?

7. Last year, it was bad news: the value of your investments plunged by 50%. This year was much better: your investments increased in value by 75%. Over the two-year period, have you gained or lost? Explain.

8. An election features three candidates: Smith, Jones, and Webb. Among the actual voters, Webb is by far the most disliked candidate; in fact, 60% of the voters oppose his election. Is it still possible for Webb to win? Explain.

9. A football league tests all its athletes for drug use, using a drug test that is 98% accurate. That is, it correctly gives a positive result for 98% of the drug users who are tested, and it correctly gives a negative result for 98% of the nonusers who are tested. Suppose that 1000 athletes take the test and 50 of these athletes are actually using drugs. What percentage of the positive tests are false positives (nonusers who test positive)?

10. The world population is currently about 6.3 billion and increasing at a rate of 1.3% per year.

    (a) If this rate remains constant, how long will it take the population to double in size?

    (b) At this rate, estimate the world population in the year 2100.

11. Suppose you are visiting an American market and see tomatoes priced at \$1.50 US per pound. Assume that 1 kilogram = 2.2 pounds and that the current exchange rate is \$1 AUS = \$0.74 US.

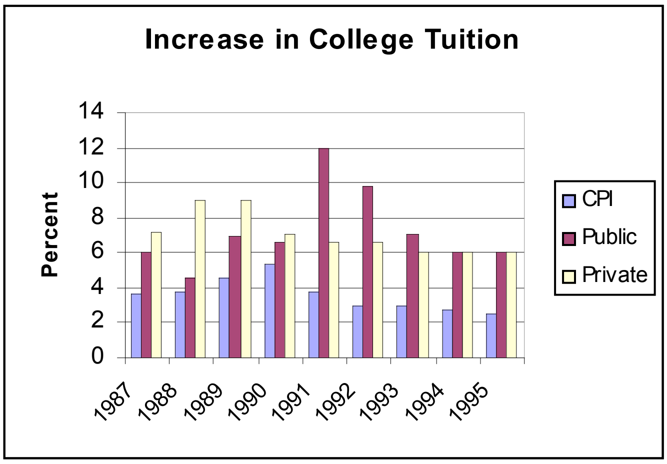
    (a) Which is worth more, \$1 US or \$1 AUS?

    (b) What is the price of the tomatoes in dollars per pound?

12. Suppose you purchase 10 tickets for a lottery in which the probability of winning any prize on a single ticket is 1 in 30. What is the probability that all 10 tickets are losers?

13. The probability of having a false result (false positive or false negative) on a mammogram is 0.07. What is the probability that in ten annual mammograms, at least one false result will appear?

14. The figure shows the increase in tuition at U.S. public and private colleges and universities between 1987 and 1995. The increase in the consumer price index (CPI) over the same period is also shown.



(a) In what year between 1987 and 1995 was the tuition at public schools the greatest? Explain.

(b) Which increased more between 1987 and 1995, tuition at private schools or the cost of living? Explain.

(c) Have there been any years in which tuition decreased at either public or private colleges? Explain.

4. Conclusion

The development of effective QL courses and programs is happening, albeit on a slow time scale. The pace will quicken if QL is given the attention lavished on more advanced mathematics courses. In the end, the development of QL initiatives should be a collaborative and interdisciplinary effort, in which mathematics departments take the lead. The result will be a rising tide of quantitative skills among all students, which is a worthy cause to be sure.

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Mathematics for teaching: some issues, some reflections

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This paper reflects upon issues concerning the mathematical background of tertiary students, whose intended vocation is mathematics teaching. Three themes form the basis of the discussion; the mathematical knowledge of undergraduates, some implications of technology for both learning and teaching, and opportunities provided by mathematical modelling as a means of enabling prospective teachers to generate examples of mathematics in use. Historical comment and research data are used to argue that intending teachers need to engage foundational ideas in mathematics in ways different from those intending to use mathematics in other pursuits. Issues raised in the literature relating to person – technology – mathematics interactions, are reviewed for insights relating to both mathematical quality and pedagogical implications. Aspects of mathematical modelling, suitable for the education of prospective teachers, are introduced and their potential to enrich the teaching repertoire considered. Creating opportunities and environment, for the provision of a quality mathematics background to support a future teaching focus, is raised as an ongoing challenge.

Concern about the pre-service mathematical preparation of teachers of mathematics appears to make periodic appearances in the public consciousness, for example when issues of shortages or unqualified practitioners gain currency. Within the academic profession it is (or should be) of constant interest, given that it not only impacts on the mathematical literacy of the workforce and the public, but on the type of student who enters undergraduate programs. Over the years, in Australia as in some other countries, the mathematical background education of teachers has been entrusted to different sections of the tertiary education sector – from single purpose teachers colleges, to faculties within Colleges of Advanced Education (responsible both for content and training in pedagogy), to Schools of Mathematics and Schools of Education in universities. Counterbalancing arguments have been provided to support the various alternatives, for example perceived advantages of a coherent group with a single vocational destination on the one hand, versus the enrichment provided by membership of a disparate group with a variety of vocational aspirations, on the other. And it would be foolish to think that matters of dollarship (rather than scholarship) have not influenced decisions presented in more altruistic terms. This paper presents a reflective discussion, based around a substantial period of time spent working in and around mathematics and mathematics education in one form or another. It has included periods spent as a teacher of mathematics on both sides of the secondary-tertiary interface, involvement in postgraduate teacher education programs, and in researching aspects of teaching and learning mathematics, again on both sides of the secondary – tertiary divide. The main substance of the paper addresses three components, felt to be of substantial contemporary importance within the content training of teachers. Of course these choices reflect to some degree personal background, values, and experience, and readers will have other priority areas of significance to add to such a discussion.

1. Reflections on Content

Comments concerning the mathematical background and abilities of students in undergraduate programs have featured in the literature of mathematics education for more than 30 years. For example [1 – 3] provide examples from an era that some today like to refer to as 'the good old days'. Writing in the context of a third-year mathematics course at the University of New

South Wales, Gray had this to say:

> Although these students had been 'exposed' to the calculus and limits for some four years – in particular they had 'passed' high school and university courses, in spite of this most of the students were incapable of formulating a simple argument in analysis, but more seriously possessed almost no intuition on the subject. To many of them mathematics was (and probably still is) a formalists dream come true, involving the use of routine algorithms applied to meaningless symbols to deduce routine answers to even more meaningless questions. Moreover they were inaccurate formalists, often misapplying their rules, theorems and algorithms... After twelve years of schooling followed by two years of university they had all but accepted the mindless mathematics that had been thrust upon them. Few enjoyed mathematics, most simply wished to get their degree and get out – as schoolteachers, whence the uncritical cycle is propagated. [3]

And from later sources:

> ... weaker students suffered from the continued misinterpretation that algebra is a menagerie of disconnected rules to do with different contexts. [4]

> In attending module after module, students tended to 'memory dump' rather than to retain and build a coherent knowledge structure... Their presumed examination strategy resulted in such a fragile understanding that reconstructing forgotten knowledge seemed alien to many taking part. [5]

So what kinds of mathematical 'knowledge' do undergraduates bring with them to tertiary courses? In a sequence of studies in London and Brisbane [6], a purpose was to identify (mis) understandings associated with different kinds of mathematical tasks.

1.1. Diagnosis of undergraduate (mis) understandings

In doing mathematics, learners may merely be required at times to complete a mechanical routine, at another level they may be required to interpret information in order to reach a conceptually based conclusion, while at a higher level again is the requirement to construct a solution. Typically this will involve the creation of new links, and interplay between concepts and procedures that must be generated as part of the solution process. With this in mind items were structured in terms of the three categories: (a) mechanical (b) interpretive (c) constructive.

1.1.1. Mechanical items

The items in this group required students to perform some standard procedure that is cued in the wording of the question.

e.g. **Item 1M** $x^2 - ax + 12 = 0$ represents a family of equations. Four members of the family are obtained by giving 'a' the values 5, 6, 7 and 8. For what values of a can the equations be solved by factorising the left-hand side?

A. 5 only B. 7 and 8 C. 6 and 7 D. 8 only E. none

The solution involves either factorising the left-hand side for the given values of 'a', or evaluating the discriminant, and selecting those values of 'a' for which it is a perfect square. In terms of operations a direct cue is provided (factorise), an activating condition that does not have to be inferred from the mathematical context, and students are tested simply on the proficiency with which the nominated procedure can be used.

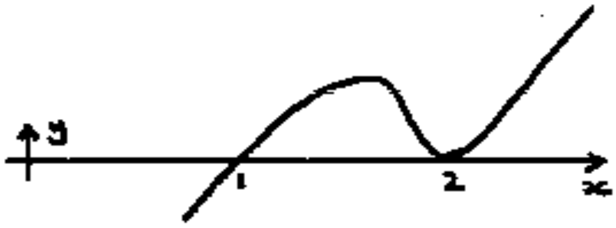
1.1.2. Interpretive items

The items in this group required the retrieval of conceptual knowledge and its application to identify a correct alternative - these items do not involve mathematical procedures.

e.g. **Item 2I** Which of the following could be the equation of the graph shown?

A. $y = (x-2)^2(1-x)$ B. $y = (2-x)^2(1-x)$ C. $y = (x-2)^2(x-1)$
D. $(x-1)^2(x-2)$ E. none of these

Reasoning such as the following is required. Since a double root occurs at $x = 2$ and a single root at $x = 1$ the equation is either $y = (x-2)^2(1-x)$ or $y = (x-2)^2(x-1)$, noting the equivalence of alternatives A and B. Since for large x the graph behaves like $y = x^3$, alternative C is correct.



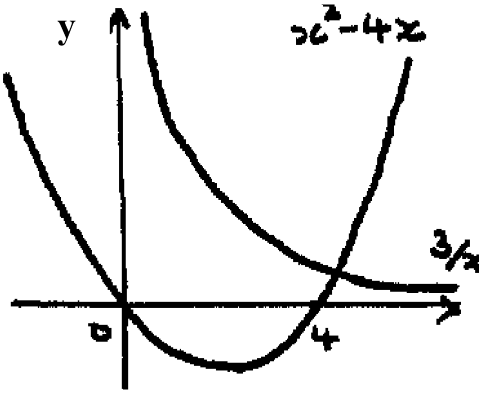
In terms of operations the student needs to draw upon knowledge about the behaviour of polynomial graphs for large x, about the algebraic meaning of single and repeated roots shown graphically, and about the equivalence of perfect squares. They must then coordinate the knowledge to eliminate alternatives systematically, which requires access to a well-developed network of linked conceptual knowledge, but no actual mathematical procedure needs to be completed. Coordination of algebraic and graphical forms is essential for success.

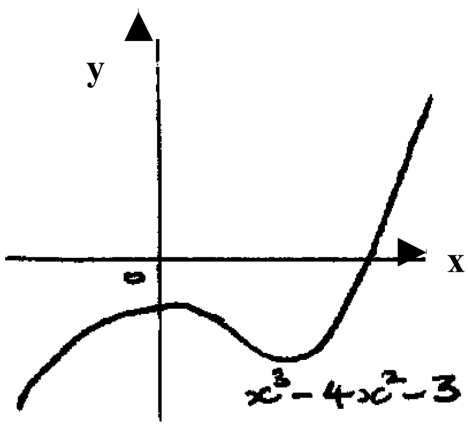
1.1.3. Constructive items

These items involve the use of both concepts and procedures in which the necessary procedures have to be introduced by the student - hence responses involve the construction of a solution rather than the selection of an alternative.

e.g. **Item 2C** The equations of two graphs are $y = 3/x$ and $y = x^2 - 4$. Obtain a cubic equation whose solution gives the x-coordinate of the point(s) of intersection of these two graphs. How many positive roots does this equation have?

The solution involves recalling that the required equation is obtained by equating $3/x$ and $x^2 - 4$ (concept), simplifying $3/x - x^2 - 4$ to provide a cubic equation in some form (procedure), sketching a graph such as one of those in the Figure below (procedure), and recognising that one intersection to the right of zero means one positive root (concept).





In terms of cognitive operations the solution involves interplay of conceptual and procedural knowledge [7]. The recognition of relevant concepts activates the formal conduct of procedures where successful use involves further knowledge of algebraic simplification rules, the shape of the respective graphs, and the interpretation of the intersection in terms of the solution to the question. It is the internal activation of procedures through the exercise of conceptual knowledge that causes constructive items to differ from mechanical items (where procedures are externally cued) and from interpretive items (which involve concepts but not procedures). The term constructive is used to indicate that the student must act to set up a solution framework rather than respond in a singular way to a precisely targeted question.

In terms of student outcomes (N = 423), the average proportion of correct responses on the respective sets of items were *mechanical* (0.41); *interpretive* (0.30); *constructive* (0.19). This confirmed the conjecture that as far as performance is concerned: *mechanical* > *interpretive* > *constructive*. And of course, competence with interpretive and constructive type activities, are essential components of the toolkit of an informed mathematics teacher.

1.2. School leaver and graduate comparisons

It is interesting to link the above outcomes across time with those from a much earlier study [8]. In this study a set of items were completed by entering undergraduates (school leavers) at three Australian universities (Queensland, Sydney, and Western Australia). At the same time, graduates entering postgraduate mathematics teacher education programs at the same universities completed the identical set of items. The items had a multiple-choice format, in which the selection of alternatives provided insight into which particular misunderstandings were driving the choices made. This detail is not provided here, but the essence of some items is presented below.

(a) $\sqrt{4} =$

(b) Noting that $\frac{d}{dx}(\cos x) = -\sin x$, evaluate $\int_{\pi}^{3\pi/2} \sqrt{(1-\cos^2 x)}dx$

(c) Sketch the graph of $y = \sqrt{(4-x^2)}$

Of interest is the combination of responses given across these three items: for example a substantial number of both graduates and school leavers answered ± 2 for (a), -1 for (b), while most (graduates anyway) knew the graph in (c) is an upper semi-circle.

The purpose here is not so much in highlighting wrong answers (although these are of interest), but in noting how conflicting answers live happily side by side without apparently creating the least curiosity or challenge. That is a significant number gave answers of ± 2, to the first one, -1 to the second one, and by implication $\sqrt{4} = +2$ at the highest point on the third one. Of those giving an answer of +2 to (a) the majority gave an answer of -1 for (b) even though the solution involved integrating a positive integrand between positive terminals. And while ± 2 was prevalent in (a) no answers to (b) contained dual possibilities.

This graphically confirms how for many students mathematics is a piecemeal collection of results, with little attention given to observing consistencies or internal contradictions, let alone seeing resolution of the latter as important. And this feature was almost as prominent among graduates of mathematics degrees as among school-leavers.

So what does diagnostic testing of these types tell us? It reinforces, among other things further comments by Gray:

> Misconceptions, misguided and underdeveloped methods, unrefined intuition tend to remain; assignments, corrections, solutions, tutorials, lectures and examinations notwithstanding. [3]

Mathematics for many (most?) is treated as a set of mechanical procedures, mathematics is viewed piecemeal, with little apparent concern for consistency across its various facets.

1.3. Implications for teacher education

Such inferences of course have implications for undergraduate teaching in general – but perhaps special meaning when considering the mathematical preparation of teachers. The severe pressures that exist within institutions make it almost impossible to provide mathematics courses that target the needs of future teachers, in ways comparable to the provision of mathematics courses for engineers or economists. But within our course structures we need to find some way to reach those with teaching intentions, in order to shake the foundations of understanding that will otherwise continue to reproduce the uncritical approaches to mathematics such as those that we have identified. This implies designing tasks and approaches that will require students to engage with fundamental ideas, concepts, and procedures, in ways that confront their understanding and past misconceptions as part of the learning process. Exemplary presentation of content alone offers no guarantee that these needs will be met, and foundational (mis) understandings substantially shifted. So we need examples and approaches that engage concepts and procedures in ways that are specifically pertinent to teacher preparation. One approach (use of vignettes) has the advantage of embedding the teaching approach in the context of recognisable classroom activity. Here is an illustration that had its genesis among a group of students studying senior level mathematics.

> Find the oblique asymptote of $y = \frac{(x^2-x-2)}{x+2}$.
>
> *Student A*: By division $y = (x-3) + \frac{4}{(x+2)}$
>
> As $x \to \infty$, $\frac{4}{(x+2)} \to 0$ so the asymptote is $y = x - 3$;
>
> *Student B*: Dividing Num and Den by x, $y = \frac{(x^2-x-2)}{x+2} = \frac{(x-1-\frac{1}{x})}{(1+\frac{2}{x})}$
>
> As $x \to \infty$, $\frac{1}{x} \to 0$ so the asymptote is $y = x - 1$;
>
> Students A and B asked for resolution

An appropriate approach to the mathematics might look something like the following:

Find the oblique asymptote of $y = \frac{(x^2-x-2)}{x+2}$.

Student A: By division $y = (x-3) + \frac{4}{(x+2)}$ so $|y - (x-3)| = \frac{4}{|(x+2)|}$

As $|x| \to \infty$, $\frac{4}{|(x+2)|} \to 0$ and hence $|y - (x-3)| \to 0$ as $|x| \to \infty$;

So the asymptote is $y = x - 3$;

Both students have used 'folksy' mathematical arguments. Student A has got away with it as far as the answer is concerned, while Student B has come unstuck in an attempt (probably unconsciously) to use the result: $\frac{\lim f(x)}{\lim g(x)} = \lim \frac{f(x)}{g(x)}$ in the absence of satisfying the requirement that both limits in the left hand expression exist (as well as the lower limit $\neq 0$). Here the limit in the numerator does not exist.

Limit theorems can be pretty dry fare; here is a genuine application (in the sense of an actual exchange between students in a real classroom) that can be used to engage prospective teachers with the relevant mathematics e.g. Resolve the difference between the students, identify the fallacy, and design an explanation. There is a special need to produce examples that challenge basic understanding across the spectrum of mathematics important as a foundation for teaching.

2. Reflections on technology

We can agree that technology is entwined with education in mathematics, powerfully, emotionally, and in some respects controversially, with respect to syllabus content, hardware preferences, and didactic choices. It is salient to recall some words of James Fey well over a decade ago, as he mused on speculative writing about the revolution predicted to follow from the application of various calculating and computing tools. In drawing attention to the paucity of actual data available to back extravagant claims he observed:

> It is very difficult to determine the real impact of those ideas and development projects in the daily life of mathematics classrooms, and there is very little solid research evidence validating the nearly boundless optimism of technophiles in our field. [9]

We note philosophical and practical issues that arise when technology is employed as a power tool, by those already knowledgeable in the mathematical field of application, versus its deployment as a learning tool within coursework where its purpose is to assist the teaching and learning of unfamiliar content. Problems compound when these two quite different purposes are confused, in particular if technology is viewed simply as a replacement or extension of hand-methods, as if no cognitive and affective implications are added by its presence. The success or failure of any teaching approach resides ultimately in the quality with which students engage the learning mediums provided, and the extent to which mathematical integrity, rather than medium specific properties is the final arbiter of understanding and quality.

2.1. Technical issues

Let us take as axiomatic that technology in the hands of expert users greatly enhances their capabilities to solve problems, and focus rather on issues emerging when technology is invoked for teaching-learning purposes. Increasing numbers of studies have targeted aspects of the student-mathematics-technology connection. Several of these have focused on properties of programs and technical issues of symbolic representation [10,11]. The former illustrates, for example, how various Maple versions gave misleading or inaccurate results, or accurate but incomplete results that are misused or misinterpreted by novice users. Examples included wrong results from software commands due to bugs in underlying mathematical algorithms, problems in dealing with equations that contain floating-point numbers, problems evaluating limits, and with integration when different commands utilise anti-derivatives and limits of sums respectively leading to different outcomes. Noss [11], demonstrates that spreadsheet generated graphical representations include both expected (traditional) and bizarre representations, showing, for example, how an 'arbitrary' change of variable from $y = 3x + 4$ to $z = 3x+4$ leads to a far from arbitrary change of output from 2-d to 3-d representation. These problem properties, inviting intellectual engagement, are generated by the mathematics-software interface independent of the presence of human learners.

Another group of studies highlights issues that emerge when students are required to actively engage with a graphical calculator or a symbolic algebra program. Zorn [12], observes how:

> we've all seen students floating untethered in the symbolic ether, blithely manipulating symbols but seldom touching any concrete mathematical ground. [12]

The issue here is on how students handle tensions between competing authorities - machine output, and mathematical integrity [13-15]. Here authors in [13] refer to student preference for calculator output over contextual reality, whereby students insisted on working with 6 decimal places on problems whose data involved using crude measuring devices, while those in [14] report on the erroneous impact of graphical representation of functions. Left alone to experiment students inferred results that were wrong - for example the number of solutions to the equation $\tan x = x$ deduced from a screen display featuring six intersections. This gap between real mathematics and the image of mathematics depicted by a screen suggests the emergence of a new 'tyranny of the screen' as an authoritative source, replacing, and possibly more insidious, than the traditional 'tyranny of the text'. In a related vein [15] identifies problems associated with the interpretation of output from symbolic manipulator (Derive) software, in which rather than building mathematical meaning from screen feedback, students' perceptions were invaded by the properties of the software. Thus in noting the output of the 'Expand' command on the square of algebraic sums, rather than focusing on mathematical regularities such as the number of terms in the sum and in the expansion, students focused on the order of terms in the expansion. This is a regularity linked only to the software, and has no mathematical significance.

Artigue [16], has organised outcomes such as the above in efforts to theorise further about interactions at the machine-mathematics interface. She develops the concept of *instrumental genesis* that incorporates two aspects - firstly the process of transforming an instrument for specific uses, and secondly the development of techniques of instrumented action that lead to effective responses to given mathematical tasks. Artigue argues that any techniques (other than trivial) require an accompanying theoretical discourse. Such discourses are well known for pencil and paper techniques where they are framed, in part, by syllabuses, text books, and associated resources, but a discourse needs to be constructed for instrumental techniques, and this calls on knowledge beyond that available solely within the standard mathematics culture.

2.2. Researching student – mathematics - technology - connections

Within an Australian context (e.g. [17]) research has been conducted into the implementation of a CAS (*Maple*) in first year undergraduate teaching. Foci include interest in the range of questions raised by students as they work with software [18], and in the links between computer-controlled processes and their mathematical underpinnings ([14,15,16,19]). The research has encompassed two main purposes: (a) to classify the range of student-generated questions that emerge when students engage with mathematical content in a symbolic manipulator environment, and (b) to identify structural properties associated with the software environment that can be identified as linking task demand and student success. Data sources included firstly, tutors' diaries, in which were entered examples indicative of the range of questions raised by students in the course of their workshop activity, and secondly outcomes from a test containing questions addressed with the assistance of *Maple* in its laboratory context. Performance was analysed in terms of the influence of two constructs labelled *syntax* and *function* respectively. Syntax: refers to the general *Maple* conventions necessary for the successful execution of commands, including the correct use of brackets in general expressions, and common symbols representing a specific syntax different from that normally used in scripting mathematical statements (such as *, ;, Pi, g:=). *Function:* refers to the selection and specification of particular functions appropriate to the task at hand, where specific internal syntax required in specifying a function is regarded as part of the *function* component. Complexity is then represented by a simple count of the individual components required in successful operation.

The success rate was given by the fraction of students ($N \sim 250$) obtaining the correct outcome where we can regard these as providing a measure of the probability of success of a student on the respective questions. A linear regression performed using these probabilities as measures of the dependent variable (success), and *syntax* and *function* as input variables found that both the *syntax* ($p < 0.05$) and *function* ($p < 0.01$) complexity measures contributed significantly to task demand, and accounted for more than 70% of variance [17].

A total of over 1300 questions indicative of the range of concerns displayed by students when working mathematically in the *Maple* environment were assembled from the tutor diaries. After eliminating problems caused by typographical errors and simple procedural issues the number of questions in which some aspect of *Maple* was unequivocally involved exceeded 80%. Specifically these involved: resolving a syntax error (27%), problem with function choice or specification (21%), stuck on mathematics (17%), procedurally stuck on Maple (22%), problem with interpreting aspects of output (13%). Apart from those requiring interpretation of text these may be generally classified as those that are software related (syntax and symbols); those that are mathematical in the traditional sense; and those that are generated by the interaction of mathematics with software (function choice and specification). Overall the intensity and scope of student questioning escalated in comparison with traditional workshop activity. So while facilitating more rapid and efficient closure to algorithmic procedures the use of *Maple* did not reduce the need for the mathematical attributes of understanding and attention to detail, as noted from the significant impact of *syntax* and *function* on success rate. *Syntax* errors penalised those who lacked sufficient care in expressing their work symbolically, while the demands imposed by *Function* were proportional to the principles and sophistication of the associated mathematics.

So in reflecting on the comments of Fey [9], we can now identify a literature that tells, on the one hand, a cautionary tale about realities and expectations of technology enhanced learning. On the other hand the same literature has begun to provide a rich base from which to theorise and test new appreciations of what is involved when students, technology, and mathematics connect in learning settings.

A different group of studies [19 – 22] help to illuminate further the distinctions noted in the findings above. These studies, while independent of each other, share a common focus in that free student comment is a major source of data. A central feature is the eliciting of student opinion about the value of using technology in the pursuit of mathematical understanding and achievement. The following direct comments (or researcher reflections), illustrate typical responses.

> They wanted to take nothing as given, retaining direct responsibility for, and detailed awareness of, every part of a mathematical process. [20]

> Some older pupils have been reported as reluctant to use the graphic calculator because they see doing so as involving a loss of their intellectual autonomy, in particular as surrendering control of the development of a mathematical argument. [21]

> I just don't understand what I'm learning here. I mean all I have to do is ask the machine to solve the problem and it's done. What have I learned? [19]

> IT is okay for drawing graphs. . . but I feel I still have to understand the principle behind the maths. I have a mental block against performing like a trained 'circus animal' and just pressing the right buttons. I need to know why? What for? What am I trying to find out? [22]

> A lot of people in higher education use computers to achieve an answer, but a lot of them feel cheated in not fully understanding the equivalent pen and paper method. . . Just using technology to achieve an aim, whilst very useful, is not understanding. [22]

So it appears that students remain very aware of how the use of information technology can impact on the quest for mathematical understanding, and this reaction is borne out in studies that researched attitudes among first-year undergraduate students in the UK and Australia learning mathematics through CAS. [23,24]

A factor analysis of correlated data between scales of motivation and confidence for mathematics and computers respectively, yielded a two-factor solution confirming that the *computer* and *mathematics* related scales occupy different dimensions, with a *computer-mathematics interaction* scale loading strongly on the computer dimension.

These outcomes were confirmed independently among students of different cohorts in different years and in different countries, and subsequent work including adaptations of these scales have been undertaken with undergraduates in Australia [25], and Spain [26]. Their findings confirmed that attitudes to mathematics and technology occupy different dimensions, so that any assessment of the power of technology to revolutionise the teaching and learning of mathematics neglects this profoundly important element at its peril.

2.3. Implications for Teacher Education

When we reflect questions emerging from the foregoing discussion onto needs associated with the preparation of mathematics teachers challenges abound, for it seems likely that central issues are at risk of falling between cracks in the educational pavement. Undergraduate programs necessarily focus on the achievement of mathematical ends, without the luxury of addressing issues raised at the mathematics-technology interface. Similarly teacher education programs, with a focus on pedagogy, have limited opportunity to address such matters even though they impinge directly on the core business of teaching. Without the development of a deep discourse involving each of the elements (people, mathematics, technology) it seems unlikely that all issues of central importance will be surfaced for analysis and debate. This field is particularly prone to prejudice (in several directions) and subjective preference, and creating space for the design and implementation of courses for teachers that combine mathematical, technical, curricular, and human components, to the level of need implied by the emerging literature in the field, has not at this point been conspicuously achieved.

3. Reflections on Mathematical Modelling

It is now more than thirty years since Henry Pollak [27], then at Bell Telephone Laboratories, challenged the mathematical education community to more seriously engage with genuine applications of mathematics. His point was that most examples in textbooks are not genuine ‘applications’, and that students leave school or university with no sense of how the mathematics they have learned can be used for purposes other than answering textbook and test questions. At the outset I want to make clear that proficiency in the techniques of mathematics is an important goal, but that it is a necessary rather than a sufficient outcome for the amount of engagement in mathematics required of students over many years of formal education.

3.1. Word problems versus modelling problems

Certainly it would be fair to say that word problems are widely construed as close relatives of modelling problems. Such links are made on the basis of semantic content, since both word problems and modelling problems are couched in verbal clothes. However in other ways the two may differ markedly, specifically with respect to meaningfulness and purpose. Modelling problems have real-world connections, and associated assumptions, that word problems often do not have. For example:

> A take-away food shop sells hamburgers, sausages, and pizzas. On one day the number of hamburgers sold was three times the number of pizzas, and the number of sausages sold was five times the number of pizzas. The number of hamburgers and pizzas sold was in total 176. How many of each type of food was sold? (2004 source)

While this problem is couched in the language of the real world there is no sense in which it represents how a vendor would make decisions essential to her/his livelihood. It does not show how mathematics can be applied to enhance understanding of real problems. The point here is that this very issue (and problem type), a major motivation behind the article by Pollak more than 30 years ago, remains an issue today. As was stated then, word problems have value in curricula and we can learn from advances made in the understanding of how students cope with them – that is they have a place. However if they are to play an assisting (rather than a subverting) role in helping students to apply their mathematics to real problems, it is essential to clarify those additional or different features that characterise examples that genuinely involve applications and modelling. Clearly this remains unfinished business. Here is a contemporary example of a problem, similar to those identified by Pollak, purporting to be an application of mathematics.

> The height of a species of hardwood is given by $y = x^2/20(1 - x/60)$, where y is in metres and x is the time in years after the sowing of the seeds. What is the latest time at which such a tree should be harvested?

We have a situation in which the tree would reach a maximum height and then shrink, eventually growing downwards into the ground. It is not only mathematically untenable, but also reinforces beliefs about mathematics having little to do with reality. Unfortunately it is all too easy to produce such examples - by taking any given formula and concocting some story around it.

Pollak also drew attention to the presence of whimsical problems:

> Two bees working together can gather nectar from 100 hollyhock blossoms in 30 minutes. Assuming that each bee works the standard 8 hour day, 5 days a week, how many blossoms do these bees gather nectar from in a summer season of 15 weeks?

"The function of such problems. . . provide comic relief in the Shakespearean sense, and probably do a lot of good – although not as applied mathematics". [27]

And some contemporary examples:

> Two meatballs roll off of a pile of spaghetti and roll toward the edge of the table. One meatball is rolling at 1.2 m/s and the other at 0.8 m/s. They fall off the table and land on a $5000 Isfahan carpet. If the table is 1.2 m high, how far apart from each other do the meatballs land? (2004 source)

> Pythagoras was sitting in calculus in the 80-degree weather of mid-April, wishing he were at the beach. While daydreaming, his terrible case of senioritis took over and his grade quickly began to plummet. When he got his final report card, he saw his grade had decreased. The amount it decreased is equal to the volume of the solid bounded in the first quadrant by $y = 2 - x^3$, revolved about the x-axis. If he started the 4th quarter with a 69, by how much did his grade decrease and will he pass the class with a 60 or better? (2005 source)

Such problems continue to abound, are arguably good fun, but provide minimal value as far as learning to apply mathematics is concerned. If they are all that intending teachers engage with, then there will be no progress along this dimension.

3.2. Models of Modelling

Theoretical positioning has substantial implications, and there appear to be currently two (at least) substantially different philosophies that influence the way that mathematical modelling in education is approached. The term philosophy is a strong one, but is used here to indicate that different belief systems lead to different priorities, and the approaches summarised in the following have distinct ontologies and epistemologies, that when unpacked lead to different teaching methodologies and emphases. They represent a further crystallisation of positions described some years ago in a major survey paper that reviewed the then state of the field [28].

1. One approach treats mathematical modelling as subservient to other curriculum purposes.

   > The curricular context of schooling in our country does not readily admit the opportunity to make mathematical modeling an explicit topic in the K-12 mathematics curriculum. The primary goal of including mathematical modeling activities in students' mathematics experiences within our schools typically is to provide an alternative – and supposedly engaging – setting in which students learn mathematics without the primary goal of becoming proficient modelers. [29]

2. The second position views mathematical modelling as a means of reaching out to provide students with abilities that are relevant to their mathematical learning, but that also

enable them, as a major goal, to learn and apply problem-solving skills to situations that arise in life outside the mathematics classroom (commonly referred to as real-world problems).

> Starting with a certain problematic situation in the real world, simplification and structuring leads to the formulation of a problem and thence to a mathematical model of the problem… It has become common practice to use the term mathematical modelling for the entire process consisting of structuring, mathematising, working mathematically and interpreting, validating, revisiting and reporting the model. [30]

The overall purpose here is to enable students to access their 'pure' mathematical knowledge for addressing problems relevant to their world, and to focus on how this can be successfully achieved. This view of the purpose of modelling is driven by convictions that it is unsatisfactory for students to 'bank' mathematical knowledge for 10, 12, or 15 years and yet be unable to 'withdraw the funds' for purposes other than answering standard questions, and performing on formal tests designed as gatekeepers to the next level of study.

3.3. Modelling problems

One of the most vivid demonstrations of the power that modelling ability conveys appeared recently in the Presidential Address of the IMA President (Pedley [31]). In this paper he described how G I Taylor using only dimensional analysis and published photographs, inferred the strength of the atomic bomb tested in the New Mexico desert in 1945. Reasoning that the blast wave radius (R), could depend only on time (t), energy released instantaneously (E), and density of air into which the wave is expanding (ρ), he inferred (using no more than school mathematics) that $R = C(Et^2/\rho)^{1/5}$ where C is a dimensionless constant. Then using a series of public photographs showing R against t, a log-log linear plot enabled C and E to be estimated. Subsequently he contacted the American authorities "I see the bomb you tested had a power equivalent to about 17 kilotons of TNT". Result – apoplexy at the communication of what was thought to be classified information!

To provide students with the ability to invoke and apply mathematics to problems in their world, there is a need to design problems that have genuine characteristics of real-world examples, and that are accessible within the mathematical resources available to the students. This implies in problem choice a certain level of authenticity, and within the solution process a capacity for problem formulation in addition to the carrying out of mathematical procedures. Here is a collection of examples, suitable for educational implementation at different levels, which are representative of classes of problems found in the real world.

1. *Design and optimisation problems:* e.g. Product design; locating positions for supermarkets, franchises, road upgrades. Student Problem: Where should speed bumps be located when designing traffic flow through a facility such as a new school or college?

2. *Prediction Problems:* e.g. What are the future needs of a community for water storage, irrigation, reservoirs etc? Student Problem: How long will water in a farm dam last for a herd of livestock, if there is no more rain?

3. *Problems that involve resolving apparent anomalies (or intuition can't always be trusted):* e.g. How shortages of goods in a supply chain can be traced to earlier decisions designed to avoid shortages.

   Student Problem: Prizes are awarded for the best bowling averages in the semi-final and the final of a local cricket competition, and in addition there is a prize for the best average when the results for the two matches are combined. The Table below shows the results for the two matches, in which Julie wins the awards for both the semi-final and final. Who wins the combined award? When will outcomes like this occur?

Match	Semi Final			Final			Combined
Bowler	Runs	Wickets	Average	Runs	Wickets	Average	Average
Julie	12	5	2.4	29	5	5.8	?
Robyn	10	4	2.5	18	3	6.0	?

4. *Problems that explain and/or improve existing practices:* e.g. Analysis of competitors' products, or redesign of existing products and methods. Student Problem: Which common errors will be detected by the EAN 13 (or some other) barcode system? Would other systems of calculating check digits be more efficient?

5. *Problems that identify (and perhaps help correct) injustices, inequalities, emerging social or personal needs:* e.g. Analysis of tax structures and wage movements for impact on different sectors of society; the credit card trap, housing loan alternatives.

   Student Problem: Is the claim on behalf of Wannamutta and Weranabe (see below) a reasonable one?

   > SYDNEY, March 30 2000 (AFP) – The ghosts of Australian bushranger Ned Kelly and the black trackers who helped police catch him in 1880 may be stirred by a Supreme Court action starting in Brisbane on Wednesday. Aborigines Jack Noble and Gary Owens (tribal names Wannamutta and Weranabe) had been promised 50 pounds each as their share of the then massive 8,000 pounds reward offered for Kelly's capture by the Victorian state government. They and three other Aborigines tracked the notorious Kelly gang to the Victorian town of Glenrowan where a daylong gun battle ensued between police and the armour-clad bushrangers... But because the trackers were Aborigines, the authorities refused to pay them and now their descendants are suing for recovery of the reward, which with compound interest plus damages is calculated at 84 million Australian dollars (52 million US)...

   http://www.geocities.com/cpa_blacktown/20000331nedkeafpfr.htm

The solutions to such problems, involve abilities including problem specification, making of assumptions, mathematical solution, interpretation, and evaluation that are similar to those required in real-world problems. The Table below indicates some of the mathematics and modelling aspects embedded in the respective problems.

Problem	**Mathematics**	**Modelling/contextual factors**
Placing Speed bumps	Kinematics	Bump design (max speed) Simulation (real car) Restrictions (non-math)
Farm dams	Pythagoras Trigonometry Integral Calculus	Evaporation rates Animal consumption Available data (distance to water line – not depth)
Cricket averages	Ratios, linear equations Integer solutions, (Diophantine equations)*	Bowling averages as rates Wickets available (10 max) Runs (bounded but unknown)
Barcodes	Divisibility, Numerical proofs (Modular arithmetic)*	Identify likely errors: single digit errors, digit reversal errors.
Ned Kelly injustice	Exponential growth	Colonial history (interest rates) Compensatory and punitive damages

3.4. Implications for teacher education

The capacity to design original problems is an enviable attribute for a teacher at any level. To be purely a consumer of the ideas of others produces the textbook dependent practitioner that has been so responsible for pedestrian teaching, and minimal compliance with efforts to add stimulus to the mathematics curriculum and its presentation. With an ability to design problems based on real-life contexts, comes the opportunity to engage with contemporary issues that are of interest to students, and that are mathematically rich. For example, the Morgan Spurlock film-documentary *Supersize Me* was aimed at the problem of junk food, and relates to issues of diet and fitness that are currently part of the educational agenda and likely to remain so.

Using the notion that: weight (today) = weight (yesterday) + excess calories consumed today converted to weight, we can set up a difference equation model of the form: $w_n = w_{n-1} + (I - 24w_0)/7800$, where 'I' represents the average (known) daily intake of calories, and the constants (that estimate calorie ~ kg equivalences, and calories consumed by various forms of exercise) are available from internet sources.

Insight into a variety of mathematically based questions can now be obtained, relating not only to Morgan Spurlock's 'experiment', but to data directly relevant to societal issues, including individuals.

But another, quite different issue, can be addressed from insights provided through modelling, as can be noted from the table above. This relates to utility arguments for the inclusion of particular topics (or not) using statements or questions like "A student these days will never need such-and-such". Arguments like these almost always refer to surface characteristics of problems, while the mathematics actually needed is often hidden. Referring to the above Table, farm dams for example looks like mensuration but involves trigonometry and calculus, cricket averages seems to be about arithmetic ratios, but involves integer solutions of linear equations, while barcodes involves far more than addition and division of integers, to wit strategies of proof by exhaustion. The only thing we can be really sure about is that the more thorough and networked the knowledge of mathematics, the better is an individual equipped to solve problems.

So what is needed to provide the necessary skills and experience to design creative problems with real world connections? This must be seen as one other major gap in our teacher preparation programs, with responsibility shared jointly by mathematical and educational providers.

At the mathematical level a combination of inhibiting factors tends to work against the provision of creative modelling expertise, with the exception of some institutions (e.g. in the U.K), which have been instrumental in maintaining the impetus generated in the early ICTMA[1] conferences. These were stimulated in large part by employers concerned about the lack of problem solving skills (as distinct from content knowledge) among graduates. One inhibiting factor is clearly the intensifying demands on a sector with diminishing resources, a sector already stretched by requirements to provide a wide range of service programs, as well as courses for mathematics specialists. A second factor is less defensible, namely lack of support among faculty due to a lack of understanding of the demands and purposes of modelling programs. Shifting the focus to the construction and formulation of problems, and critical report writing, produces syllabus content that is less obviously traditional tertiary content than a listing of theorems or advanced techniques. Yet the intellectual demand of such programs is high, as illustrated by a recent comment within a short course involving the construction of models appropriate to senior secondary mathematics: "I've got a PHD in pure mathematics but I'm finding this incredibly challenging".

At the educational level, again a combination of time demands and personnel issues are at work. Courses in mathematical pedagogy fight for time increasingly in crowded programs, so decisions involve choices among topics that really should be essential in such programs. And not all those engaged in teacher education are equipped to create or grasp opportunities for mathematical inventiveness should they arise, nor are they necessarily convinced that students are capable of benefiting from such intentions. Regarding the latter, there seems to be a common experience that is shared by those with a belief in the contribution that skills in mathematical modelling can provide. This is the sense of awe experienced when the latent mathematical power of students is unleashed in ways that were never envisaged – but this will never be fostered through teaching hampered by a belief that students are only capable of

[1] International Conference on the Teaching of Mathematical Modelling and Applications

performing what has been demonstrated and practised.

4. Final reflections

In the foregoing we have noted implications for teacher education within the three thematic areas selected for discussion. With respect to conceptual understanding of foundations, to the intellectual (as well as practical) demands of the people-mathematics-technology nexus, and with respect to enabling connections between learned mathematics and the real world of students, we have argued that specific needs exist for those who intend to teach. What is not clear is where, and how, these special needs can best be met, and the balance to be achieved between common content relevant to many fields, and dedicated teacher oriented coursework. However we can be certain of one thing, design of an ideal 'syllabus' alone will be no more successful than past efforts that remained satisfied with the production of a content 'blueprint'. Somehow we need to enlist the collaboration of our most gifted and insightful communicators to fashion, share, and deliver activities needed for the important and challenging focus of 'learning mathematics for teaching'.

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From a Logical Point of View: an Illuminating Perspective in Teaching Statistical Inference

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Offering perspectives in the teaching of statistics assists students, immersed in study of detail, to see the leading principles of the subject more clearly. Especially helpful can be a perspective on the logic of statistical inductive reasoning. Such a perspective can bring to prominence a broad principle on which both interval estimation and hypothesis testing are designed, and so can unify these seemingly disparate techniques in students minds. In this paper I show how to construct an illuminating perspective over this unifying principle, that experience shows is valued by intellectually-lively students. To aid readers attracted by the idea of employing this perspective in their teaching, the paper includes a skeletal version of a recommended line of presentation.

This paper can be found in the *International Journal of Mathematical Education in Science and Technology*, **36**(7), pp.801–811.

The transition from embodied thought experiment and symbolic manipulation to formal proof

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This presentation considers how different students attempt to make sense of formal mathematical proof at university based on their previous experiences. These experiences include thought experiments based on their fundamental human embodiment of ideas and experience of manipulating symbols in arithmetic, algebra and symbolic calculus. A theoretical framework is proposed which suggests three mental worlds of mathematics – embodied, symbolic and formal – each with its own distinct way of constructing mathematical concepts and verifying their truth.

1. Introduction: three distinct mental worlds of mathematical thinking

When students begin to study formal proof at university, they already have a wealth of preceding experience on which to build. In mathematics there is the use of visual diagrams, dynamic images and thought experiment on the one hand and the use of symbols of arithmetic, algebra and the calculus on the other. I suggest that formal mathematics builds on a combination of embodied and symbolic thought (Tall, 2004), (figure 1).

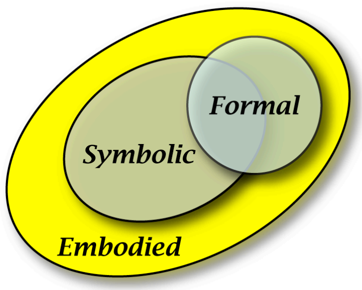


Figure 1: Formal mathematics building on embodied and symbolic thought

The terms 'embodied', 'symbolic', 'formal' are here used with specific meanings described briefly as follows:

- Embodiment refers to *conceptual embodiment* in which we reflect on our sensory perceptions and imagine relationships through thought experiment.
- Symbolism here refers not to symbols in general, but to those symbols used in mathematical calculation and manipulation in arithmetic, algebra and subsequent developments. These arise through *actions* on objects (such as counting, sharing, evaluating) which are symbolised and manipulated as concepts (such as number, fraction, algebraic expression). A symbol used dually to represent process (such as addition) and concept (such as sum) is called a *procept* (Gray & Tall, 1994).
- Formalism refers to the formal theory defining mathematical concepts as axiomatic structures whose properties are deduced by formal proof.

Tall (2004) theorizes that this categorization of mathematical thought involves three substantially different worlds of mathematics, each with its own way of building concepts and formulating relationships that lead from thought experiment and calculation to deductive formal proof:

- An object-based *conceptual-embodied world* reflecting on sensory perceptions to observe, describe, define and deduce properties developing from thought experiment to Euclidean proof.
- An action-based *proceptual-symbolic world* that compresses action-schemas into thinkable concepts operating dually as process and concept (procept), leading to proof by calculation and manipulation
- A property-based *formal-axiomatic world* of set-theoretic definitions and logical deduction leading to formal proof.

These will be referred to as *embodied*, *symbolic* and *formal* in the remainder of this paper.

Each world has its own development of deductive argument. For instance, in the embodied world, we can see that addition is commutative by re-arranging 5 objects as 3+2 or 2+3. In the symbolic world, an individual can *calculate* that the two sums give the same answer. In the formal world, $x + y = y + x$ is true *because it is an axiom*.

Furthermore, each world has its own manner of development in sophistication. The embodied world is based on sensory perception where what is perceived is then analyzed, described, defined and verbal arguments are developed to formulate inferences typified by Euclidean geometry. The symbolic world shifts from a focus on action to increasingly sophisticated procedures and on to the conceptual structure of arithmetic and the generalized symbolism of algebra. The formal world reverses experience. Instead of analyzing existing concepts to determine their properties, it *begins* with selected properties as axioms and constructs other properties of the structure through formal proof.

Each world also has its own special use of language. In the embodied world, language is used in increasing sophistication to describe properties of objects, then to categorise and define. In the symbolic world, the symbolism has a new part of speech, the procept, acting dually as process and concept in a special way that extends everyday language use of verb participles such as 'writing' in 'I am writing' to gerunds acting as nouns, as in 'writing is a mode of communication'. In the formal world, language modifies its role once again, using set-theoretic technical terms to define concepts in a special tone of meaning that Alcock & Simpson (1999) call 'the rigour prefix'.

Formal thinking necessarily builds on the students' prior experience of embodiment and symbolism. For instance the notion of vector space has an embodied foundation in physical transformations of objects in space represented by arrows in two and three dimensions and a symbolic foundation as n-tuples in $\mathbf{R}^n$. This is typical of cognitive development in which embodiment offers an insightful meaning into a mathematical concept in 2 and 3 dimensions, but requires the symbolism of $\mathbf{R}^2$ and $\mathbf{R}^3$ to imagine generalizations to $\mathbf{R}^n$ and subsequently to the formal definition of a vector space as a set-theoretic abelian group with operations by a field of scalars.

The same occurs in many areas, so that identities such as $x(y + z) = xy + xz$, $x^2 - y^2 = (x + y)(x - y)$ can have simple embodiments for positive values of x, y, z but soon become more complicated if these variables take on positive or negative values of different sizes.

In general, therefore, the trend in elementary mathematics is for successful students to move from embodiment, which is meaningful in simple cases, to symbolism that is powerful in more sophisticated contexts (Krutetskii, 1976; Presmeg, 1986; Gray, Pitta, Pinto, Tall, 1999). However, embodiments continue to have powerful effects on meaning in formal mathematics that can be beneficial in some instances and deceptive in others.

2. An example

We begin with an example of a mathematics student attacking the proof of a theorem that can be approached by a variety of different methods (Mejia & Tall, 2005). Grad is a competent student at a high-ranked university who had completed three years of study and found mathematics difficult. What is insightful from the way he operates is the way his thought is based on embodied and symbolic experience to build proofs.

Grad (a pseudonym) was given the following task (based on a problem from Raman (2002)).

Task:	Determine whether the statement below is true or false. Explain your answer by proving or disproving the statement. *The derivative of a differentiable even function is odd.*

As he read the statement out loud, Grad drew a parabolic shape in the air with his finger (figure 2) and thought for a few seconds. He continued as follows:

> First the even function. . . . I don't think the derivative can be even [. . .]. It's symmetric to the y-axis (gesturing with his hand to show a vertical axis) . . . effectively, I'm talking about two dimensional . . . so it's (err) . . . quadratic function (draws a parabola in a form similar to on the desk with his finger) . . . so the derivative is decreasing all the way (traces the parabola again) . . . so it can't get the same value twice, so it must be odd, so from that it's definitely not even.

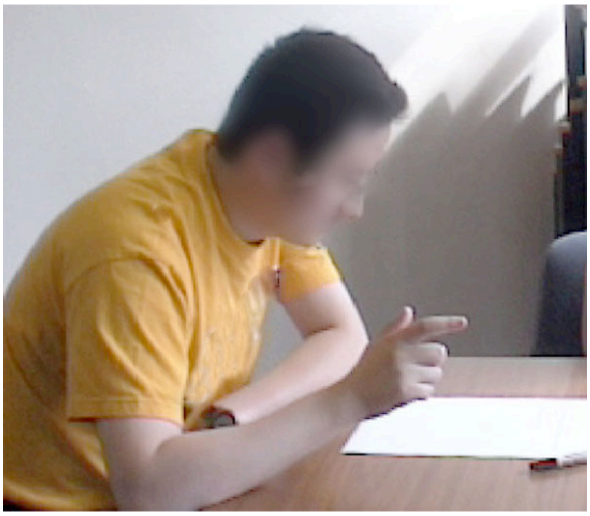


Figure 2: Grad imagines a parabola

In this excerpt he appears to shift from the general notion of even function ("symmetric to the y-axis") to a more specific even power, (perhaps $y = x^2$) and deduces that its derivative can't be even so it must be odd. This deduction is false for a general even function, but happens to be true for even and odd powers of x that seem to be his current focus of attention. Without further discussion, he concludes, saying, "generally I think it's true, but [laughs] I'm not so sure."

Analysing Grad's use of embodiment (through enactive drawing) and symbolism (through focusing on even and odd powers), we see that both lead to him making deductions that are true in specific cases but which do not hold in a general formal proof.

When asked for a proof of the statement, he uses the two-sided definition of derivative, and manipulates it to give his version of the proof (figure 3). The use of the two-sided derivative would be justified because the statement claimed that the function is differentiable, but Grad did not seem to be aware of this. His manipulation lacked fluency but he was able to give a proof that contained the fundamental essentials.

$$f(x) = f(-x)$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h} = 0$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

$$=$$

$$f'(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x-h)}{2h} = \lim_{h \to 0} - \frac{f(x+h) - f(x-h)}{2h}$$

$$f'(x) + f'(-x) = \lim_{h \to 0} \frac{0}{2h}$$

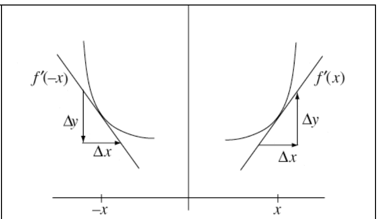
$$= 0$$

Figure 3: Grad’s symbolic solution

Grad was then shown a number of pre-prepared responses to the problem, from which we focus on three:

Response A:

If $f(x)$ is an even function it is symmetric over the y-axis. So the slope at any point x is the opposite of the slope at $-x$. In other words $f'(-x) = -f'(x)$, which means the derivative of the function is odd.



Response B:

Want to show if $f(x) = f(-x)$ then $-f'(x) = f'(-x)$.

$f'(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h}$ by the definition of the derivative.

$f'(-x) = \lim_{h \to 0} \frac{f(x-h) - f(x)}{h}$ since f is even.

Let $t = -h$

$f'(-x) = \lim_{t \to 0} \frac{f(x+t) - f(x)}{-t}$

$f'(-x) = -\lim_{t \to 0} \frac{f(x+t) - f(x)}{t}$

$-f'(x) = f'(-x)$, as required.

Response C:

Given $f(x)$ is even, so $f(x) = f(-x)$. Take the derivative of both sides.

$f'(x) = -f'(-x)$ by the chain rule. So $f'(x)$ is odd.

Grad considered the embodied visual proof (A) to show the best understanding and considered that a school pupil giving this response should be given a mark of 100% while a university student would probably get less than 50% because it needed further explanation. He thought response (B) was more formal and very convincing, but still in need of more explanation, so the mark given would “depend on the marker”. He felt Response (A) and (C) were the best arguments, suggesting that Response (C) would get full marks, because it is more convincing and has “. . . less steps, the less mistakes you can make . . . less assumptions made . . . yeah, straightforward.”

3. The role of prior experience and the notion of 'met-before'

Grad is just one individual responding to the notion of proof in his own way. However, *all* of us respond to new ideas based on our prior experiences. In Tall (2004), I introduced the notion of 'met-before' as a *current* cognitive structure that arises from previous experience and which is evoked to make sense of a current situation. What matters here is not the actual previous experience, but the way in which it becomes part of the student's conceptual structure affecting current thinking.

Met-befores are essential in curriculum building, and curriculum builders necessarily sequence topics so that later topics build on earlier ones. However, such sequences are usually conceived in a logical way, building new mathematical structures on previously introduced mathematical ideas. In practice, the student builds on his or her own personal met-befores in ways that may not fit with the intended new developments.

4. Different approaches to proof based on embodiment and symbolism

The shift to the formal world initiates students into the culture of mathematicians who base their public communications in terms of formal definitions and proof. Students trying to make sense of this new culture must build on their experience of embodiment and symbolism.

Pinto (1998) found that, when introduced to the formal limit concept in analysis, some students built explicitly on their previous embodied images in an attempt to *give meaning* to the formal idea from their current knowledge structure. She called this a *natural* approach. Others attempted to gain insight by focusing on the definition and the way it is used in formal proofs to *extract meaning* from the definition to give a *formal* approach.

Chris (a pseudonym) used a natural approach to build from his embodied imagery. He imagined the definition arising from the picture of a graph of a sequence approaching a limiting value L by noting that given any desired error ϵ, there is a value of N such that the values of the sequence to the right of N lie in the range $L \pm \epsilon$. Chris was able to build from his imagery to build relationships about limits, continuity in various contexts that both make sense to him and also satisfy the mathematical community. In some instances, such as the constant sequence $1, 1, 1, \ldots$, he could sense that it did not fit in with his met-before that a sequence *approached* a limit, nevertheless, he sufficiently confident with the formal theory to see that it fitted the formal definition (Pinto & Tall, 2002).

Ross, on the other hand, approached the task *formally*, repeating the definition until he could say it in full detail and then carefully reading proofs to see how they were constructed logically. Indeed, he made sense of the notion of limit from his experience of considering the convergence of sequences. In the case of the constant sequence $1, 1, 1, \ldots$ he thought about the *speed* of convergence and remarked that some sequences converged faster than others, and that the constant sequence converged the fastest of all. While the natural thinker Chris adds the constant sequence to his imagery as a special peripheral case, the formal thinker Ross places it *centrally* in his concept of limit.

Many students made only partial attempts at natural or formal approaches. Cliff tried to make sense of the ideas based on his embodied imagery. For instance, his image of continuity built on the met-before that a continuous graph is drawn without taking the pencil off the paper. When the lecturer proved that an arbitrary function defined on the integers is continuous, Cliff did not believe it to be true because the graph consisted of disconnected dots over integer values. He failed to overcome the conflict between his embodied imagery and the formal theory.

Another student, Rolf, built upon his experience working with symbolic calculations and saw his initial task to gain fluency at using the definition of the limit of a sequence. This entailed working with specific sequence, for example, $a_n = 1/n^2$, and for a specific numerical value of ϵ, say, 10^{-6}, then he could calculate the value of $N = 10^3$ for which $|a_n| < 10^{-6}$ when $n > N$. However, this strategy is not sufficient to give the limit concept its full logical meaning. A problem which proves difficult for such procedural students is to show that if $a_n \to 1$, then for n beyond some value N, the terms a_n must be bigger than $\frac{3}{4}$. Not knowing the formula for a_n in this problem means that it is not possible to carry out a *numerical* calculation to find N (Pinto 1998). This can mean that students who base their approach on numerical thinking can face a serious obstacle when attempting to make sense of formal arguments.

Weber (2004) refined this analysis by a qualitative case study on a particular analysis lecturer and his students. He found that the lecturer began with an initial *logico-structural* teaching style in which he guided the students into constructing a sequence of deductions to prove a theorem. He divided his working space on the board into two columns, with the left column to be filled in with the text of the proof and the right column as 'scratch work'. He wrote the definitions at the top of the left-column and the final statement at the bottom, then used the scratch work area to translate information across and to think about the possible deductions that would lead from the assumptions to the final result. Later in the course, he became more streamlined in his proofs, working in a more sequential *procedural* style, writing the proof down in the left column and using the right column to work out detail such as routine manipulation of symbols. At a later stage, he used what Weber termed a *semantic* style, teaching topological ideas building on visual diagrams to give meaning, then translating this embodiment into formal proof.

His students learning approaches were analysed into three types, building on the theory of Pinto:

- a *natural* approach involved giving an intuitive description and using it to lead to formal proof,
- a *formal* approach where students had little initial intuition but could logically justify their proofs,
- a *procedural* approach where students learnt the proofs given them by the professor by rote without being able to given any formal justification.

The term 'natural' corresponds to that of Pinto in terms of giving meaning from intuitive (embodied) knowledge, 'formal' now refers to those who are *successful* in following a formal approach and 'procedural' refers to those who attempt to learn the formal proofs by rote without either embodied or logico-structural meaning. Of the students mentioned in Pinto's research, Chris is successful in giving embodied meaning to formal theory via a 'natural' route. Ross is successful in a 'formal' approach in extracting meaning from the definitions and the logical structure of theorems. Cliff is prevented from making sense of the formal procedures because they conflict with his embodied imagery. Rolf attempts to extract meaning from the definitions based on his symbolic experience and remains 'procedural' in the sense of Weber. In essence Weber's 'procedural' route is taken by both Cliff and Rolf, but there is a difference: Cliff experienced a sense of conflict because it contrasted with his embodied ideas, Rolf was happy to learn procedures by rote. Neither of these students went on to develop a formal approach.

Weber's study has further data that shows how students can vary in approach in different contexts according to their previous experience, including the way in which theory has been presented to them. Six students interviewed after the course all responded in a natural manner to a topological question (where topology had been taught in a semantic way building from visual imagery). In two other questions about functions and limits taught in a logico-structural manner, only one student responded naturally. The other five responded to a question on functions with 4 being classified as formal and 1 procedural, and to a question on limits where 2 were formal and 3 procedural.

5. From formal definitions back to embodiment and symbolism

We now have evidence that students moving towards formal proof have a variety of ways of building on their previous experience of embodiment and symbolism as they attempt to make sense of formal proof as shared by the community of mathematicians. Tall (2002) analyses in general how previous experiences provide concept images for thought experiments that may support formal proof. In parallel, concept definitions that arise are used as a basis for formal proof of a succession of theorems. A natural approach builds formalism on intuitive embodied imagery, supported by experiences in calculation and computation. A formal approach focuses less on embodiment and more on the logical structure, (figure 4).

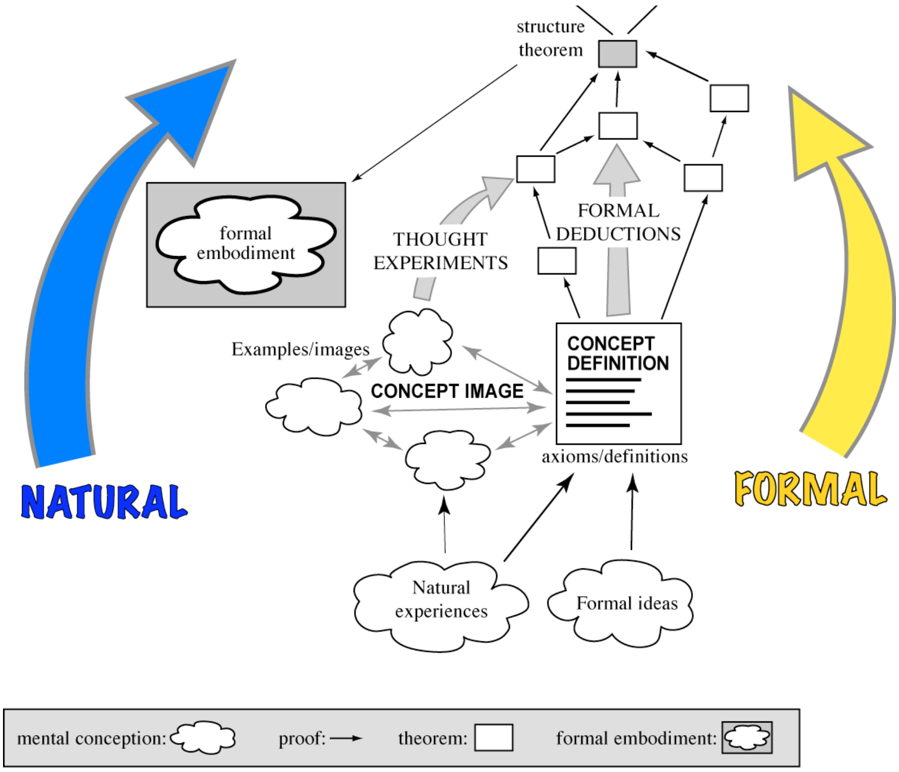


Figure 4: From embodiment to formalism and back again (taken from Tall, 2002)

However, as successive theorems are proved by formal deduction, there may come a special type of theorem called a structure theorem that gives an insight to the structure of the axiomatic system itself. Typical examples of such structure theorems are as follows:

- An equivalence relation on a set A corresponds to a partition of A;
- A finite dimensional vector space over a field F is isomorphic to F^n;
- Every finite group is isomorphic to a group of permutations;
- Any two complete ordered fields are isomorphic (to $\mathbf{R}$).

In every case, the structure theorem tells us that the formally defined axiomatic structure can be conceived an embodied way, often with corresponding manipulable symbolism. For instance, an equivalence relation on a set A – axiomatized as reflexive, symmetric and transitive –corresponds to an embodiment that partitions the set. Any (finite dimensional) vector space is essentially a space of n-tuples that can (in dimensions 2 and 3) be given an embodiment and (in all dimensions) can be handled using manipulable symbolism. Any (finite) group can be manipulated symbolically as permutations and embodied as a group of permutations on a set. A complete ordered field specified as a formal axiomatic system corresponds precisely to the symbolic system of infinite decimals and to the embodied visualisation of the number line.

Thus, not only do embodiment and symbolism act as a foundation for ideas that are formalized in the formal-axiomatic world, structure theorems can also lead back from the formal world to the worlds of embodiment and symbolism. These new embodiments are now far more sophisticated, with their structure built from formal definitions and proof.

Such embodiments may be too unwieldy to use in practical ways to prove theorems. For instance, the structure theorem that every finite group is isomorphic to a group of permutations does not really help us solve problems symbolically for larger groups because the calculations using permutations become so unwieldy that other techniques need to be developed to prove more sophisticated theorems. However, the structure theorem *does* provide an insight that illustrates the interrelationships within the framework of three worlds of mathematics.

The new embodiments may provide a springboard for imagining new developments and new theorems. However, as with all embodiments, they may have incidental properties that mislead.

For instance, the embodied interpretation of the real numbers as a number line using Dedekind cuts gave generations of mathematicians the belief that the irrationals 'completed' the real line geometrically by 'filling the gaps between rationals'. Consequently it was widely imagined that there is no room for more elements, so that infinitesimal ideas prevalent in earlier thinking were treated with mistrust. Using the formal notion of ordered field, it is a simple matter to consider fields containing the real numbers (such as the field of rational functions) and defining an appropriate order relation to construct an ordered field containing infinitesimals (Tall, 2002). What is important to note is that such ordered fields are *not* complete.

In a course which I taught (see Li and Tall, 1995), I worked very carefully to introduce a group of pre-service mathematics teachers to the idea of limits of sequences being calculated as accurately as required on the computer, introducing the ϵ-N definition of limit and the completeness axiom that an increasing sequence bounded above tends to a limit less than or equal to the bound. Half the students in the class refused to believe that the completeness axiom is true. They had not entered the formal world where concepts are specified as definitions. For them, definitions related to the world they know and they did not believe the axiom was true in their world. It was not. For their world contained imagined infinitesimal quantities and in such a world, the completeness axiom is *false*. For them there is a barrier between the worlds of embodiment and symbolism that they 'know' from their experience and the world of formal mathematics where concepts are 'defined' to satisfy set-theoretic properties.

6. Embodied aspects of formal definitions

Formal definitions are based on generative properties that are appropriate for building rich conceptual structures from minimal starting points. As mathematicians we often build 'proof-generated' definitions, such as the definition of compactness in topology, formulated to be a good starting point for proving theorems. Students shifting from the embodied and symbolic worlds with no understanding of the formal axiomatic method, are likely to build not only on the ideas that are presented to them, but also on their met-befores that they carry in their minds from earlier embodied and symbolic experience.

Embodiment is a natural mode of operation for human beings. They start from everyday experiences of counting, measuring, sorting, ordering, sharing, moving, categorising. In his famous lecture given at the turn of the twentieth century, Hilbert (1900) asserted:

> To new concepts correspond, necessarily, new signs. These we choose in such a way that they remind us of the phenomena which were the occasion for the formation of the new concepts. So the geometrical figures are signs or mnemonic symbols of space intuition and are used as such by all mathematicians. Who does not always use along with the double inequality $a > b > c$ the picture of three points following one another on a straight line as the geometrical picture of the idea "between"?

Such embodied ideas not only provide inspiration for formal theories, as met-befores they may also cause obstacles in understanding.

Definitions are built up from individual parts that I will call 'definitional elements'. For instance, an equivalence relation has three definitional elements:

R : $a \sim a$ for all a;
S : $a \sim b$ implies $b \sim c$;
T : $a \sim b$ and $b \sim c$ implies $a \sim c$.

Another possible definitional element might be:

$$\mathbf{T}^* : a \sim c \text{ and } b \sim c \text{ implies } a \sim b.$$

It is easy to prove that the definition **RST** gives the same structure as **RST***. However, the definitional elements **T** and **T*** are not the same; they have different meanings in themselves and in other systems. For instance, **T** is a definitional element for an order relation ($a < b$ and $b < c$ implies $a < c$) but **T*** is not (since $a < c$ and $b < c$ does not imply $a < c$).

7. Students constructing their own definitions

Asghari (2004) investigated how individuals who had not met the concept of equivalence might write down their own rules to formulate the structure found that an entirely unexpected law arose. His problem concerned a mad dictator who restricted travel between the ten cities in his country, so that a 'visiting-city' that one is allowed to visit must obey two conditions:

1. When you are in a particular city, you are allowed to visit other people in that city.
2. For each pair of cities, either their visiting-cities are identical or they mustn't have any visiting-cities in common.

The problem was for his officials to formulate valid visiting laws, which they demonstrate on a 10×10 grid (figure 5). While the diagonal and reflection in the diagonal occurred often, the transitive law was more opaque and several respondents suggested an alternative that might be called 'the box law' (figure 6).

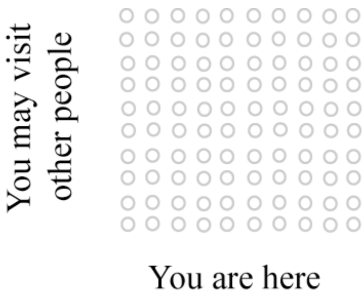


Figure 5: The problem grid

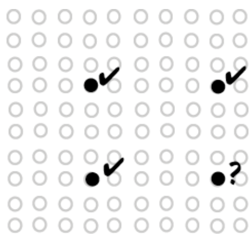


Figure 6: The 'box law'

The box law essentially says:

> If three corners of a box (with horizontal and vertical sides) are in the relation, then so is the fourth corner.

It is left to the reader to find the relationship between this box law and the formulations **RST** and **RST*** for an equivalence relation mentioned earlier. It is also interesting to see what the 'box law' means in formal terms. What it says is if two columns (or two rows) have an element in common, then the two rows have *all* their elements in common. The elements marked in the rows and columns of the equivalence relation correspond to the equivalence classes involved and two equivalence classes are either disjoint or identical.

This data shows that what students bring to formal thinking may involve sound intuitions. However, such intuitions may not be recognised and linked to the standard approach to the formal ideas. On the other hand, when students are presented with formal definitions as used by mathematicians, they may interpret them using previous knowledge in a way which is not consonant with the formal theory.

8. Students embodying standard definitions in inappropriate ways

For many years, at the University of Warwick, students were first introduced to a 'foundations' course focusing on the development of the formal elements of mathematics. Over all this time, the topic that was consistently considered the most difficult by the students was a section that began by introducing the notion of relation and then moved on to specific relations including functions, order relations and equivalence relations. We were mystified why a simple idea like an equivalence relation should provoke such a reaction when it involved only the universal quantifier "for all" with none of the apparent difficulties coordinating multiple quantifiers in analysis.

Chin (2002) investigated the situation and it became apparent that the notion of relation was embodied very differently from the notion of equivalence relation. Whereas a relation from A to B has a natural representation as a subset of $A \times B$ and this is inherited by the concept of function from A to B, many students see an equivalence relation on A arising in the set A, rather than being represented as a subset of $A \times A$. He realized that the reflexive law **R** is easily embodied as the diagonal of $A \times A$ and the symmetric law **S** as reflection in the diagonal, but the transitive law **T** is altogether more subtle (figure 7).

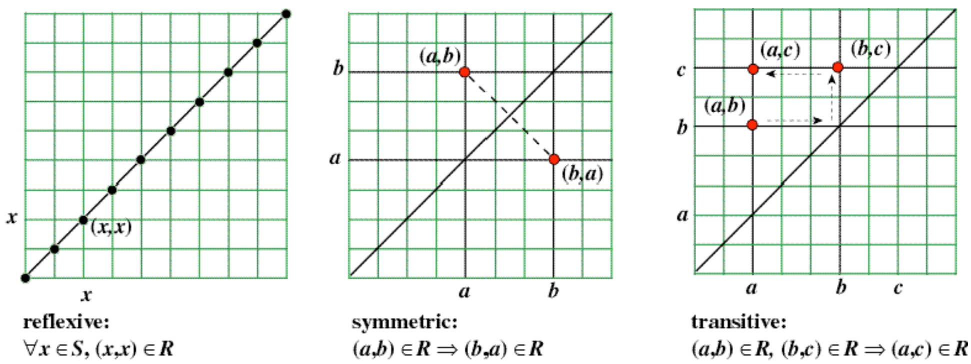


Figure 7: Visual representations of the axioms for an equivalence on a set R.

Chin studied the development of 15 students over the first two years. Their marks for the first year were widely distributed – three over 80, four between 70 to 79, four between 60 to 69, one between 50 to 59, three between 40 to 49 (where an honours degree pass mark is 40 and a first class degree is 70).

He asked the students the following question:

$$A = \left\{(x, y) \in \mathbf{R}^2 | 0 \leqslant x \leqslant 10, 0 \leqslant y \leqslant 10\right\}. \text{ Is } A \text{ an equivalence relation on } \mathbf{R}?$$

In the first year *none* of these 15 students responded positively to this question, even though they had been taught an equivalence relation on $\mathbf{R}$ is a subset of $\mathbf{R} \times \mathbf{R}$. Several wrote explicitly that they did not understand the question, (figure 8).

In the second year, only *one* gave a satisfactory response, (figure 9).

Chin's work revealed that while it was usual for a student to have a mental picture for a partition (as a set A broken into disjoint subsets) they were far less likely to have a mental picture of an equivalence relation as a subset of $A \times A$.

Different kinds of relations – functions, order relations, equivalence relations – have different embodiments. A function $f : A \to A$ lives in $A \times A$ as a graph, but order relations and equiv-

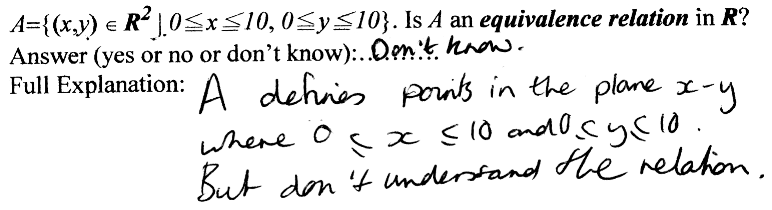

$A=\{(x,y) \in \boldsymbol{R}^2 \mid 0 \leqq x \leqq 10,\ 0 \leqq y \leqq 10\}$. Is A an ***equivalence relation*** in $\boldsymbol{R}$?
Answer (yes or no or don't know): Don't know.
Full Explanation: A defines points in the plane x-y where $0 \leq x \leq 10$ and $0 \leq y \leq 10$. But don't understand the relation.

Figure 8: Typical response of a first year student (Chin & Tall, 2001)

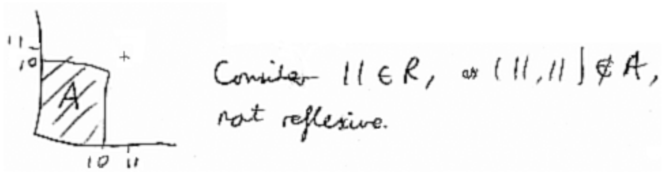


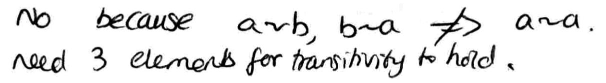
Figure 9: The single correct response in the second year (Chin & Tall, 2001)

alence relations live in the set A. The embodiments of order relation and equivalence relation are fundamentally different. The embodiment of an order relation met before by students is usually in the form $a < b < c$ where a, b, c are three points ordered from left to right. In arithmetic we would rarely write $5 \leqslant 6$ because we *know* 5 is *less* than 6. Likewise, we would not write $5 \leqslant 5$ because $5 = 5$. Thus in arithmetic we use the *strong* relation $<$ rather than the weak relation $\leqslant$. Hence, although the transitive law '$a \sim b$, $b \sim c$ implies $a \sim c$' has the same format in the case of an order relation and an equivalence relation, the (strong) order relation $<$ has the property that the three elements a, b, c are *different*.

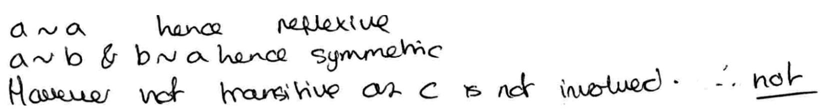
Chin found that many students found the following question difficult:

> Let $X = \{a, b, c\}$ and the relation $\sim$ be defined where $a \sim b$, $b \sim a$, $a \sim a$, $b \sim b$, but no other relations hold. Is this an equivalence relation? If not, say why?

Formally it is not an equivalence relation because the reflexive law fails for $c \sim c$. However, a quarter of respondents (68 out of 277) asserted that this is not an equivalence relation not because of the reflexive law, but *because the transitive law fails*. Typical responses included:


No because $a \sim b$, $b \sim a$ $\not\Rightarrow$ $a \sim a$.
need 3 elements for transitivity to hold.

and


$a \sim a$ hence reflexive
$a \sim b$ & $b \sim a$ hence symmetric
However not transitive as c is not involved. ∴ not

Both of these students, and others in interview, asserted that the transitive law needs three *different* elements, a, b, c (Chin & Tall, 2002). This may arise in part through previous experience with strong order relations.

The difficulties students meet in their encounter with the three types of relation (function, order relation, equivalence relation) may relate in part to the very different underlying embodiments that they bring to bear when they perform thought experiments to solve problems.

9. Reflections

The discussion presented here has shown how prior experiences (met-befores) from the worlds of embodiment and symbolism subtly affect formal meanings. Professional mathematicians depend on embodiment and symbolism to inspire their choice of theorems to prove and use symbolic manipulations in their proofs. Conversely, structure theorems may lead back to more sophisticated levels of embodiment and symbolism.

Students learning about formal proof bring their prior experience to bear in different ways. Some build on their embodied experience in a natural manner to give meaning to the formal theory and some of these are successful. Some find their embodied ideas conflicting with the formal theory and find it difficult to make sense of the formalism. Others take a formal logico-structural approach and work successfully at the formalism; others fail to complete such a formal programme, attempting to rote-learn the procedures that are given in the definitions without making sense of the full impact of the formal theory.

This presents teachers of mathematics at university level with choices how to help students make sense of formal proof. Simply presenting the theory in a logical order and hoping the students will make sense of it will work for some. But we are now teaching a much wider spectrum of students including many who fail to understand a logical sequence of presentation and base their thinking on inappropriate met-befores. To make sense of formalism many students need to realise how their prior knowledge – which worked in perfectly well in previous contexts – may need re-thinking in the new context. Learning about the subtle coercive effects of met-befores gives a powerful new meaning to Ausubel's famous dictum from the opening pages of his monumental book on psychology (1968):

> "The most important single factor influencing learning is what the learner already knows. Ascertain this, and teach him accordingly."

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Dinner Paper

The Romance Of Numbers – The Secret Life Of A Statistician

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It was almost nearly forty years ago in March 1967 that I decided to abandon my tranquil life as an actuarial clerk and venture into the unknown as student at Sydney's third and newest university. It was the year that Macquarie University opened its doors to undergraduate teaching with around 1200 students, only about 5% of the total enrolment today.

This was an unusual year since there had been no school leavers the year before, there being a changeover between the old five-year Leaving Certificate and the six-year Higher School Certificate. In the whole university there were only 11 subjects on offer in the first semester, 15 in the second semester and 11 that could be done over the full year for a total of 36 subjects.

The very first lecture was in a unit called *Structure and Properties of Matter*, an exposition in physics that had around 85 students of whom just two were female. Like a number of other units offered that year, its colourful name was matched by the wonderful subject *Conspectus of Mathematics* that also disappeared without a trace after its first offering. The next year saw the introduction of *Man and the World of Life* followed by *Witchcraft, Sorcery and Magic*. But I had a passion to major in chemistry and the flexible structure of the university enabled me to undertake subjects in mathematics, physics, chemistry, mechanics, statistics and even French in my first year. The subject *Introduction to Computing* was particularly amusing since the computer wasn't yet installed and the unfortunate lecturers had to resort to drawing pictures of what one looked like. However, it ceased being funny when it finally arrived and we had to grapple with punch cards and turnaround times of several hours to see if the computer program we had written actually worked properly.

These were the days before pocket calculators so even the smallest calculation had to be performed using logarithm tables or a slide rule. A number of professors wore their black robes when lecturing and soon found themselves covered in white chalk and so to avoid looking like a penguin the practice was abandoned. Those first few years of Macquarie were a time of tremendous growth with other buildings soon sprouting up all around, the first HSC students arriving the next year in 1968 and of course the Vietnam War was in full swing. Students seemed became more militant and by October 1969 a 'tent city' was erected in the quadrangle. About sixty students lived there for weeks in protest at the lack of student housing on campus.

There was also more of a clear demarcation between students and staff with 'staff only' toilets and a 'staff only' car park manned by a person in a white coat and whose role it was to keep pesky students out. This, of course, included me. Mind you, parking was also free in those days and there was never any trouble getting a spot. Unlike today.

In a short space of time my passion for chemistry had faded and at the completion of first year I decided that mathematics was far more interesting and so I changed to a maths major, finishing my degree at the end of 1969. The following year was a strange one since I decided to do mathematics honours, but being the only such student every subject was a reading course with no lectures or alternatively I was sent off to Sydney University to do one of theirs. Just to make life even more difficult, Macquarie also had the novel approach (soon abandoned) that I also undertake a written exam covering all four years of every undergraduate maths course I had done. And that was a nightmare.

Some of us went straight into jobs in our major field of study, a few into higher degree programs and others did things apparently unrelated to anything they had studied here. Two of my closest friends from that group ended up being television weathermen while many others became schoolteachers. One unusual fellow went off to be a professional gambler but we didn't hear from him again. As for me, I decided to carry on with my mathematics and completed my Masters degree at the University of Minnesota in the USA. But it wasn't long before I changed yet again, abandoning mathematics and doing my doctorate there in the emerging field of *Operations Research* which is basically trying to solve the problems of the world using scientific method.

At the end of 1973 with a nice new shiny PhD, no money after now being a full-time student for seven years and still stuck in the USA, it was time to decide what to do with my life. Luckily for me this was during the Whitlam years back in Australia and universities were seemingly actively recruiting lots of staff. Among several options of academic jobs, fortunately one was actually from Macquarie to take charge of their program in operations research. It was a deal that I was delighted to accept and turned out to be one of the best moves I have made.

The first ten years of my academic life were singularly uneventful, but then things brightened up with an entrepreneurial statistics professor who felt, with little justification, that statisticians were capable of almost anything. And so it was that he contacted Channel 9 in Sydney and proclaimed that he could predict the result of the March 1983 election that pitted the incumbent Malcolm Fraser against the newcomer Bob Hawke. Believing it to be a cakewalk with the new fangled computer wizardry and mountains of data, the idea was to use statistics to forecast the result of each electorate as the votes were counted. There was a buzz of excitement that 'the computer' would be able to predict the result before the rival channels and the advance publicity said it all. But life has a strange way of working out.

From a statistical point of view it was a disaster. A bug in the program saw the computer predicting a landslide victory to the Liberal Party even late into the evening when rival channels had declared their cause lost. As indeed it was. The headlines in the following day's press were most unkind, a typical example being "Haywire Crystal Chip Was the Star of the Show".

The fiasco had given me pause for thought that maybe statistics wasn't of much use after all when it predicted so poorly in such a public manner. Was there anything I could do to save the day? Less than six moths later it occurred to me that perhaps it was time to redress the situation. I was a keen sports fan and one of my great interests was rugby league with my favourite team being Parramatta. As luck would have it, they were playing their nemesis Manly in the Grand Final in September and I contemplated whether statistical models might be used to predict the winner.

Having number crunched important factors such as scrums, penalties, home ground advantage etc in past matches, I finally hit upon the notion that Parramatta would be victorious, but with no real conviction. Having gone this far, I rang the sports editor of the Sun Herald newspaper and explained to the editor that I was a statistician and I could use my expertise to predict the winner of the upcoming rugby league grand final and whether they would like to make a note of my prediction in their paper.

Surprisingly, the answer was yes, provided I could also give them the final score for each team along with all the reasons written in layman's terms. Having no shame, I said that this would be no problem and went to work looking at further evidence including the season's tackle counts, tries scored and a number of other statistics that might be useful.

Manly were the hot favourite to win, but I certainly wasn't going to tip them, fearlessly predicting that my team Parramatta would get home by a score of 18 – 14. It was a short time after submitting the article that it dawned on me that I was facing the same ridicule as the learned professor before me. I hoped that it would be buried somewhere in the sports section that a lunatic academic had made an outrageous prediction and that it would be quickly forgotten.

My worst fears were realised, however, on the morning of the match when I opened the Sun-Herald and spread in very large type across pages 4 and 5 were emblazoned "It's Parra by 4 points!!". My tables had been enhanced in size by 300% and on arriving at the match I was mortified to see plastered at all the ground entrances the placards that screamed *'Computer tips major upset'*.

There was no trouble in my being inconspicuous since nobody knew what I looked like but that day became a most important turning point in my academic career. I was always going to either look like an idiot or a genius. As it turned out, Parramatta didn't win by 18 – 14 as I forecast, but they did win by 18 – 6. Although largely dumb luck on my part, it opened the door to intriguing possibilities. The next morning I received a call from the *Sun* newspaper expressing their astonishment that I had not only correctly predicted the upset but was spot on with the winning team's score. The magic had returned to statistics after all. I was immediately offered a regular weekly half page column as a tipster for *every* match played in the next season. This was a task that I did for each of the following five years.

During my first year as a columnist in which I had surprising success, the NSW State Government had introduced legal betting on football. It was known as *FootyTAB* and was conducted by the TAB of NSW. At the beginning of 1984 they were looking for a television presenter and I was invited to go for a screen test. I was astonished to be given the role (I later learned that I was the only one who had been asked) and my role was to give live on-air predictions and other statistical information as a professional statistician on football telecasts on Channel 10. I had absolutely no experience at all in television, but I relished the task for eight years making over 300 appearances. All the while I was a full-time academic but when until the competition expanded in 1992 to include a number of interstate teams I found that I could no longer keep up with the travel required and reluctantly called it a day.

During all those years I hoped that I had achieved something for the profession and gone some way to resurrecting battered pride from the 1983 election debacle. But more media work was to come. It was in 2000 that I was interviewed at Macquarie by a reporter from the Good Weekend section of the *Sydney Morning Herald*. She happened to mention that she was a columnist who was soon quitting her role and was not aware of any replacement. Seizing the moment, I rang her boss and asked if there might be a vacancy for a riveting weekly column written by a statistician. I was astonished to hear that they would welcome a sample column or two and after duly despatching these the next day, two weeks later saw the birth of my Number Crunch column that is now in its sixth year. Packed full of absorbing statistical information, it also appears in the *Melbourne Age* and has an Australian weekly readership or around two million.

More was to follow. On a visit to London in mid 2004 I took the opportunity of dropping in, without appointment, to the UK Mail on Sunday and showed them some of my *Number Crunch* columns from Sydney. Fate struck again since as luck would have it they had become a little weary of their existing columnist on their prized letter page. The result was I was offered the role of writing a new column called *Statistically Speaking* based on UK data that would have a weekly readership of six million in England, Scotland and Wales. Now in its second year it has gone from strength to strength. So much so that it has also led to a regular weekly radio spot on BBC Radio Scotland in which I present a statistical quiz and answers for listeners.

My career as a statistician has been a source of tremendous satisfaction and I am most grateful that I have been able to combine it with an academic career. Macquarie has one of the largest statistics departments in Australia and with a first year unit having over 2000 students each year it is easy to see that statistics is difficult to ignore.

I am sure that there are many others in academia who have the same job satisfaction as I enjoy, but mine is not one that I would swap readily. The excitement and enthusiasm that each of us can engender in our field of study can be quite infectious and lead to challenging and interesting courses for the students. It seems that maybe statistics might be of some use after all.

Full Papers

Mathematics and Dumping Lectures? Another Perspective on the Shift Towards Learner Pragmatism

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A provocative report by Dolnicar [1] proposes that a shift towards learner pragmatism “defines the reality” of the current tertiary learning environment. It identified a group of so-called “pragmatics” (17% of the students investigated, and mainly Commerce and IT students) who claimed that they attended lectures almost solely to gain essential course information, as against enjoyment or to learn. This group is reported to have claimed the lowest levels of lecture attendance, yet delivered “the highest grade point average of the students in the study”. Are Mathematics students pragmatic in the sense that they only come to lectures to obtain essential course information, not to learn or enjoy? Are Mathematics lectures valuable in that they have an effect on performance? I present data from 85 Mathematics and Engineering students in an Algebra & Calculus course in Australia, in which all resources were readily accessible outside of lectures. Students attending class two thirds of the way through the course achieved statistically higher levels of performance on all but one of the course assessment items (significant at the .05 level) than those not in class at that point. “Strategic” learning styles, which may characterise the “pragmatics” described in [1], yielded only small non-significant correlations with performance. Lecture attendance has diminished very little in this Mathematics course, despite full and easy access to all course information and materials, online and in hardcopy. Hence there was little evidence of the pragmatism reported in [1] for Commerce and IT students. A follow-up study indicated that mathematics attendees regarded lectures as an efficient and companionable way to meet new material and build understanding and confidence. Lectures were motivating, enjoyable, kept students on track, provided a sense of community and common purpose, and clearly suited those who prefer learning by hearing, seeing and asking, rather than reading. The strong relationship between attendance and mathematics performance indicates the value of explicit teaching. Clearly ICT alternatives need quality resourcing and careful support.

1. Introduction: Students and face-to-face lectures

Awareness of tertiary students’ changing goals, needs, preferences, and perceptions, is vital if educationists are to respond quickly and appropriately. Certainly we are in a period of change that may have an effect on learner pragmatism. Pursuit of studies directed towards professions, rather than general studies, impacts on students’ learning choices and approaches. Advances in internet access have impacted strongly on the way we gather and manage information, and claims of short attention spans (as low as 10 to 15 minutes) are being fuelled by a plethora of fast technology-based activities for both leisure and learning.

Questions arise about the value of one or two-hour lectures, no matter how stimulating and interesting. Explicit face-to-face teaching is offered on the grounds that it is effective and efficient to resource. However, ICT’s and multimedia offer increasingly attractive packages for transmission of information, with full flexibility. Moreover, arguments about internet access no longer hold, with CD’s providing an inexpensive and convenient alternative. Burning questions are:

- Are the benefits (real or perceived) of lectures largely lost on a body of Generation Y pragmatists?

- How can we tell if we are wasting time and resources designing and delivering classes to today's undergraduates?
- If the demise of lectures is inevitable, should we not advance the transition period by investing in the resourcing of technology alternatives?

Whatever the current trends and philosophies, there is substantial and documented evidence across a wealth of educational literature that teacher-centred learning has a strong positive effect on student performance. Hattie's meta-analysis [2], for example, confirms that quality face-to-face teaching has a strong positive effect on students performance. That study investigated the educational effects on performance of a full range educational interventions reported in 160,000 studies of more than three decades of education research covering more than 50 million students across all disciplines and levels. The highest effects were associated with quality direct instruction, and quality feedback and remedial help, with motivation to improve. Clearly this study does not include the effects of burgeoning experience of new pedagogies. Educational technologies provide a host of new opportunities, and online stimuli are replacing the role of teacher, and the effects of these are probably being felt most keenly at tertiary level.

What do reasonably recent studies reveal about undergraduate lectures? Nothing surprising:

- That student variables (affect, cognition, behaviour, motivation) affect what they get out of lectures [3]
- That expressive enthusiastic teaching behaviours relate to higher levels of student achievement [4]
- That the lecturer communicating enthusiasm for the topic is effective in lectures [5],[6],[7],[8]
- That meaningful context, and applying information in the lecture, are effective [9],[5], [10]
- That large classes may cause some barriers [11]

And what do reports reveal about the reasons why students attend lectures? A range of responses:

- That enjoyment is the main factor for attendance, followed by concerns about difficult material [12]
- That some attend to acquire current information [8], and assessment information [6]

What about disciplinary differences?

- That indeed there are some, at least for lecture attendance levels and motivation [12]
- That science use lectures as a way into their reading, whereas arts students use them to help interpret what they have read [13]
- That science students value logical and structured lectures more highly than arts students [14]

Observations of discipline differences are not surprising to those who teach Mathematics, and these pointers highlight the need for caution when interpreting and generalizing findings from other disciplines.

2. Is there growing learner pragmatism in our undergraduate classes?

Are changes in our student body resulting in a creeping learner pragmatism that affects learning attitudes and preferences? Are students prioritizing fast and flexible access of core information over deeper learning goals? Certainly access to technology, pursuit of studies for professional rather than academic reasons, and learners accommodating work and family commitments, may be powerful agents for change. Do we still need lectures, or should flexible online delivery and ICT's take their place? In a recent study of reasons why undergraduates attend lectures, Dolnicar [1] collected data from 612 students in Australia. Students were invited to say if each of 12 listed reasons applied to them or not. I summarise and group the data that emerged below:

Highest frequencies of agreement were associated with “information” gathering:

- to find out what they are supposed to learn (78% agreement)
- not to miss important information (72%)
- to find out about assessment tasks (59%)

Moderate frequencies were related to learning:

- make sure I learn fundamentals (45% agreement)
- easier than learning it myself (43%)
- make knowledge meaningful (39%)

Expectation:

- expected to be there (30% agreement)

Lowest frequencies for a mixed range of factors:

- work on problems (22% agreement)
- find out real world application (21%)
- enjoy them (21%)
- find out latest thinking (20%)
- enthuses me (17%)

Dolnicar notes the relatively high frequency of responses on gaining essential information, compares this data with studies from the 1970's, and claims there is a shift towards lectures being viewed more as a means of, rather than learning and enjoyment, and claims that the findings reveal a “shift towards pragmatism” that now “defines the reality of the tertiary learning environment”.

In particular, students labeled “pragmatics” were those who reported attending lectures to get the information they needed to succeed in the subject. Pragmatics (17% of those surveyed) were found to be among the younger students, more typically Australian than Asian, and “over-represented” in the Commerce and Informatics faculties. They reported the “lowest lecture attendance while achieving the highest grade point average of the students in the study”. In contrast, students labeled “idealists” (7% of those surveyed) were wholly and genuinely enthusiastic about attending lectures. Typically, idealists were found to be “mature aged students with work experience and more frequently in the arts subjects surveyed”. Between these two extreme groups, other clusters identified were 11% were averagely motivated across the range of reasons offered, 15% who reported that lecture attendance also gave them the course fundamentals, 14% who only attended lectures not to miss relevant information, and 11% who attended for every reason but pleasure. However, Dolnicar's claims were established from data gathered from a sample of predominantly Commerce, Arts and IT students. Indeed, the reliability and generalisability of these provocative findings may be affected by a number of factors:

- Discipline bias in the sample: 294 Commerce students, 156 Arts, 71 Informatics, 53 Health and Behavioural Science, 33 Engineering, and 5 Science.
- Courses sampled largely opportunist, and completion of the questionnaire voluntary.
- No indication of course level. Numbers suggest that Science and Engineering courses sampled may have been at a higher level than the Commerce and Arts courses.

- Timing of the questionnaire is not offered. Students may have different priorities at different stages of a course.

Particularly provocative is the finding for mainly Commerce students that links "lowest" levels of attendance with "highest" levels of performance for a group of Commerce and IT students. This finding may be discipline-specific, even institution-specific. Some of the effect may be explainable by the fact that top performers may be bored in ill-pitched lectures. But a growing trend of that kind would challenge well-established strong relationships between quality face-to-face teaching and performance: as evidenced by Hattie broad meta-analysis [2], for example. Discipline bias alone makes the findings in [1] well worth testing for Mathematics, Science and Engineering students, who were strongly under-represented.

3. This Study: Why do mathematics students want explicit face-to-face teaching, and do they benefit?

The underlying questions were: Are mathematics students pragmatic in their views about the usefulness of lectures? Do they want explicit face-to-face teaching? If so why? And do they benefit from large-group lectures? I tested the following questions in on-campus cohorts of mainly Science and Engineering students. Lectures were voluntary. All course materials and resources were fully and readily available outside of lectures. All students had the texts, Study Guide, course rules, and the assignments, and files containing all lecture slides and weekly exercises were posted on the course website some days before class. Hence there was no pressure of any kind at all to attend lectures.

1. Why do mathematics students attend lectures?
2. Did those who attended lectures perform better in the course than those who did not?
3. What were the performance levels of those students whose learning attitudes and approaches are well aligned with those that define Dolnicar's so-called "pragmatics"?

4. Method

Qn 1: I invited all 30 students present in an Algebra & Calculus II class roughly two-thirds into the course, to write a few sentences (anonymously) on why they come to mathematics lectures when they have full access to all the materials and lecture slides. This data was investigated for primary and common themes.

Qn 2: I used data captured in Semester 1, 2005, for research into learning approaches and behaviour in an Australian undergraduate Algebra & Calculus I course. To identify those students who attend a substantial number of the course lectures ("attendees"), I identified those present in class (85) roughly two thirds of the way through the course. While opportunist, this sample was clearly representative of the population committed to attending lectures. An independent samples T-test was used to compare the mean performances of that sample with those not present, for each of the 5 assessment items in the course.

Qn 3: No equivalent data was available to identify so-called "pragmatics", that is, those students who attended lectures to "*find out what they are supposed to learn, not to miss important information*, and *to find out about assessment tasks*". However, these three characteristics are well aligned with those that define Strategic learners, viz *organized studying*, *time-management*, and *alertness to assessment demands*. And data on strategic approaches, alongside *deep* and *surface* approaches, had been captured late in the course using the ASSIST scales (see Tait, Entwistle & McCune, [15]). Thus correlations and scatterplots were examined to establish whether the mathematics group contained a substantial number of *strategic* learners who were also high mathematics performers.

5. Findings

Qn 1: Students reported and listed a range of reasons for attending lectures. The primary reasons were:

- to gain understanding of concepts and processes via explanations and examples;
- learning style: hearing and seeing, rather than reading;
- to ask questions and hear what others ask;
- motivation, and the stimulus to keep up;
- enjoyment;
- a sense of belonging and purpose;
- to establish and keep in contact with the lecturer.

Qn 2: Attendees at a lecture two-thirds of the way through the course performed substantially better on all course assessment items than non-attendees. T-tests revealed significant differences between the group means on all assessment items but Assignment1. Table 1 gives the mean scores on all assessment items for the two groups. Variations in group size is due to some students not completing all assessments. Note that * indicates the difference was significant at the 0.05 level (2-tailed); ** indicates that the difference was significant at the 0.01 level (2-tailed).

Table 1: (SPSS) Group Means for the Five Course Assessment Items

	0 = non-attendees 1 = attendees	N	Mean	Std. Deviation	Std. Error Mean
Assignment 1	0	30	70.6833	14.79369	2.70095
	1	50	75.5900	16.10827	2.27805
Midsemester Mark **	0	35	18.0714	10.48258	1.77188
	1	46	24.8804	11.80216	1.74013
Assignment 2 *	0	34	74.2206	25.97119	4.45402
	1	48	84.6667	17.83683	2.57452
Examination *	0	35	87.1714	40.28811	6.80993
	1	50	105.6600	39.73425	5.61927
Final mark *	0	34	54.3020	20.45663	3.50828
	1	50	63.5970	19.87536	2.81080

See Figure 1 for the distributions of overall performance levels attendees and non-attendees. Very few non-attendees scored in the top performance bracket, only 3 achieved over 80% compared to 10 attendees.

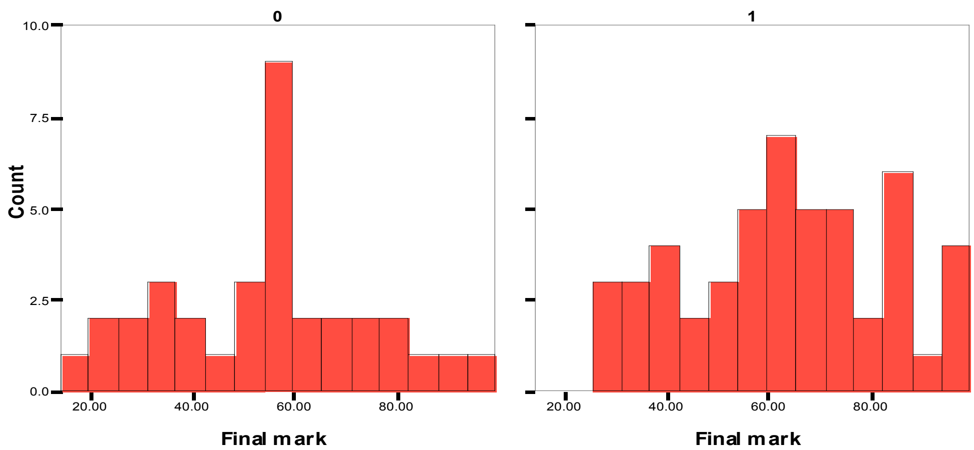

Figure 1: Course performance levels for Non-Attendees (0) and Attendees (1)

Qn 3: There was no evidence at all that students with highly strategic approaches to learning performed better on any of the course assessment items. Correlation coefficients were close to 0 and not significant. For example, Figure 2 show scatter plots for 46 attendees later in the course (not the same students as those analysed above.) These plots offer no evidence that high performance on the examination was related to highly *strategic* approaches; or to the subscale *alertness to assessment demands*, in particular.

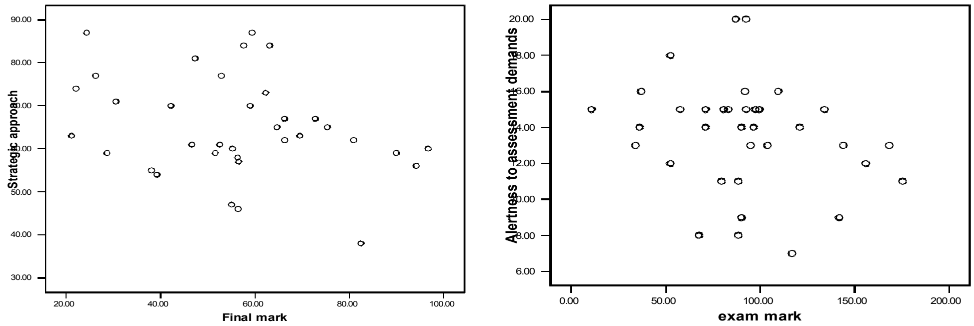

Figure 2: SPSS Scatter Plots of Exam Performance and Approaches to Learning for 46 student present at a lecture at the end of the course

6. Discussion and Conclusions

The views and behaviour of lecture attendees investigated in these mathematics courses presented little evidence of the kind of "pragmatism" reported by Dolnicar, who found students (predominantly in Commerce and IT) attended lectures with low levels of enjoyment, to gain essential information. I argue that levels of pragmatism of that kind vary with discipline, content and study-goals, and are strongly affected by the effectiveness, commitment, and approachability of the lecturer.

The attendees studied here regarded lectures as an efficient, companionable, and convenient way to meet new mathematical material, build understanding of the concepts, and develop the confidence to perform challenging tasks. Lectures were reported to be motivating, keep many students on track, gave many a sense of community and common purpose, and clearly suited the learning preferences of those who said they learned by hearing, seeing and asking, rather than reading. Gaining information per se was not found to be a primary reason why students attended lectures. However, that purpose is implicit in the ascribed learning role.

Students' belief in the value of attending lectures may also have paid dividends. A strong positive relationship was evident between performance and lecture attendance. This is in line with a wealth of studies (see Hattie [2]) that indicate that face-to-face teaching and strong individualized feedback are generally highly effective for performance. Only a few students performed well without attending lectures, and relationships between mathematics performance and strategic approaches to learning were very low. *Strategic* approaches to learning, as defined in [15], are associated with *organized studying*, *time-management*, and *alertness to assessment demands*, three characteristics that seem closely associated with the attitudes ascribed to "pragmatics". Thus this data suggests that the finding in [1], that pragmatics are high achievers but low attendees, may not be robust.

Certainly, lecture attendance has diminished only a little in the Mathematics courses described. It remains in the region of 60–65% despite easy full access to all course information and materials, online and in hardcopy, and generous support for independent learning in the course by both email and phone. Particularly interesting is that a substantial number of students are second-years with a strong background of enforced collaborative work and online problem-solving. Yet they attend voluntary large group lectures willingly and good-naturedly, and voluntary online discussion has declined in the wake of enforced participation. Does that convey a preference for face-to-face communication, or simply falling back on old habits?

To conclude, findings on the strong relationship between lecture attendance and performance of mathematics students in Science and Engineering, are in contrast with the findings reported in [1]. Rather, they add weight to the value of lectures and the wealth of studies and meta-analyses like [2] that demonstrate the high effects of face-to-face teaching and individual feedback. The fact that students with online and collaborative learning experience, and access to all essential course information elsewhere, still enjoy, support and attend lectures, and that many prefer face-to-face teaching, confirms the value and place of explicit teaching within the high-tech educational arena. While we welcome ICT's and the internet as a powerful force for change, it is clear that online mathematics courses are not a quick cheap alternative. Generous high quality guidance or face-to-face or online teaching, and remedial feedback to students on their individual mathematical understanding and progress are vital factors underpinning educational success, and electronic courseware and feedback must be resourced strongly and appropriately.

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Using case studies as bridges in teaching mathematical modelling

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This study focuses on the concept of growth upon growth, mathematically known as compound interest. The paper first illustrates students' grasp of this concept on entering a mathematical modelling course, based on responses to a question posed on this concept. How their capabilities change after some weeks of tuition with case studies as an aid is investigated. The conclusion is made that case studies as a way of embedding mathematical modelling into practical situations and of improving understanding are only partly successful.

1. Introduction

We begin this paper by describing the nature of mathematical modelling as well as the use of case studies for teaching purposes.

1.1. The nature of mathematical modelling

Mathematical modelling can be viewed as the process of describing phenomena in terms of mathematical equations [5]. What distinguishes a mathematical model from, say, a poem, a song or a portrait is that the mathematical model is an image or picture of reality painted with logical symbols instead of with words, sounds or watercolours [2].

Mathematical modelling differs from other more theoretical mathematical fields in that beyond understanding the mathematics itself, one also needs to know how and where it fits into the real world. Because mathematical modelling is embedded in reality it cannot be taught without applications. This is, for example, not the case with a course in analysis or even calculus. As Pollack [9] aptly states:

> Thus success cannot be measured entirely through assessment in mathematics itself, or in terms of mathematical preparation for the next level of courses. We must also look for the ability to examine in a mathematical way situations in everyday life, on the job, and as a citizen. Mathematical modelling, in a sense, is the link between mathematics and reality. Case studies offer a possible link.

1.2. Case studies as a teaching method

Using case studies as an accepted method of teaching is a fairly recent development. In fact, until the early 1990s the case study literature in science is virtually non-existent, although it is claimed that the formal use of stories, now called case studies, was introduced as a way of teaching into Harvard University's law and business schools around a hundred years ago [6]. The use of case studies became more commonly used as more faculty realised the inadequacies of the lecture method and began to seek novel methods of instruction. Best practices and guidelines to the general use of case studies has been reported on in literature, see [4] as an example.

Case studies are used frequently in teaching in business, law, and medical schools but are still less commonly used in science and mathematics in particular. Quoting the website of the

National Centre for Case study Teaching in Science [8]

> ...the use of case studies holds great promise as a pedagogical technique for teaching science, particularly to undergraduates, because it humanizes science and well illustrates scientific methodology and values.

A sound awareness of real-life issues such as of marketing, finance, communications and interpersonal relations are seen to be important for engineering students and this is best leaned by participating in a case study method of instruction [10]. The rationale is that real-world problems allow students to vicariously experience situations in the classroom that they may face in the future and thus help bridge the gap between theory and practice. A warning [10] is that careful shielding of a university from the activities of the world around us is the best way to chill interest and to defeat progress. Celibacy does not suit a university. It must mate itself with action, according to them.

Related to the present study, Townend [12] investigates the use of integrated case studies and mathematical modelling in teaching mathematics to engineering and computer science students. He proposes that case studies take on a fully integrated central role as vehicles for whole group discussion from which the students 'discover' the necessary mathematics which is taught subsequently. Not only is the 'carrot' of the application then central to their learning, but the need for the mathematics being taught is also clearly demonstrated.

Theall [11] quotes reasons for incorporating real-life situations into instruction of which the foremost is that applications of theoretical material in real-life situations make content easier to understand and that the relevance of content is demonstrated by real-life examples. He also states relevance as a major component for motivating students.

2. The students and the course

Vazsoni [13] relates an anecdote from a personal experience on approaching Von Neumann for help on a problem in game theory. Von Neumann is claimed to have said: 'Tell me what you know; I will help you build a bridge to what you need to know'.

Students taking the course in mathematical modelling on which the study is based and to be discussed, come from a school background where a procedural and rather clinical approach is followed and where real life applications are rare. As Vazsoni [13] rightly says, 'we cannot wait until our students come with the 'right background', we must take a proactive attitude and include in our approach the math upgrade of our students'.

We have to determine which von Neumann bridges need to be built, and this effort requires insight into the level of knowledge and understanding of students entering the course. In short, standing in the shoes of Von Neumann, one could find that it is not so easy to know which bridges to build and how to go about building these.

We refer to a course in mathematical modelling taught at first year level to 120 students. For half the students the course is elective and of those the vast majority offer as reason for enrolling that they expect it to be beneficial to their careers. Proposed careers are mostly in the field of Information Technology or in the Financial Mathematics and Actuarial Science field. The group is academically strong with close on 40% of students entering the course with an A symbol in the final examination of secondary school and less than 10% with the bottom requirement of a D symbol. When asked what they would like to see as the result of taking this course and given a few options, almost half of the students would like to have a sound mathematical foundation in mathematical modelling and understand theoretical aspects. More than half the students would like to have enough knowledge to be able to construct a mathematical model but also were keen to use software where possible. In this group are a few students who took the subject called Additional Mathematics in secondary school, which includes components such as calculus and financial mathematics.

In summary, these students form an academically strong, career-orientated group with high ideals on entering the course.

The mathematical modelling course itself consists of three parts. The first is a section on using difference equations to model real-life situations, focussing on financial applications such as mortgages and annuities and on population growth models. The second part of the course deals with model fitting such as polynomial fits and linearising data to obtain an appropriate fit. The third part of the course focuses on differential equation models, again for modelling population growth, continuously compounded interest rate models and applications of the heat equation. The approach to the course is problem-based and hands-on. A customised textbook [7] has been developed, offering a more practical than theoretical approach with interesting problems from a wide background. Technology is used extensively, Matlab in particular, to perform tedious calculations, to explore with and to provide functionalities such as for curve fitting, plotting etc.

The course kicks off with a section on modelling with linear difference equations. This aspect involves equations of the form $y_{n+1} = ay_n + b$ where a and b are constants and a is of the form $1 + r$ where r is the growth percentage. For some initial value y_0 a solution can be iteratively generated. A solution formula is also available, namely

$$y_n = \frac{b}{1-a} + \left(y_0 - \frac{b}{1-a}\right) a^n.$$

Applications of modelling with linear difference equations include annuities, mortgages and population models.

3. The study

We turn our attention to the simplest case: $y_{n+1} = ay_n$ for which the solution is $y_n = a^n y_0$. The underlying principle is that of compound interest. If you invest money, say R1000 at 12% per annum, compounded monthly, then the model is $y_{n+1} = 1.01y_n$, $y_0 = 1000$ where n is measured in months and where y_n represents the value of your investment after n months.

It would seem unfair to expect of students, other than those who had Additional Mathematics at school, to be armoured with the concept of compound interest and how to do calculations as required for this simple case, and without using difference equation model notation. It does not form part of the school syllabus.

Yet, more fundamental than actual compound interest calculations is the notion of growth upon growth. If an amount increases by 20% and then again by 20% then the total increase is not 40%, it is more than 40%. Most reasonably educated citizens invest money, take out a bond, read about the increases in social phenomena and need to deal with this concept in some or other form.

What is the understanding of the concept of growth upon growth of this group of students who are about to start with mathematical modelling? The following question (Question A) is posed early in the course in a relaxed classroom situation but as an ad hoc ‘class test’:

A population increases from 25 to 27,5 million over a period of five years. What is the average percentage increase per year?

This is the type of data one frequently comes across in newspapers and magazines. The level of difficulty of this question should not be underestimated. Built into the question is

- percentage calculation
- the notion of growth upon growth
- the calculation of a geometric mean instead of an arithmetic mean.

3.1. Responses

Responses to this question naturally divide students into three groups according to their ability to deal correctly with the question, shown in Figure 1.

The **first** group consists of students who follow the correct reasoning to obtain:

$$27.5 = 25a^5$$

and to come to an answer of $a = 1.019$, rendering a figure for the yearly increase of 1.9% (to 1 dec). This group consists of only 18% , and the vast majority of this group had Additional Mathematics as a school subject where they took a financial mathematics module and so have dealt with this notion before.

The **second** group of students all commit to the same faulty reasoning. They do the following calculation (in one form or another):

$\frac{\frac{27.5-25}{25}\times 100}{5}$ to come to a figure of 2% per year. Amongst these are students (roughly a quarter of this group) who cannot quite get it 'right' but who followed the same thinking, either offering the total percentage increase (10%) as an answer or by calculating the average absolute increase (27.5 25)/5 = 0.5, often simply writing it as a percentage: 0.5%. This group forms 57% of the total.

The **third** group (25%) consists of students who cannot deal with the problem at all. To quote a few popular answers:

1. 25/27.5 = 90.09% for five years or 18.1819% per year
2. 25000000 = r27500000 and so r=11% over 5 years or 2.2% per year.
3. 27.5 25 = 2.5/100 = 0.025%
4. (25 000 000 + 27.5 000 000)/2 = 25 000027.5 = 12500013.75

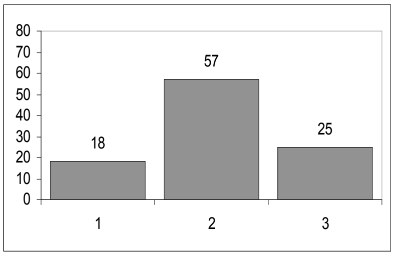

Figure 1: Percentage distribution of responses to Question A

Was the question misleading? Was the reference to average percentage unfairly tempting for simply dividing the difference by 5? For this reason the question was reformulated and posed to another group of students, a group of Engineering students, comparable in size (75) and symbol distribution.

'A population increases by the same percentage per year. Over a five-year period the population increases from 25 to 27.5 million. By what percentage did it increase every year?'

Results show a similar distribution, but with numbers for the first and third groups reversed (see Figure 2). Again, almost everyone belonging to the first group had Additional Mathematics at school level. The results also verify that the question was fair.

3.2. Discussion

The first group of students can adequately deal with the concept of growth upon growth, but this ability is not due to what they learned from the regular school syllabus but mainly because of their exposure to compound interest when doing Additional Mathematics. One student in this group, not having taken Additional Mathematics, needs to be mentioned specifically. He simply gives the answer "It must be less than 2%" showing the type of insight and thought process required. This is exactly the type of 'mathematical common sense' referred to by Alsina [1], an adequate mixture of number sense and practical skills.

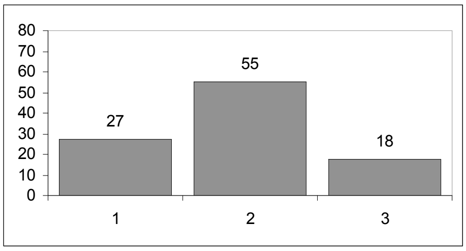

Figure 2: Percentage distribution of control group

For the second group the faulty reasoning to the question does not come unexpectedly but the fact that this group comprises 57% does come as a surprise. Yet, the exact size of this group gives no rise for concern and is irrelevant because this course offers the opportunity to rectify the misconception. What is worrying is that the concept investigated here is part and parcel of investing money and taking out a mortgage and this result would insinuate that a large part of the population are not au fait with growth upon growth calculations. In defence of these students it has to be pointed out that they had limited exposure to the financial side of society. One would, however, have expected students to realise that the interest rate must be less than 2%, as mentioned by the one student quoted above. A grasp of the concept of growth upon growth is lacking. The results for the second group show that there is a (large) group of students who are lacking in knowledge. A Von Neumann bridge needs to be built from their basic knowledge on percentages to the notion of growth upon growth and how to deal with this notion mathematically, before starting with the actual course material. This group of students do not have adequate understanding of the notion of growth upon growth although they can calculate percentages.

Results of the third group are disturbing. These students, a quarter of the group, could not deal with the underlying basic mathematics such as percentage calculations. This group of students are neither quantitatively literate nor mathematically literate on this particular issue. The question is whether it is possible to build a Von Neumann bridge to where they are?

3.3. Case Studies

The lack of knowledge and understanding displayed in the second and third groups certainly needs addressing and this could be done in a number of ways. In the mathematical modelling course under discussion a premium is placed on developing case studies, supporting DuDehart's opinion [3]:

> We need to find a way to apply what is learned in formal mathematics courses to everyday 'real life' situations, circumstances, scenarios, case studies.

Case studies emulate the societal situations that students in mathematical modelling should be prepared for as closely as possible. We view case studies as the Von Neumann bridges of mathematical modelling. Any interesting story has universal appeal and an interesting story that applies the mathematics at the order of the day in a course is valuable for learning.

Developing case studies requires creativity and time, but the investment is compensated for by a certain creative satisfaction as well as appreciation from students. With a hunch and resources such as the internet (and a good search engine) as well as newspapers and magazines the only real constraint is time.

For the section on difference equations referred to above and for the simplest model $y_{n+1} = ay_n$ a case study was developed on the Iranian population that shows an interesting growth pattern.

A short synopsis is given:
The Iranian population maintained an annual growth of 2% per year for the first half of the century. After World War II, however, the average growth percentage started increasing, calculated as 2.9% between 1956 and 1966. It then dropped to 2.7% for the next ten-year period after which it suddenly shot up to an astounding 3.7% on average for the period 1976 1986. In terms of population dynamics this is a particularly high growth rate. Why the sudden rise? Prior to the revolution in 1979, the government had promoted a family planning programme. However, following the revolution, the new government ceased all official involvement in family planning. If this growth rate had been maintained the population would have stood at 92.7 million in 2004 whereas it only numbered 66.7 million. Since then the growth rate has decreased sharply due to family planning laws and for the period 1991 to 1996 the population increased from 55.8 million to 60 million and from 1996 to 2001 the population increased from 60 million to 66.1 million people.

The case study developed around the basic information related here includes quotes and graphs, similar to a newspaper article. The goal of the case study is to develop a difference equation model of the type $y_{n+1} = ay_n$ for the Iranian situation, based on the rich and interesting data given.

One the one hand (for the first part of the data before 1991) percentage growth is given for different periods and for each period a difference equation can be constructed. The second part of the data (after 1991) corresponds to the question posed at the start. In this case the solution formula $y_{n+1} = a^n y_0$ applies. The fact that average percentage growth cannot simply be calculated by the total growth percentage divided by the number of years is discussed in a real-life context and students are alerted to the implications of the common faulty reasoning. By linking this notion to a concrete case study, the idea is enforced and so mathematical literacy is cultivated here as a by-product.

Case studies form an integral part of this course and are prominently included in the assessment process. In every term test at least one question is directly based on a case study.

3.4. Assessing again

In order to investigate students' mathematical development in this regard, a question on the Iranian population was included in the first term test, but now with the possibility of using a difference equation to solve the problem:

Question B: *The Iranian population grew from 60 million in 1996 to 66.1 million in 2001. At what rate on average, measured in percentage per year, did the population increase over this period?*

The results again divide students naturally into the three groups named above, again based on their understanding displayed in the question (see Figure 3).

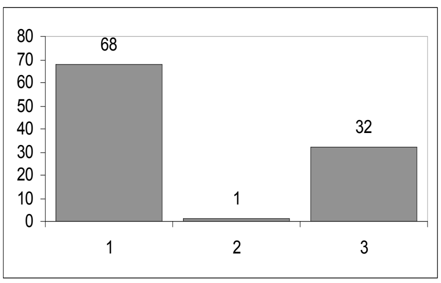

Figure 3: Percentage distribution of performance in Question B

The first group consists of students who have the answer completely right or made some trivial calculation error (or mistook the period for 6 years) and this now comprises 68% of students.

The second group of students who follow the faulty reasoning of the majority of students previously now remarkably consists of only two students (1%). This second group has, in other words, almost completely dissolved.

The third group of students who cannot do the problem now comprises 31% of the total. Of these students half made some futile effort in the near right direction, mostly trying to construct some sort of difference equation and the others had it totally wrong or could not even start. Although it is theoretically possible that most of the third group students have moved over to the first group, it is a highly unlikely scenario and not considered here.

3.5. Discussion

We start our discussion by looking at the second group. The group has dissolved in two ways. Most of the group of students who followed the faulty reasoning at first have now joined the group who have it right. The Von Neumann bridge provided was adequate. Others, a smaller group, have moved to the third group but they at least have realised that they should do the problem differently to before, although unsuccessful in their efforts. They know that they don't know and in that sense their mathematical competency for this concept has improved. They know what is the wrong way of doing it although they are not quite able to do it right.

Is the figure of 31% for the third group disappointing? That a large group of students are still left in the dark, is indeed disappointing, but it should be pointed out that the performance in this question correlates well with the overall performance (0.79) and the average of 61.4% in this question is significantly higher than the 54.4% for the full paper when a t-test is performed. These are students who experience difficulties in general with the subject. The step for most students who could not even calculate percentages to becoming mathematical competent regarding growth upon growth is simply too big. The bridge we built did not reach far enough to fetch them. These students need to follow a basic bridging or remedial course.

4. In conclusion

This study shows that for an academically strong group such as the one under discussion, there are a number of students who come to university not only lacking in mathematical competency but also in basic numerical competency. The lack of understanding in students with such strong backgrounds, and who are pursuing careers in information technology or the financial sector is of obvious significance. Although case studies do serve a purpose for teaching mathematical modelling, the underlying assumption that students are mathematically equipped enough to cope with the basics of a case study, is not sound for the group under discussion. A command of the (often simple) mathematics embedded in real life is often assumed but just as often lacking. Case studies as a way of embedding mathematical modelling into practical situations and of improving understanding are only partly successful. Indications are that some students find the step towards mastering mathematical modelling too big. The study shows that too large a sector of students (and society) is lacking in dealing calculations of growth upon growth.

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Understanding multiple interpretations of the notion of solution in linear algebra

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The first concepts in most linear algebra courses that university students encounter focus on the fundamental subspaces of a matrix and the relationships between them. In this paper we report on the results of a questionnaire designed to assess students' ability make links between these concepts. The questionnaire was given to three groups of students from three institutions, all of whom have completed a first year course in linear algebra. Most of the students were not able to link the different equivalent formulations of the same underlying concept, but there is some evidence that suggests that the ability to do so improves somewhat by the time they are in their second course.

1. Introduction

"Linear algebra is a circular subject"
"I felt that we kept going around in circles"

The first quote appears in the introduction to the recent book "Transform Linear Algebra" by F. Uhlig [1] the second is from an interview with one of a group of Engineering students from Edith Cowan University. Uhlig, while first acknowledging that the subject contains an overwhelming number of interconnected concepts, refers to Linear Algebra as "this magic mathematical place" that is ripe for exploration and the awakening of intuition. The student, on the other hand, was a lot less enthusiastic about the subject and admitted to having felt pretty lost.

The research reported in this paper sets out to look at how university students handle certain notions in linear algebra that would seem to constitute part of the "circularity" of the subject. More specifically, we wanted to ascertain whether students can link the notion of the solution vector $\mathbf{x}$ or the homogeneous system $A\mathbf{x} = 0$ to the following equivalent interpretations:

1. $\mathbf{x}$ is a vector satisfying a set of constraints (defined by the equations)

2. $\mathbf{x}^T$ is a vector that is orthogonal to the rows of A

3. $\mathbf{x}$ is a vector that expresses a certain linearity relation among the columns of A

4. $\mathbf{x}$ is a vector that belongs to the kernel of the linear transformation induced by A

2. The Research Questions

A very common phenomenon in mathematics is that the same mathematical object (or concept) is interpreted from different perspectives. This already occurs in the early years of schooling, where the "signifier" $\frac{a}{b}$ could signify such things as ratio, a rational number, a fractional number, or division; later, 'x' can signify a variable or an unknown. The multiple interpretations endow the object with both richness (in terms of its connectivity to other concepts) and flexibility; a particular interpretation may prove a lot more useful for dealing with a given mathematical problem situation. From a learning point of view, understanding a mathematical concept/object therefore entails more than an ability to operate within each context. It also requires interpreting information culled from one perspective within another perspective; a fluency in shifting points of view. At all levels of instruction, achieving such fluency has been problematic for learners.

Linear algebra is one subject that is particularly fraught with multiple points of view. This apparent "circularity" is further compounded by the fact that, in general, a typical linear algebra course will include several modes of description of the basic objects and operations namely:

- the abstract mode using the language and concepts of the general formalized theory, including: vector spaces, subspaces, linear spans, dimension, operators, and kernels,
- the algebraic mode using the language and concepts of the more specific theory of R^n, including: n-tuples, matrices, rank, solutions of system of equations, and row/column spaces,
- the geometric mode using the language and concept of 2- and 3-dimensional spaces, including: directed line segments, points, lines, planes, and geometric transformations [2] .

These modes of description co-exist, are sometimes interchangeable but they are not equivalent. Put together, they make up for an extensive vocabulary and terminology. Because these multiple modes of description have often been proved to overwhelm beginner linear algebra students, proponents of linear algebra reform recommended that the first linear algebra course be matrix based [3] so the trend nowadays is to emphasize the algebraic mode rather than the abstract mode. While such an approach removes the study of the general theory of vector spaces from the first course, it still borrows most of the notions and language underpinning the abstract theory.

A particular concept, such as a solution of a homogeneous system of equations has equivalent (at least in the finite-dimensional case) notions that are framed in different contexts and use different vocabulary. Usually, students deal successfully with the notions and standard problems that fall within a specific context, particularly if the problems require them to draw on some relevant procedural knowledge (e.g. decide if a set of vectors is linearly independent; find the null space of A; find all vectors which are orthogonal to a set of vectors). At a procedural level, students know that answering such questions reduces to solving a homogeneous system of equations. Conceptually, however, students may fail to understand that the notions are actually equivalent, i.e. that a solution to a problem within one context simultaneously gives answers to problems framed in a linked context. Thus, the theoretical framework for this research draws from the cognitive analysis of what constitutes mathematical understanding (see, for example, [4]) and the emphasis on the importance of the ability to make conceptual connections between various notions which have are represented by the same object.

More specifically, in this study we asked

1. Does students' understanding of equations and their solution go beyond the procedural? Do they understand how the four different interpretations of the solution (space) of a homogeneous system are linked? Can they translate a fact stated within one of the modes to equivalent statements within the other modes?

2. Is there a "maturation factor"? Will students at the end of a second linear algebra course attain a more cohesive understanding of the equivalent interpretations of a solution space than students in the end of a first linear algebra course?

3. The questionnaire and the students

There were two, slightly different, versions of the questionnaire. The first one (Version A in Figure 1) was pre-tested at Edith Cowan University (ECU) with a group of Engineering students who have completed their first linear algebra course 3 months earlier.

Suppose that $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$ and consider the 4×3 homogeneous system of linear equations

$$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Is the column vector $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ a solution to the above system? Explain.

(b) Is the column vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ a solution to the above system? Explain.

(c) Can the first column of the matrix *A* be written as a linear combination of the second and third columns of *A*? If your answer is 'yes', show by an example, how to do so; if your answer is 'no', explain your reasoning.

(d) Give an example of a non-zero vector that is orthogonal to the first row of the matrix *A*.

(e) Is it possible for a row vector to be orthogonal to all the rows of the matrix A? If your answer is 'yes', give an example, if your answer is 'no', explain your reasoning.

(f) Are the columns of *A* linearly-independent? If your answer is 'yes', explain your reasoning, if your answer is 'no', give an example of a non-trivial linear combination of the columns of *A* which yields the zero vector.

(g) If $T : R^3 \to R^4$ is the linear transformation defined by $T(x_1, x_2, x_3) = (x_1 + 2x_2 + 3x_3, 2x_1 + 3x_2 + 4x_3, 3x_1 + 4x_2 + 5x_3, 4x_1 + 5x_2 + 6x_3)$ then $T(0,0,0) = (0,0,0,0)$. Are there other vectors (x_1, x_2, x_3) such that $T(x_1, x_2, x_3) = (0,0,0,0)$? If your answer is 'yes', give an example, if your answer is 'no', explain your reasoning.

Figure 1: Questionnaire version A

After examining the results of the questionnaire and interviewing 6 of the respondents, the questionnaire was modified slightly (Version B) and comprised of four questions. (see Figure 2). However, one of the experimental classes (**G3**) received Version A . The variations in the two versions did not influence the results of the study. The way that **Q.III** was split up in Version A provided, fortuitously, some extra data about students' compartmentalised learning, which we will discuss in the sequel.

The questionnaire was administered to 154 students in three classes in three different universities. In all cases the students had completed at least one one-semester course in linear algebra. The students in these courses included mathematics, computer science, engineering, actuarial mathematics, physics, economics, and education students reflecting the wide clientele that is typically enrolled in a linear algebra course.

The three groups were as follows:
G1: 40 students from the University of Technology of Sydney (UTS) who had just completed the first linear algebra course (text: Lay's Linear Algebra and its Applications [5]). They were a

All of the following questions refer to the **same** matrix $A = \begin{bmatrix} 1 & 0 & 1 & 2 & -4 \\ 0 & 1 & -2 & 0 & 1 \\ 1 & 1 & -1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 5 \\ 0 & -3 & 6 & 2 & -9 \end{bmatrix}$

Q.I Consider the homogeneous system of linear equations $A\mathbf{x} = \mathbf{0}$

Given $[1,1,1,1,1]^T$ and $[-3,1,1,3,1]^T$, which of the these two column vectors is a solution to the above system? Explain.

Q.II (a) Give an example of a non-zero row vector that is orthogonal to the first row of the matrix A

(b) Can a non-zero row vector be orthogonal to all the rows of A? If YES, give an example of such vector; if NO, explain your reasoning.

Q.III Are the columns of A linearly independent? If YES, explain your reasoning; if NO, give an example of a non-trivial linear combination of the columns of A which yields the zero vector.

Q.IV Does the kernel of A contain any non-zero vectors? If NO, explain your reasoning; if YES, give an example of such vectors.

Figure 2: Questionnaire version B

mix of second semester first year and first semester second year students, mostly mathematics majors, except for ten students who were doing the subject as an elective. The questionnaire (Version B) was embedded as part of the final exam.
G2: A class of 47 students from Concordia University comprised of a mix of statistics, mathematics, actuarial, and computer science students who were attending the second half of a two semester course in linear algebra. (The second half of the course covered inner products, nilpotent matrices, and Jordan Canonical Forms, using Schaum's Outline in Linear Algebra [6] as a reference). The questionnaire (Version B) was completed on a voluntarily basis in the last 15 minutes of a class, near the end of the course.
G3: 67 students from ECU, mostly engineering students, with a scattering of science and secondary mathematics education majors who had just completed their first linear algebra course also using [5] as the text. The questionnaire (Version A) was embedded as part of the final exam.

4. The results

4.1. Overall results

The main focus of the analysis was to look at how answers to **Q.II**, **Q.III**, and **Q.IV** were linked to **Q.I**. Specifically, we were interested in the following links:

$L_{1\to 2}$: $\mathbf{x}$ is a non-trivial solution of $A\mathbf{x} = 0 \Rightarrow \mathbf{x}^T$ is a non-zero vector orthogonal to the rows of A

$L_{1\to 3}$: $\mathbf{x}$ is a non-trivial solution of $A\mathbf{x} = 0 \Rightarrow$ columns of A satisfy a linear dependence relation with a set of weights given by the components of $\mathbf{x}$

$L_{1\to 4}$: $\mathbf{x}$ is a non-trivial solution of $A\mathbf{x} = 0 \Rightarrow \mathbf{x}$ is a non-zero vector in Ker(A) (or Null(A))

Establishing the above links is predicated on students having answered **Q.I** correctly. In fact, twenty respondents did not conclude that one of the two columns vectors given in **Q.I** is a solution to the homogeneous system $A\mathbf{x} = 0$, so for the purpose of the analysis we were left with 134 questionnaires [**G1** = 35, **G2** = 38, **G3** = 61]. It is worth noting that 39 students of the 134 students who successfully answered **Q.I**, did so by finding a (reduced) row echelon form matrix (REF) rather than checking whether the given vectors satisfy the matrix equation.

Overall, $L_{1\to 2}$ was established by 51 students (38%) with the group percentages being [40%, 58%, 25%] . Out of the whole group of 134 students, only three mentioned explicitly the relation Null(A) = (Row(A)) in their answer to **Q.II**. The link $L_{1\to 3}$ was the least apparent, with just fifteen students (11%) able to link the dependence of the columns of the matrix A to the result from **Q.I** ([11%, 18%, 7%]) with 60 respondents resorting to the REF. For $L_{1\to 4}$, 41 students (30%) directly linked the existence of a non-zero element of the kernel to their answer to **Q.I** ([23%, 39%, 30%]) while 35 others either reduced to echelon form or used the reduction they had already obtained in answering **Q.III**.

4.2. Detailed Analysis

We now proceed to give a more detailed, question-by-question, and class-by-class analysis of the results. For each question, we consider the following categories (see Table 1):

Table 1: Abbreviations for categories used in the analysis of questionnaire responses

Abbreviation	Description
L	linkage with **Q.I**
U	unsubstantiated answers (whether correct or not)
NR	No response
REF	using row reduction on A (whether it led to a correct or incorrect conclusion)
Eq	setting up equations
Z	the kinds of answers that would usually merit a mark of zero
O	others

Q.III is analyzed in two parts: (a) establishing that the vectors are dependent, and (b), finding a linear dependence relation of the columns. The overall responses of the three groups are summarized in Table 2.

Table 2: Characteristics of questionnaire responses of the 4 groups

	Q.II	Q.III(a)	Q.III(b)	Q.IV
G1: N=35				
L	14 (40%)	4 (11%)	4 (11%)	8 (23%)
U	7 (20%)	2 (6%)	0 0	8 (23%)
Eq,	1 (3%)	0 0	1 (3%)	0 0
REF	0	17 (49%)	0 0	13 (37%)
NR	2 (6%)	1 (3%)	25 (71%)	3 (9%)
Z	10 (29%)	9* (26%)	4 (11%)	2 (6%)
O	1 (3%)	2 (6%)	1 (3%)	1 (3%)
G2: N=38				
L	22 (58%)	7 (18%)	7 (18%)	15 (39%)
U	5 (13%)	12 (32%)	0 0	4 (9%)
Eq,	2 (5%)	1 (3%)	0 0	0 0
REF	0	7 (18%)	0 0	1 (2%)
NR	7 (18%)	5 (13%)	27 (71%)	15 (39%)
Z	2 (5%)	3 (8%)	0 0	2 (5%)
O	0 (0%)	3 (8%)	4 (9%)	1 (2%)
G3: N=61				
L	15 (25%)	4 (7%)	3 (5%)	18 (30%)
U	11 (18%)	5 (8%)	10 (16%)	11 (18%)
Eq,	3 (5%)	7 (11%)	0 0	0
REF	0	36 (59%)	34 (58%)	13 (21%)
NR	8 (13%)	1 (1%)	4 (7%)	7 (11%)
Z	16 (26%)	5 (8%)	7 (11%)	6 (10%)
O	7* (11%)	3 (5%)	3 (5%)	6 (10%)

*These include the six students that gave the zero vector as a one which is orthogonal to all the rows.

The first question was answered properly by all but five of the 40 students in group **G1**, though twelve of these students performed row reduction on A to get their answer. Half the students also resorted to using a REF in order to establish the linear dependence of the columns of A ("existence of free variables" argument) and none of these students came up with a linear dependence relation among the columns (in some instances, they mistakenly gave the general solution of $A\mathbf{x} = 0$ in terms of its basic solutions as an answer). Only four students linked the linear dependence of the columns to the existence of a non-trivial solution ($L_{1\to3}$), and these same four students used the non-trivial solution to also produce the linear dependence relation among the columns of A.

Students in **G2**, (with four exceptions) answered **Q.I** by simply evaluating $A\mathbf{x}$ for the two given vectors. They were relatively successful (58%) in establishing $L_{1\to2}$. (In fact, a second cohort group of thirteen students from another section who answered only **Q.I** and **Q.II** also had $L_{1\to2}$ established by five of the nine students who answered **Q.I** correctly.) For this group, as for **G1**, the two parts of Q.IIIwere essentially answered simultaneously. Those seven students who linked the linear dependence of the columns to the existence of a non-trivial solution also wrote down the linear dependence relation. These same seven students, in fact, established all the links. The large number of NR's in part III(b) is due to the fact that ten students concluded, for a variety of reasons (or no reason at all), that the columns of A were linearly independent so they did not need to answer the second part; five others who answered that the columns were dependent but did not write any linear relation in the columns. Four students found, by inspection, another linear dependence relation, involving the first three columns of A.

Recall that **G3** (N=61) is the only group that used Version A of the questionnaire, so III(b) actually appeared prior to III(a), and was separated by **Q.II**. Nearly all students answered **Q.I** correctly though 21 students first row reduced A. This group was quite well versed with the fact that a linear relation in the columns of A can be culled from a REF so this approach was taken by 40 students to answer **Q.III(b)**. Interestingly, fourteen of these students concluded, erroneously, that the first column could not be written as a linear combination of the other two. The most common reasoning for the "no" answer was that in the REF, the three columns were non-zero hence linearly independent, a consequence of inappropriately transferring properties of the rows in a REF to the columns. Of the 36 students who used a REF for **Q.III(a)**, only nine students directly linked the two parts of the question, i.e., argued that since the first column is a linear combination of the other two, the columns were dependent. The majority switched to the "free variables" criterion to ascertain dependence of the columns. Furthermore, ten students had inconsistent "yes" answers for both parts of **Q.III** and another eighteen students had virtually inconsistent "no" answers for both parts. (the exception to inconsistent answers would have been had columns 2 and 3 of A been linearly dependent).

5. Interpretation of the data

Straightforward answers to the research questions posed in this paper would be that most students of linear algebra are not able to link different but equivalent conceptions of the same underlying object, but that the ability to do so improves somewhat by the time they are in their second course. Certainly, the raw data support these conclusions: the ability to link the results of **Q.I** to the other questions by the different groups varied between 11% and 38% of the total population of the study. The students enrolled in a second linear algebra course (**G2**) had a higher rate of establishing each of the links than the overall population ([58%, 18%, 39%] compared to ([38% 11%, 30%]). They also had the largest number of individuals who established all the three links (seven of 38). However, there were some factors that should be taken into account and that could have influenced the results.

Two groups (**G1** and **G3**) answered the questionnaire as a question on their final examination. It was a decision taken by their instructors who felt (and as it turned out, correctly) that students would take the questionnaire more seriously. Consequently, students in these two groups were more careful with their calculations (particularly with **Q.I**). They also tended not to leave a question unanswered (hence many replies fell into the Z-category). They might have been operating in an "exam-mode" reverting to the tools that they were most familiar with, namely, row reduction and checking for free variables (and, probably assuming that their knowledge of these procedures is what is being tested). The general tendency of linear algebra students to be "practical" rather than "theoretical" thinkers discussed in [7] certainly must get amplified under the conditions of an examination. In particular, **G3** had been taught the fact that row reduction, while changing the columns of the matrix, respects the linear relations among the columns. Consequently, this group had more correct (though unlinked) answers to **Q.III**(b) than the rest of the groups. However, their knowledge of the connection between linear combination and linear independence was so fragmented and compartmentalised that they could not even link the two parts of **Q.III**. Not only did they use different procedures for each part of the question, but they also came up with inconsistent answers in 28 of 37 cases.

On the other hand, students in **G2** appeared to be less interested in completing the questionnaire to their best of their ability. The questionnaire was given near the end of a 2-hour evening class and many students were probably just anxious to go home. Consequently, over half of the students did not give any answer to **Q.IV** involving Ker(A) (a concept that they were well acquainted with particularly since they just finished a class dealing with nilpotent operators). Such factors probably skewed the overall results.

Systems of linear equation and their solutions are the early focus of a first linear algebra course. There is, justifiably, a lot of emphasis on row reduction and any mention of a system of equations and their solution immediately triggers REF, without even questioning whether row reduction is necessary. This was particularly evident with **G1** and **G3** (where 12 of 40 and 21 of 67 respectively used REF to answer **Q.I**). On the other hand, the students enrolled in a second linear algebra course used row reduction very sparingly, a behaviour that was consistent with the orientation of the second half of a linear algebra course.

Q.II, on whether a non-zero vector can be orthogonal to all the rows of A, evoked the largest number of Z-type responses. These included a frequent assumption that A itself had to be an orthogonal matrix, hence a square matrix; another, that it had to do with linear dependence or independence of the rows of A. Also, confusing linear dependence with the notion of a scalar multiple (in **Q.III**) was prevalent among all but the more experienced **G2**.

6. Conclusion

The pedagogical difficulties involved in teaching linear algebra have been discussed extensively in recent years (e.g. [8, 9]) Sensible recommendations have been made about teaching the first course on the subject, particularly the already mentioned recommendation to emphasise matrices, $\mathbb{R}^n$, and applications, rather than the overarching theory of vector spaces. But even taking such sensible steps a matrix-based course still consists of a large web of interconnected ideas. If we look at a central object, a matrix A, and its roles within an introductory course, we note that:

1. The matrix A can represent a system of linear equations, a linear operator, a change of basis, or be itself an element of a vector space (of matrices).
2. With A we can associate: row space, column space, null space, which have various relations among them or their dimensions which are, in turn, related to the range and kernel of a linear operator.
3. The matrix A can be "massaged" by elementary row operations, and these operations leave some things invariant but not others.
4. Matrix multiplication $A\mathbf{x}$ involves taking dot products of rows or linear combination of columns.

This is quite a handful for students to juggle. Perhaps then, the expectation that they will be able to attain "conceptual understanding" by easily linking several seemingly disparate contexts is unrealistic at the beginning level. Certainly, the results of this study, notwithstanding its limitations, point this out. However, generally speaking, students are not given enough practice and problems of the type that we have asked in the questionnaire. More often, linear algebra instructors tend introduce many new topics, emphasize procedures, point to connections, but not make the web of connections sufficiently explicit and an actual object of study. We feel that this lacuna in the teaching of linear algebra should be addressed.

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Teachers' context preferences for mathematics in grades 8 to 10 in South Africa

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Mathematical Literacy deals with contexts. This paper deals with the contexts teachers would prefer to deal with in Mathematical Literacy. Quantitative and qualitative analyses of data obtained through a survey indicate that teachers accord high priority to deal with health promotion and socio-economic issues. They least prefer issues which have the potential to inculcate negative behaviours prevalent in environments of low socio-economic status. Notwithstanding that the contexts teachers prefer should be considered in the design of learning activities for in-service courses for Mathematical Literacy, it is concluded that the design of such learning activities cannot be driven by the interest of teachers only nor can their interest be ignored. Thus there is a need for a balance between the contexts deemed appropriate by teachers, parents, learners, and others involved in the Mathematical Literacy enterprise.

1. Introduction

It is generally accepted that learners should graduate from schools sufficiently mathematically literate to deal with mathematically-based issues in their lives and as productive and responsible future citizens. In this regard Romberg [1] refers to a shift in school-going mathematics as one where '....the emphasis is on mathematical knowledge put into functional use in a multitude of different situations and contexts in varied, reflective and insight-based ways'. Similar sentiments are expressed in the Programme for International Student Assessment (PISA) where 'Mathematical literacy is defined in PISA as the capacity to identify, understand and engage in mathematics, and to make well-founded judgements about the role that mathematics plays in an individual's current and future private life, occupational life, social life with peers and relatives, and life as a constructive, concerned and reflective citizen' [2]. What is clear from these exemplifying conceptions and purposes of mathematical literacy is the functional use of mathematics in contexts. This functionality is primarily seen as using mathematics to address issues exterior to mathematics. Generally, these extra-mathematical issues are selected by curriculum, learning resource and test designers. The issues thus represent the interests and intentions of these stakeholders in curriculum affairs. Essentially there is nothing wrong or sinister about this although there are historical examples where mathematics curricula were differentiated to serve particular class and other interests. But it has to be accepted that in modern society curriculum, learning resource and test designers do their best to steer clear of such differentiation and seek contextual issues that are such that justice is done to the goals of mathematical literacy as enunciated above. However, teachers are also curriculum stakeholders and very little is known about their preferences of the situations and contexts they want to deal with in mathematical literacy. A search of research bases and web trawling with the sentences and phrases given in table 1 rendered zero hits highlighting the paucity of research in this regard.

Ascertaining such preferences is not easy but some indication can be obtained by surveying, amongst other ways, the contexts that teachers would likely want to deal with in mathematical literacy. The results of such a survey are discussed and reflected upon below.

Table 1: Sentences and phrases used in data base and web-trawl search

Teacher interest in context for mathematical literacy
Perceptions of context for mathematical literacy by teachers
The use of context in mathematics by teachers
Teachers' motivations for using context in mathematics
Teacher preferences of contexts to be used in Mathematical Literacy
Perceptions of teachers of the use of context in school mathematics
Attitudes of teachers towards the use of contexts in school mathematics
Interest of teachers in the use of contexts in school mathematics
Teachers, contexts and mathematics
Mathematics in context by teachers
Views of teachers of mathematics in context

2. Instrumentation, sampling and methodology

The survey is part of a larger study, the Relevance Of School Mathematics Education (ROSME), concerned with the contextual interests of learners, teachers and parents. ROSME was inspired by the Science and Scientists project [3] and its extension, Relevance Of Science Education (ROSE) project [4]. The instrument to survey the contextual situations teachers would prefer was extracted from the survey instrument developed to assess the contexts learners would be interested in. The items of the last-mentioned instrument were identified by mathematics educators from South Africa, Zimbabwe, Uganda, Eritrea and Norway, a group of mathematics teachers from South Africa and one from South Korea [5]. The teacher instrument comprise of twenty items given in Appendix A. Teachers were required to give their preference on a four-point scale with 1 indicating “Strongly disagree” and 4 indicating “Strongly agree”. In addition to expressing their preference for all the items, teachers were also requested to select three of the items they would definitely want to deal with and three they least likely agree with that should be dealt with in mathematics. For these items they had to provide reasons for their choices.

The sample was a convenient one and data were collected by the researcher, his colleagues and by practicing teachers involved in a master's degree course in Mathematics Education. The researcher and his colleagues collected data from teachers attending in-service teacher education courses at the university. The students administered the questionnaire at their own and surrounding schools. Two hundred questionnaires were distributed. The total number returned and other demographic data are presented in table 2. Given the demographics of students attending the university and the areas where the teacher data collectors teach, it is reasonable to assume that the responses were representative of teachers teaching in schools serving learners from low socio-economic status urban and peri-urban areas.

3. Analysis and Findings

The data were analysed using descriptive statistics. As shown in Appendix A, means were calculated and the items ranked according to them. Perusal of the means indicate that the differences between the items choices are not significant. Only the last-ranked item is below the acceptable cut-off point of 2.5 for a four point scale. The choice of 2.5 as cut-off is linked to the neutral point where it is statistically reasonable to pitch the point of movement from a position of non-agreement to agreement and vice versa [6].

The item which teachers ranked highest is that which they deemed would prepare learners to study mathematics at tertiary institutions. Ignoring the third ranked item which in a sense also deals with the future studies at tertiary institutions then the next three items teachers prefer learners in grades 8 to 10 should deal with are linked to socio-economic issues. The three lowest ranked items are those generally deemed as undesirable for school-going young people to be involved with.

Table 2: Demographic Data of Teachers

SEX				Missing	TOTAL
Female		Male		2	
60		87			149
AGE				14	
<30 years	30 to 40 years	41 to 50 years	>50 years		
9	80	40	6		149
TEACHING SUBJECTS				5	
Mathematics	Language	Other Subjects	Primary		
36	25	70	13		149
TEACHING EXPERIENCE				6	
<10years	10 to 20 years	>20 years			
29	89	25			149

Figure 1 gives the percentage of teachers who indicated the items they definitely want (most definitely prefer) and with which they least likely agree (least likely prefer) learners should deal with in their mathematics classes. In this figure, “Future studies” is the item “Mathematics that will help learners to do mathematics at universities and technikons”; “Health” to the item “The use of mathematics in issues about health such as mathematics used to prescribe the amount of medicine a sick person must take; mathematics used to describe the spread of diseases such as HIV/AIDS”; and so forth. The concern with issues, both desirable and undesirable, of social import is also borne out in this overall response of teachers’ expression of the items they would most and least prefer learners should deal with in their mathematics classes.

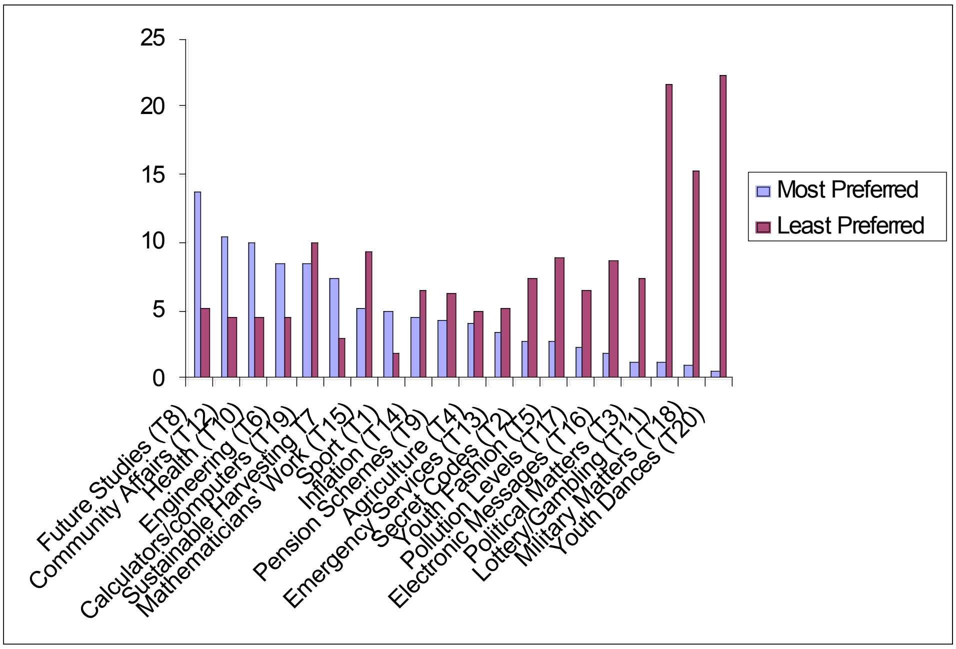

Figure 1: Percentage of teachers indicating most and least preferred items

In order to identify the items for qualitative analysis, the subjects taught by the teachers were clustered as Mathematics, Languages and Other Subjects. Since there were also some primary school teachers who completed the survey and they generally are not subject specialist teachers, their subjects were classified as a single group, Primary Subjects. The five items ranked highest overall (see Appendix A), those ranked by the different subject groupings and those appearing in the list of items definitely preferred were compared and the common items extracted for analysis. This procedure rendered 'The use of mathematics in issues about health such as mathematics used to prescribe the amount of medicine a sick person must take; mathematics used to describe the spread of diseases such as HIV/AIDS' and 'The mathematics to assist in the determination of the level of development regarding employment, education and poverty of their community' as the items ranked highest (table 3).

Table 3: Five highest ranked items across the different categories (Y = item appears amongst five highest ranked items, N = item does not appear amongst five highest ranked items)

	Overall	Math	Lang	Other	Prim	Most Pref
Engineering	Y	Y	N	Y	N	Y
Health	Y	Y	Y	Y	Y	Y
Community Development	Y	Y	Y	Y	Y	Y
Emergency services	Y	N	Y	Y	Y	N
Inflation	Y	N	N	Y	N	N
Sport	N	Y	N	N	N	N
Pollution	N	N	Y	N	N	N
Sustainability	N	Y	Y	N	Y	Y
Secret codes	N	N	N	N	Y	N
Calculators & computers	N	N	N	N	N	Y

A similar procedure was employed to obtain the least preferred items by taking the five lowest ranked items for the same six categories. 'Mathematics involved in military matters', 'The mathematics of a lottery and gambling' and 'Mathematics linked to rave and disco dance patterns' (table 4) emerged as the least preferred items. The reasons provided for the choice of these five items were subjected to qualitative analysis.

Table 4: Five lowest ranked items across the different categories (Y = item appears amongst five lowest ranked items, N = item does not appear amongst five lowest ranked items)

	Overall	Math	Lang	Other	Prim	Least Pref
Electronic messages	Y	Y	N	Y	Y	N
Political matters	Y	N	Y	Y	N	N
Military matters	Y	Y	Y	Y	Y	Y
Lottery/gambling	Y	Y	Y	Y	Y	Y
Youth dances	Y	Y	Y	N	Y	Y
Secret codes	N	Y	Y	N	N	Y
Youth fashion	N	N	N	Y	N	N
Mathematicians' Work	N	N	N	N	Y	N
Sport	N	N	N	N	N	Y

4. Discussion

Why do teachers accord the high preference to health and community affairs? Explanations from a variety of angles can be provided. As mentioned before the teachers are teaching in schools primarily situated in low-socio-economic status urban and peri-urban areas. In these areas the undesirable consequences of social deprivation are highly visible and these issues do have an impact on the teaching and learning activities in schools. Thus at a surface level, the visibilization of issues of health and community development seem to be important to teachers. At another level the highly preferred issues also receive high media (print and electronic) attention as pressing needs to be addressed in contemporary South African society. This awareness seems to have filtered down to teachers.

Beyond these visibility levels of the issues are the teachers' own expressions for the ranking preferences. As stated above these were ascertained by requesting the responding teachers to select from the list of items on the questionnaire the three items they would definitely want learners to learn about in their mathematics classes and provide reasons for their choices. The reasons provided were qualitatively analyzed using primarily the compare and contrast method and exhaustive analytic induction. As with any inductive qualitative analysis process, although various procedures are used to ensure validity and reliability, the labeling of categories is a function of the conceptual gaze and domain-specific experience of the analyzer. Declaring one's conceptual gaze and domain-specific interests and experience is line with the assertion of Miles and Huberman [7] that 'It is good medicine forresearchers to make their preferences clear'. In this study the conceptual gaze and domain-specific experience of the primary analyzer are the interest in school mathematics teacher education and teaching; applications and modelling of school mathematics of particularly social issues and the socio-political dimensions of mathematical education and its consequences for learners from poverty-stricken and marginalized environments.

The major categories emanating from the analysis of the motivations provided by teachers for their preferences are: competent and productive citizenship; coping; enhancement of mathematics teaching; strengthening of consciousness of local issues and utilization of knowledge locally. The categories, their descriptions and exemplar statements are given in Appendix B. These categories are the reasons offered by the teachers for the choice for use of contexts dealing with health and community affairs in the teaching of Mathematical Literacy.

In the same way as teachers' reasons for the essential topics they prefer to be included as contexts for dealing with Mathematical Literacy were determined, their reasons for the topics they least agreed with for incorporation in Mathematical Literacy were analysed. The categories generated appear in Appendix B.

The majority of statements given for the least preferred items were linked to the 'Potential for the inculcation of behaviours which might result in negative consequences' (30%-number of statements link to the category as a percentage of the total number of statements of all categories for the least preferred contexts). Combining this with 'Illegality of contexts for the cohort of learners' (5%) and 'Inappropriateness of contexts for socio-economic development' (8%) then it is reasonable to assert that teachers do not 'take easily' to contexts which they perceive as having the potential of 'strengthening' the ways and habits which burden societies in low socio-economic status areas. Given the strong preferences expressed for issues related to health promotion and socio-economic development, the reasons provided for the contexts that teachers least prefer for treatment in Mathematical Literacy are in direct opposition to such. It is not only the potential for the contribution of perceived untoward behaviour that concerns teachers, there is concern for pedagogical and didactical handling of the contexts. These are discernable in the categories 'Perceived non-mathematical nature of contexts' (14%), 'Lack of readiness of learners to deal with the contexts' (12%) and 'Impediments to teaching' (20%). In fact the pedagogical and didactical dimensions have a marginally higher percentage (46%) than the avoidance of issues which burden societies in low socio-economic status environments (43%).

5. Conclusion

From both the most and least preferences forthcoming from teachers, it seems that they attached high moral value to what they want to deal with in Mathematical Literacy. Their own observations, experiences and environmental knowledge of their teaching contexts seem to drive them to, in addition to deal with whatever Mathematical Literacy deals with, want to have the functional use of mathematics in the sense of wanting to equip learners to engage in actions to improve their own lived conditions. These actions are not only related to issues of future relevance such as the definitions of Mathematical Literacy seem to imply by its reference to future productive citizens. Rather the issues are more immediate ones such as the current state of development of a community. As alluded to above Mathematical Literacy is a new inclusion in the school curriculum. Teachers, particularly in late developing countries, are not well-versed with Mathematical Literacy and its teaching as a curricular orientation. More and more, in South Africa at least, a demand placed on tertiary institutions to design and offer in-service courses in Mathematical Literacy for teachers. Given that Mathematical Literacy deals with contexts and given that teachers do have opinions about the contexts they would prefer to deal with, it is imperative that these contexts should be taken into account when such in-service courses are designed. Not attending to contexts teachers prefer might lead to inclusions of situations which are at variance with the values teachers want to inculcate as Zevenbergen, Sullivan, and Mousley [8] (2002) report. They found that a group of Australian teachers did not find the context of a seeming police identification parade suitable and appropriate as inspiration for an open-ended mathematical activity that dealt with averages.

A caveat, however, is that decisions about what to include and exclude as contexts is not the preserve of only one curriculum stakeholder group. Parents, the educational bureaucracy, designers of learning materials and learners are also important stakeholders in the curriculum endeavour. For example, by asserting that 'our [teachers'] ideas about the real world were somewhat different from our students' ideas.' Lesch [9] brings to the fore how teachers can come up with contexts which learners will not find relevant.

What is thus required, I contend, is a careful and sensible balance between the contexts all curriculum stakeholders prefer learners should deal with in Mathematical Literacy.

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Appendix A
Means and Standard Deviations for Items: Teachers' Contexts Preferences

Item	N	Mean	Std. Deviation
Mathematics that will help learners to do mathematics at universities and technikons	138	3.49	0.737
The use of mathematics in issues about health such as mathematics used to prescribe the amount of medicine a sick person must take; mathematics used to describe the spread of diseases such as HIV/AIDS	139	3.47	0.716
The mathematics in making bridges, airplanes and rockets	140	3.43	0.721
The mathematics to assist in the determination of the level of development regarding employment, education and poverty of their community	140	3.38	0.694
The placement of emergency services such as police stations, fire brigades and ambulance stations so that they can reach emergency spots in the shortest possible time	140	3.30	0.784
The mathematics of inflation	137	3.29	0.778
To do their mathematics with calculators and computers	139	3.29	0.801
Mathematics that will help learners to understand how decisions are made about the sustainable harvesting of natural resources such as the amount of fish that can be caught during a season or the amount of trees that can be cut in a forest.	140	3.29	0.752
Mathematics involved in determining levels of pollution.	139	3.24	0.687
The mathematics involved in making pension and retirement schemes	139	3.18	0.783
The mathematics involved in learners' favourite sport	139	3.15	0.658
The kind of work mathematicians do	139	3.14	0.886
The mathematics involved in agriculture such as deciding the number of cattle to graze in a field of a certain size	136	3.13	0.778
The mathematics linked to modern clothes and shoes young people like	140	3.02	0.800
The mathematics involved in secret codes such as pin numbers used for withdrawing money for an ATM	139	2.99	0.847
Mathematics involved in the sending of messages by SMS, cellphones and e-mails	140	2.96	0.781
The mathematics involved in political matters such as the allocation of seats for parliament given to political parties after an election	140	2.86	0.819
Mathematics involved in military matters	140	2.79	0.718
The mathematics of a lottery and gambling	140	2.55	0.892
Mathematics linked to rave and disco dance patterns	140	2.40	0.863

Appendix B
Categories for most preferred items resulting from the qualitative analysis

Category	Description	Examples of Quotes
Competent and productive citizenship	Development of citizens who will participate knowledgeably in civic and political affairs and to make contributions in the future to the economic development to the country.	Will make them responsible and informed decision makers taking responsibility for their actions They are the future of this country and the leaders of tomorrow. It is important that they know about these things in order to make correct decisions and laws to eradicate the problems we are currently experiencing.
Coping	Management and improvement of the personal lived conditions of the learners	To improve lifeskills and style. This is what the learners really need to do for their lives and careers.
Enhancement of mathematics teaching	Useful as motivational mechanisms and vehicles for the teaching of mathematics and broadening learners' vision of mathematics.	Interesting and stimulating for learners- give them a broader vision of the use of mathematics. To bring maths in the contexts of their daily lives.
Strengthening of consciousness of local issues	Development of awareness of crucial societal issues and what the contributors to these issues are.	Make learners more aware of what is happening in their community and make them more aware of disease Always be aware of how fast HIV/AIDS can spread. Must be aware of community needs. It makes them aware of important critical implications of certain health issues. Learners expectations and aspirations are very low because of various factors eg poverty and socio-economic (se) problems etc. So they become what the current system and their se position dictate to them.
Utilization of knowledge locally	The application of their mathematical knowledge as an explanatory and motivational mechanism to take actions	The items are very important to alleviate the problems that hamper the community. By understanding the maths of these issues they would be able to do better decisions regarding these issues.

Appendix C
Categories for least preferred items resulting from the qualitative analysis

Category	Description	Examples of Quotes
Perceived non-mathematical nature of contexts	Does not comply with what is perceived as acceptable as mathematics.	Mathematics is a global subject and these three items don't have the substance for global standards and application. Lottery and gambling are not mathematical concepts nor rave and dances. It adds little value to the deeper values of the study of maths
Lack of readiness of learners to deal with the contexts	The learners that are being referred to are not at intellectual, emotional and experiential levels of development to deal with the topics.	Too abstract for their level of thinking. The learners need to wait until they reach university to do that kind of work. These are issues that will not help learners to develop intellectually. Learners might be too young to handle the responsibility of this kind of knowledge.
Impediments to teaching	Dealing with these issues will deflect from what is expected to be taught and need additional time.	Most of them enjoy the dance and forget about the patterns they suppose to learn. It will take too much time doing it in the classroom It's going to overload them. Other learning areas (subjects) can deal with that issues.
Illegality of contexts for the cohort of learners	The topics are limited and have the potential of causing discord.	We don't want to sketch a scenario that rave and disco is normal to attend for children They must wait until they are 18 years old.
Inappropriateness of contexts for socio-economic development	The topics will counter the need to address pressing socio-economic issues	Because these items do not contribute to the development of the country. Increase poverty.
Potential for the inculcation of behaviours which might result in negative consequences	The behaviours accompanying the topics are not in accordance with values being striven for.	If you want to instill positive value these might be the opposite effect. This normally encourages learners to do the wrong things It would lead to other social problems. Rave and disco are places where drugs are mostly sold.
Personal affair	The topics belong to the private life of individuals.	Is very personal Gambling is a matter of personal interest.

Components of learning and assessment in linear algebra

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Linear algebra provides theory and technology that are the cornerstones of a range of cutting edge mathematical applications, from designing computer games to complex industrial problems, as well as more traditional applications in statistics and mathematical modelling. The challenges of balancing theory, practice and computational work across mathematical and statistical applications are considerable, particularly given the diversity of abilities and interests in typical cohorts. In this paper we consider two such cohorts in a second level linear algebra course in different years. The course objectives and materials were almost the same, but some changes were made in the assessment package. In addition to considering effects of these changes, the links with achievement in first year courses are analysed, together with achievement in a following computational mathematics course. Some results that may initially appear surprising provide insight into the components of student learning in linear algebra.

1. Introduction

The linear algebra unit (MAB312) under focus here is a second level course taken by students in mathematics degrees; double degrees of mathematics with information technology, business, law or electrical engineering; science degrees, usually majoring in physics/mathematics but some combining mathematics and biotechnology in maths/science double degrees; and some education and postgraduate courses. MAB312 is an important unit for mathematics and statistics students proceeding to a variety of areas ranging from operations research, statistics and financial mathematics to mathematical modelling, computational mathematics and visualisation. Much of modern linear algebra is closely associated with, and applied in, computational mathematics and complex or large-scale industry or research problems, and MAB312 is a prerequisite for the computational mathematics strand. It is particularly popular with students combining mathematics with information technology. Graduates making their mark in areas such as the computer games industry or visualisation have identified the pivotal roles played in their careers by the skills developed in this unit.

Focus on mathematicians and statisticians as educators [1] is often on first level units or on maths anxiety, but it is of vital importance to the professions and to society that the learning and teaching in second, third (and beyond) levels, receive the same attention. In all areas of mathematics and statistics, a challenge for tertiary teachers is finding a balance between theory and practice that is appropriate and relevant for the learning cohort. As indicated above and discussed in [2] and [3], this balance is very pertinent in modern linear algebra because of the wide-ranging nature of both theory and applications, the close links with computational mathematics, and the diversity in typical cohorts. Cohort achievement bimodality has been of particular concern in second level linear algebra. Examination of the literature suggests the balance of theory and practice is an issue in student learning in this area, and initial discussion with students and staff indicated that student background and engagement play key roles. Changes were made in the continuous assessment in 2005, aiming to maintain student engagement throughout the semester. Hence the motivation for the analysis in this paper includes the effects of the continuous assessment, the balance of theory and practice, and the predictors of achievement from first level units.

2. The cohorts

MAB312 is not a service unit, with all students enrolled in the unit either taking a mathematics degree course or a major or comajor in mathematics, with a considerable proportion of the cohorts heading towards double degrees or honours, and either research or highly quantitative, rewarding careers in industry, business or government. However the cohorts can still be somewhat bimodal for a variety of reasons. For example, the two cohorts considered in this paper are the 2003 and the 2005 classes. In 2003, 65 students enrolled in the unit, including 12 education students of whom 6 withdrew or failed, while in 2005, 96 students enrolled in the unit, including 20 science (mostly physics) students of whom 12 withdrew or failed. All had the formal prerequisites and the formal background knowledge for the unit, but, as discussed below, analysis shows that the best first level predictor of achievement in the unit is an introductory unit in probability and distributional modelling.

The overall rating of the unit on the 2005 student surveys consisted of 14% giving a rating of satisfactory, 34% a rating of good, and 50% a rating of very good (2% did not respond to this question).

In 2004 there were 92 students enrolled in the unit. This paper focuses on 2003 and 2005 because the teaching strategies were similar in those years, with the main difference being changes in the continuous assessment. In 2004 the development of material in the unit was slowed in an attempt to reduce the bimodality problem. As is often found in mathematics, this was not in the best interests of any of the cohort, and 2005 focussed on changing the continuous assessment and providing extra support as needed through the programs of the Maths Access Centre [4].

3. Content, pedagogy and learning experiences

3.1. Student background

The aims, objectives, structure and materials of the unit are informed by

- knowledge of the range of skills and confidence of students entering the unit
- knowledge of the range of skills and capabilities students need to take from the unit into further study, and, ultimately careers
- feedback from current and past students, staff and employers
- literature on teaching linear algebra such as [5]

The formal prerequisites are a first level calculus unit (MAB111) and a first level introductory linear systems and analysis unit (MAB112), with the brief synopsis of the latter being

> linear systems and matrices; vector algebra; coordinate systems; introduction to abstract algebraic systems; complex numbers; first and second order differential equations.

Entry to these first year units is via advanced mathematics in senior school or an equivalent first level unit. Alternative prerequisites are the first level engineering mathematics units which are also taken by science students majoring in physics.

Most of the MAB312 cohorts have also taken an introductory unit in modelling with probability and distributions (MAB210), an introductory unit in computational mathematics (MAB220), and an introductory data analysis unit (MAB101). MAB101 is taken by a very large cohort of all science and other students and has a non-mathematical emphasis, so is not considered here. It is usually assumed that MAB112 above is the main prerequisite for MAB312, but the analysis reported in this paper identifies MAB210 as key, so its brief synopsis is given here, namely:

> probability; independence; system reliability; using conditional probability in modelling; introductory Markov chains; random variables; special distributional models; Bernoulli

> process; Poisson process; exponential; introductory queueing processes and collecting data from them; expected values; distribution function; goodness-of-fit tests; measures of dependence; introductory bivariate and correlation properties; conditioning arguments.

Thus the content has little direct bearing on MAB312, but the emphasis in MAB210 is constructivist and problem-based learning, focussing on applying and bringing together mathematical techniques in new contexts, concepts and models. It is this synthesis of techniques and problem-tackling with new contexts, theory and applications that appears to be the common thread linking these unlikely partners.

3.2. Content

The four main topic areas of MAB312 are: systems of linear equations and matrix algebra; vector spaces; inner product spaces and eigenvalues and eigenvectors.

The first topic includes revision and continues with systems of linear equations and an introduction to Maple; general matrix properties; Gaussian elimination; homogeneous linear systems; the general solution of a linear system; properties of inverses; powers of a matrix; determinants and their properties and a brief introduction to eigenvalues and eigenvectors.

In topic 2, the general concepts of vector spaces are introduced along with the notion of linear combination and span, linear independence, basis and dimension; rank and nullity; subspace construction and an introduction to linear transformations.

In topic 3 inner product spaces are discussed by introducing inner products; orthogonality; orthonormal bases; followed by the Gram-Schmidt process and QR-decomposition; orthogonal projections; best approximation and least squares solutions; data fitting and function approximation.

Finally, in topic 4 the diagonalisation of a matrix is presented along with computing powers and functions of matrices.

The examples and learning experiences are motivated by higher level needs in the fields of statistics and computational mathematics, and by applications based on experience with industry problems. As advocated in [2], the lecture material and examples form a coherent structure that has been derived from experience in research and teaching with this fascinating topic [6], with the blend of theory, problems, applications and computational practicals oriented to balance across the current and future needs of the whole cohort. Theory is included where it introduces or illustrates mathematical techniques of value to students in their higher level units, or of value in mathematical thinking [7], [8]. Full lecture notes are provided, and an appropriate text such as [9] or [10] is used to supplement the lecture notes, examples and exercises. Lecture notes and supporting materials are available via the Online Teaching (OLT) system. Provision of full notes frees the class contact hours for focus on key aspects, examples, demonstrations of the use of Maple in examples, questions and discussion. This is our standard practice in mathematics and statistics and students greatly value use of class contact hours for guidance, emphasis, facilitation of understanding, demonstrations and discourse, developing a true 'learning community' [11].

Of four contact hours per week, one is a computer practical in which students use Maple to explore linear algebra concepts and applications. An optional student-driven support session is also offered under the auspices of the Maths Access Centre. Students are free to use such support sessions whenever and however they wish.

4. Assessment

> If learning really matters most, then our assessment practices should help students develop the skills, dispositions, and knowledge ...... [11]

> Students study more effectively when they know what they are working towards... Students value assessment tasks they perceive to be 'real' [12]

4.1. The 2005 cohort

The assessment for the unit was split into 40% continuous assessment and 60% final examination. The objectives of the continuous assessment are to facilitate learning and development of skills and understanding. The continuous assessment consisted of three quizzes totalling 16% and two Maple group assignments totalling 24%. Each quiz comprised three questions, the first being a series of ten true/false answers and two other questions based on application of the relevant lecture material. Students were given five days to complete the quiz and submit their answers online. The solutions and results were also posted online. The quizzes spanned the semester and student feedback (both informal and formal via standard unit evaluation surveys) indicated that they were a popular assessment item and perceived by the students as excellent assistance in encouraging student engagement. The Maple group assignments had three students per group. All groups attempted the same assignment and had two weeks to finalise the work and submit their Maple worksheets online. The questions given in the assignments were extensions of the weekly practicals, but cast within interesting case studies in linear algebra. Examples included formulating mathematical models based on matrix systems and interpreting general solutions where free variables play an important role in the model; exploring techniques in information retrieval; or formulating least squares problems in coordinate meteorology. Complete Maple solutions were posted online and groups received a results spreadsheet highlighting any errors or flaws in their thinking. Student feedback indicated that this was an enjoyable component of the unit; in 20% of the surveys, respondents chose to add comments on “enjoying the assessment/unit”. Overall the continuous assessment was done very well and students scored highly. Responses to questions about assessment on the student surveys were

The assessment tasks are clearly related to what I am expected to learn

NA	SD	D	N	A	SA
2%	0	0	7%	29%	62%

I have been provided with guidelines or criteria which give me a clear explanation of how individual assessment will be marked

NA	SD	D	N	A	SA
2%	0	9%	20%	35%	35%

I understand the requirements of the overall assessment program

NA	SD	D	N	A	SA
2%	0	0	12%	38%	49%

NA=not applicable, SD=strongly disagree, D=disagree, A=agree, SA=strongly agree

In the final examination students were asked to attempt four from a total of five questions. Each question consisted of two parts, the first part based on theory and the second based on applications similar in style to those covered over the semester. As discussed above and in [7], theory in this area is important, but it was expected that more students would focus on the application components than the theory. However more than half the class attempted both parts of the questions, with many successful completions. Naturally, overall the scores on the final were nowhere near the level attained during the continuous assessment, but the interest in the analysis below lies in which continuous components were the best predictors of the scores.

4.2. The 2003 cohort

In 2003 a slightly different assessment strategy had been adopted, which consisted of three Maple group assignments, each of value 7%; a mid-semester exam contributing 15%; and a final examination contributing 64%. The final examination was similar in style, format and level to that of 2005. The style of the Maple group assignments were consistent with those used in 2005, however the major difference of using a mid-semester exam rather than quizzes proved a key indicator to overall student results when comparing 2003 and 2005.

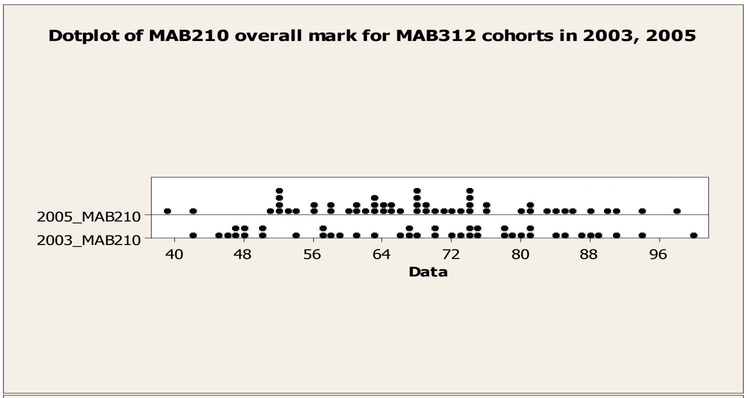
5. Analysis of data

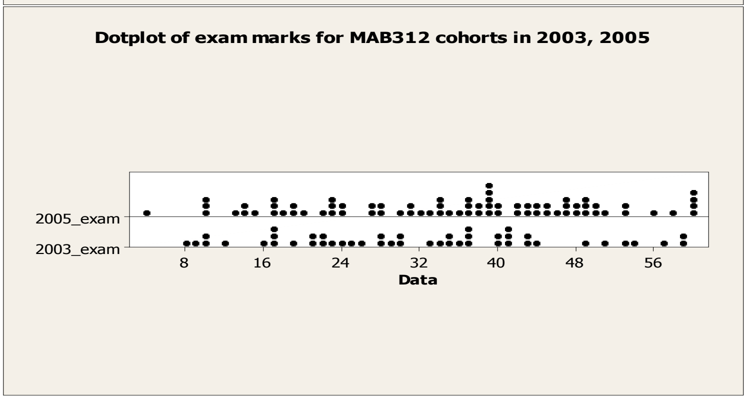
5.1. The continuous assessment

The 2005 student surveys reported above were taken before the exam period, demonstrating student approval of the continuous assessment. The changes in the assessment from 2003 to 2005 were introduced with a view to helping students manage their learning throughout the semester. Because of the generally very high marks in the continuous assessment contributing 40%, the overall marks in 2005 tended to be higher than in 2003. The questions of interest are: did the changes in the continuous assessment program affect the students' learning, for either better or worse; and which components of the continuous assessment were the better predictors of performance in the end of semester exam?

Analysing student results requires considerable care because of the many possible interdependencies and confounding or hidden variables, particularly when comparing results of different cohorts. For example, simple correlations between pairs of variables tend to be at best difficult to interpret, and at worst considerably misleading. Also in this particular case, the overall results in 2003 and 2005 should not be compared because of the different continuous assessment programs. However the end of semester exams in 2003 and 2005 were similar in construction and level, and can be used for comparison purposes. Examination of the cohorts revealed that in 2003 there were ten students in MAB312 in a particular course (Bachelor of Education) but none in 2005 because of a change in the course structure. The remainders of the cohorts were reasonably similar between the two years, except for a small number of advanced physics students in 2005. If the ten BEd students are included in the 2003 cohort, a simple 2-sample t-test gives evidence of a higher average exam mark in the 2005 cohort, but not if these students are omitted.

Although dotplots have the disadvantage of presenting frequencies rather than relative frequencies, they also retain detail that can be hidden by plots that group data. The dotplots below (with the BEd students omitted) show a similarity in location and spread of MAB210 marks for the cohorts, but a tendency for a greater proportion of students in the 2005 cohort achieving above 50% in the MAB312 exam. This provides reassurance that the change in the continuous assessment program is not detrimental to performance, and appears to assist in learning.





Best subsets methods were used to analyse relationships between continuous assessment components and the exam. In 2003, the Maple group assignment 2 and the mid-semester tests were the best predictors of the exam marks (and hence of the total performance), explaining 55% of the variation and with a CP-Mallows index of 1.5. In 2005, quiz 2 and the Maple group assignment 1 were the best predictors of the exam results, explaining 25% of the variation and with a CP-Mallows index of 1.3. The residual plots showed no problems with the models, and the unusual observations had 0 for one of these components of assessment - important data points to retain. In both years, despite some differences in the continuous assessment arrangements, the best predictors of exam achievement were a combination of a theoretical task and a practical task, with each involving key conceptual aspects.

2003: The regression equation is

```
exam = - 1.51 + 3.41 Maple_assign2 + 1.15 mid_sem

Predictor         Coef  SE Coef      T      P
Constant        -1.512    7.326  -0.21  0.837
Maple_assign2    3.409    1.192   2.86  0.006
mid_sem         1.1546   0.1628   7.09  0.000

S = 15.7381   R-Sq = 56.6%   R-Sq(adj) = 55.0%
```

2005: The regression equation is

```
Exam = 4.29 + 0.320 quiz2 + 0.289 Maple_assign1

Predictor          Coef  SE Coef     T      P
Constant          4.289    6.810  0.63  0.531
quiz2           0.31968  0.08991  3.56  0.001
Maple_assign1    0.2892   0.1347  2.15  0.035

S = 13.3941   R-Sq = 26.4%   R-Sq(adj) = 24.5%
```

5.2. The first level units as predictors

Analysis of the relationships between achievement in MAB312 and in the first level units, as described in 3.1 above, produced some unexpected but enlightening results. As described, the prerequisites are MAB111 (calculus) and MAB112 (introductory linear systems), with the content of the latter closer to the background one would expect for MAB312. A number of the MAB312 cohorts have first year engineering mathematics as alternative prerequisites, either because they are physics majors or because they changed from engineering to mathematics. Hence analysis considered both scenarios, namely, scenario 1 with strictly MAB111 and MAB112, and then scenario 2, grouping the appropriate engineering mathematics unit with each of MAB111, MAB112 to give 'MAB111' and 'MAB112'. For the 2005, but not the 2003, cohort, the Grade Point Average (GPA) on entry to MAB312 was also available for inclusion in the analysis.

For the 2005 cohort, for both scenarios, the single best predictor of achievement in MAB312 of GPA, MAB111 (or 'MAB111'), MAB112 (or 'MAB112'), MAB210 (probability and distributions) and MAB220 (introductory computational maths) was MAB210, explaining 47% and 44% respectively of the variation in scenarios 1 and 2. The best two (and the significant) predictors in scenario 1 were MAB210 and GPA, explaining 52% of the variation; in scenario 2, they were MAB210 and 'MAB112', explaining 51% of the variation. Omitting GPA and considering just the first level units, the best two (and the significant) predictors in scenario 1 were MAB210 and MAB111, explaining 54% of the variation.

For the 2003 cohort (with no entry GPA available for analysis), the single best predictor in scenario 1 was MAB112, but MAB210 in scenario 2. The best two (and the significant) predictors were MAB210 and MAB112 or 'MAB112' in both scenarios, explaining 61% and 58% respectively of the variation.

5.3. The first level units and MAB312 as predictors for computational mathematics

Analysis of achievement in the second level computational mathematics unit, for which MAB312 is a prerequisite, is possible only for the 2003 cohort at present. For that cohort, the best (and the significant) predictor in either scenario was MAB111 or 'MAB111'. In the absence of information for this cohort on GPA at entry to level two units, it is highly likely that this unit is representing GPA.

6. Discussion

Under both continuous assessment programs, a test-type component and a Maple group assignment component combined as best predictors of an exam comprised of both theory and applications but with no actual Maple use, providing support of the claims in the literature, such as [2] and [3], that both theory and practice contribute to overall learning and understanding in linear algebra. The lecturer's concerns about the very high marks in the 2005 continuous assessment program are reflected by only 25% of the variation in exam marks being explained, but this challenge of how to grade the continuous assessment can be tackled with confidence in the program's facilitation of student learning across the theory and practice components of the unit.

The link between MAB210 and MAB312 is surprising until it is recognised that MAB210 consolidates foundation skills across both calculus and algebra in problem-based learning that requires students to synthesize prior and current learning in situations combining theory and applications, and requiring at least some dissection of problems into smaller segments. Another interesting link is that the best predictor components of continuous assessment of MAB312 were on topics requiring considerable conceptual learning; this is also a major aspect of MAB210.

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Laplace Transform across boundaries

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Originally, the Laplace Transform was created to solve particular mathematical problems and several decades later, it has become widely utilized in many disciplines in science and engineering. From an educational view point, previous research concerning the Laplace Transform involved second and third year university students, solving real-life problems, in their chosen fields. There appears to be little research where this opportunity is given to first year students. This paper will describe the introduction of the Laplace Transform in a first Calculus course. Three groups of students will be compared, where two groups received the traditional lectures and tutorials whilst the other group will be introduced to the Laplace Transform. The outcomes of this innovation and experience led to several conclusions and recommendations for other similar first year engineering courses.

1. Introduction

The Laplace Transform method reduces the solution of a linear ordinary differential equation (O.D.E.) to an algebraic procedure [1] and may also be helpful for other applications such as linear partial differential equations (P.D.E.), [2]. This powerful mathematical tool has been employed in many disciplines including Mechanics [3], Chemistry [4], Electromagnetism [2], Reactors Design [5], Pharmacokinetics [6] and Environmental Engineering [7].

In teacher training courses, the Laplace Transform is one of the mathematical concepts used when solving real-life problems, where emphasis is directed to "Modeling" [4] and/or "Applications" [8]. Recently, a whole course devoted to the Laplace Transform and its applications, was given to university mathematics educators (i.e., teachers and researchers from several Argentinean universities) [9].

In previous research, the Laplace Transform is used by second and third year students to solve real-life problems. There appears to be little or no research where this opportunity is given to first year students. This paper will describe the introduction of the Laplace Transform (at a preliminary level) in a first Calculus course.

Three groups of students will be compared at the end of semester where two groups received the traditional lectures and tutorials whilst the other group was introduced to the Laplace Transform. The outcomes of this innovation and experience led to several conclusions and recommendations for other similar first year engineering courses.

2. Rationale and preliminary considerations

Several disciplines at the University of the Republic of Uruguay (the only state university), require that students are able to solve certain elementary O.D.E. by the end of first semester. Some subjects at university such as physics, chemistry and economics require a deeper expertise of O.D.E. theory, by this time hence Mathematics educators are requested by other departments to teach O.D.E. methods at the beginning of the year. The difficulty lie in the fact that students are only taught about derivatives and Riemann integral (in Calculus course) and vectorial spaces (in Linear Algebra course) in the first semester. These students do not yet possess the knowledge and mathematical maturity to learn about O.D.E. theory and methods.

The "typical approach" is to postulate different solutions for several kinds of O.D.E., then a verification of these solutions by differentiation. By using this teaching methodology, the other disciplines' demands are satisfied and the syllabus is strictly adhered to, however, this is not very motivating for student or lecturer.

This is a paradox, because O.D.E. is a mathematical tool with a wide range of applications, which is known to be closely related with motivation [10].

So, the question is: "can this problem be solved without making dramatic changes in the syllabus?" To answer this question, the following points were considered:

i) The Laplace Transform can be studied at this level as a simple parametric improper integral. This is not new for the students because they use to study other parametric integrals such as the harmonic one:

$$\int_{1}^{+\infty} \frac{1}{x^s}\,dx$$

In this case, they arrive to a well known classification (the integral converges if and only if) which depends of the parameter ( ) values.

ii) A list of exercises for simple improper integrals can include the next integral

$$\int_{0}^{+\infty} F(t)e^{-st}\,dt$$

for different functions , allowing the student to construct a self-made Laplace Transform table which can be employed later.

iii) The first two subjects taught in typical O.D.E courses are "Separation of Variables" and "First Order Linear O.D.E" [11]. It is important to note that the second one is strictly related with the Laplace Transform. In fact, if the following O.D.E is considered:

$$\begin{cases} y' - sy = F(x) \\ y(0) = 0 \end{cases}$$

the solution is easily computed as:

$$y(x) = \left[\int_{0}^{x} F(t)e^{-st}\,dt\right] e^{sx}$$

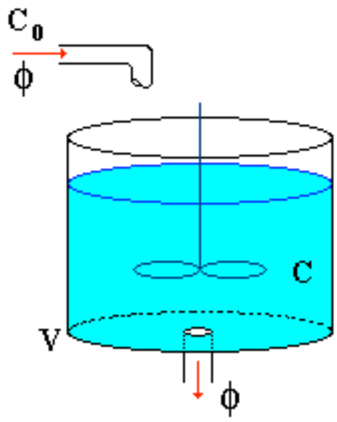
where the expression in brackets is an integrating factor. The Laplace Transform can be regarded as a limit of the integrating factor when , again, the Laplace Transform is related with previous concepts thus it is introduced as a complementary technique rather than another topic.

iv) This approach states a strong relationship between Laplace Transform and O.D.E solving methods, making it reasonable to spend more time in analyzing a few properties (linearity, translation, transformation of derivatives, etc.). These properties and a Laplace Transform table, like (ii) or a more complete version [12], can be used to solve Linear O.D.E of Second Order (or even higher), without postulating solutions.

3. An experience in first semester Calculus courses.

As a result of this analysis, an innovative approach is proposed in the first semester Calculus course: while two of the three lecturers taught in the same manner as previous years, the other decided to use this Laplace Transform based approach. Several examples of applications of this "new" mathematical tool were presented during this new style of lecture. Moreover, several modified advanced versions of those examples were used later in teacher training courses ([4], [8] and [9]).

For example, a simple problem proposed to first year students was related with salt concentration in a tank, like in the following figure:



The input of this system is a solution of salt in water with a volumetric flux ϕ (L/s) and a concentration C_0 (g/L), being V (L) the tank volume. If ϕ and V are constants, then:

a) Propose a Differential Equation to express the variation of salt concentration in the tank versus time.

b) Solve the equation proposed if (i.e., there is only water in the tank at t=0).

c) Compute the quotient

$$G(s) = \frac{L\{C(t)\}}{L\{C_0(t)\}}$$

The solutions for first two parts are very simple: Part (a) is:

$$\frac{dc}{dt} = \frac{\Phi C_0 - \Phi C}{V}$$

Part (b) is:

$$C(t) = C_0 + ke^{\frac{\Phi}{V}t}$$

Solving part (c) it is more convenient to put the O.D.E. in the following form:

$$V\frac{dc}{dt} = \Phi C_0 - \Phi C$$

Applying the Laplace Transform to this equation we get:

$$V\left(sL\{C\} - C(0)\right) = \Phi L\{C_0\} - \Phi L\{C\}$$

Rearranging:

$$L\{C\}(Vs + \Phi) = \Phi L\{C_0\}$$

Then

$$G(s) = \frac{L\{C(t)\}}{L\{C_0(t)\}}$$

can be computed as

$$G(s) = \frac{\Phi}{Vs + \Phi}$$

Finally, it is important to mention that, if an average time

$$\frac{V}{\Phi} = \tau$$

is considered, then

$$G(s) = \frac{1}{1 + \tau s}$$

This equation gives the Transference Function of an ideal chemical reactor: the continuously stirred tank reactor (CSTR).

This example illustrates how first year students can be introduced to technological issues, strongly related with their careers, through very simple problems were the Laplace Transform plays an important role.

Final examinations were identical for all the students whichever style of teaching they experienced. In the final two weeks of semester, the Education Department carried out a teacher evaluation, in the form of a questionnaire to the students. It is important to mention that this "official" evaluation was not designed to assess this internal experience, but it did provided interesting results that will be mentioned later in the paper.

4. Results

The Education Department evaluates teachers, courses, assessment procedures (exams, etc.) by asking for students' opinion through questionnaires. This questionnaire is composed of 25 questions using a Likert scale as follows:

a) Total agreement 10
b) Partial agreement 7.5
c) Indifference 5
d) Partial disagreement 2.5
e) Total disagreement 0

From this set of twenty-five questions, seven are particularly relevant and important to evaluate the innovative approach. These questions are:

1) The pitch and pace of class can be followed by the students
2) The examples presented in the classroom illustrate the courses' main concepts.
3) Relationship with other subjects is established.
4) An applied approach is developed, giving examples and applications connected with real-life problems and professional practice.
5) Students are motivated in this course.
6) Students feel comfortable and enjoy classes.
7) Final exams and assessment problems and exercises can be solved using knowledge obtained in class.

Table 1 compares the average scores for the seven questions in both the innovative experience (group A) with the other two traditional groups (group B and group C). It is important to mention that students had been randomly placed in one of the three groups.

Group A teachers' assessment can be compared in two different situations: in 1993, before the innovative experience, and after the innovative experience, in 1997. This comparison can be observed in the following table:

Table 1: Comparison between ”innovative” and ”traditional” groups.

	1	2	3	4	5	6	7
A	9.27	9.52	8.87	9.03	8.39	8.47	8.75
B	7.88	8.00	7.50	6.44	6.50	7.31	7.56
C	6.94	6.48	4.22	4.19	5.16	5.58	4.90

Table 2: Group ”A” teachers’ assessment before and after the innovation.

	1	2	3	4	5	6	7
A (1993)	8.45	8.87	8.20	7.61	8.63	8.57	-
A (1997)	9.27	9.52	8.87	9.03	8.39	8.47	8.75

A comparison of the teaching style is possible before and after the innovative experience, in 1993 and 1997 respectively, since Lecturer A taught equivalent students both times.

There are other things that must be commented:

a) Question 7 was not included in the questionnaire in 1993, so there is no result for that year.
b) The mean scores in questions 5 and 6 diminished slightly, this may be due to the dramatic increase in the number of students between 1993 and 1996.
c) It follows from Table 2, that the first four questions showed an important mean score increase. Moreover, if questions 1 to 6 are considered, there exists an average increase of 6.65 %, which is very important, because this improvement was obtained teaching the same syllabus plus an extra subject (the Laplace Transform, and several applications).
d) There were no significant changes in these scores after 1997.
e) Finally, it is important to mention that the final examinations included O.D.E. exercises that could be solved using or not the Laplace Transform.

All these results were reinforced by several students’ comments about different topics as applications, motivation, etc. Several examples are the following sentences:

> Student I: ”I never thought that Maths had so many applications related with my career”
>
> Student II: ”Teachers help us in solving real-life problems, but they don’t do all the task...we must work hard...this is the best way to learn”
>
> Student III: ”All Maths courses should be like this”

Another consequence of this innovative approach was the dramatic improvement in comprehension as students had their second experience with O.D.E. and Laplace Transform. In fact, tests in fourth semester course showed better performances in students post-1997 with the best results obtained by students who participate in group ”A” (i.e., the innovative group) in 1996.

Finally, teachers involved in this innovative experience (i.e., four different professors, considering those of group “A” in first semester and those who teach in fourth semester) achieved the best results in motivation and students’ performances.

5. Conclusions

The Laplace Transform is an important mathematical concept with well known applications in Engineering, Physics, Chemistry, etc. There is an equally important role for the Laplace Transform to play in Mathematical Education.

This fact is observed by several other authors, though usually, this mathematical tool is exploited only in higher level university courses and not so commonly introduced in first semester courses. This technique for solving O.D.E. is useful as the motivation of the students is high. It also provides the opportunity to develop mathematical models, related with the core subject of different areas.

This paper mentions only an experience in chemical engineering courses. This new approach may be utilized in other areas where there is an increasing number of applications of the Laplace Transform such as economy, environmental sciences, etc. This may be the next area to explore in mathematical education.

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Preliminary investigation into aggregating grades in a first year course with mixed assessment types

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This paper provides an overview of the types of assessment and the methods used to produce an overall grade in a first year university mathematics course. Due to the nature of the open-ended problem solving assignments used in assessing this course, criterion-based grading is applied to these items of assessment. As a consequence all other items of assessment in the course are graded rather than scored to match university assessment regulations. The processes used in grading rather than scoring assessment items and the problems encountered in aggregating grades to produce an overall grade are discussed. Reliability of the examination result appears to be improved by aggregating scores/marks rather than grades.

1. Introduction

Assessment is an integral part of the learning process. Used effectively it encourages students to engage in their learning, influences their approach to learning, measures their level of understanding or attainment, ranks them, and can provide feedback on their progress [1, 2]. The assessment process results in an overall grade which in some way is an aggregate of all the summative assessment items for a course, reflecting 'a range of factors including student effort, student ability, the quality of the teaching, the design of the assessment, and the implementation of the assessment procedure' [1]. The design and implementation of an assessment item includes the decision about whether to use scores, marks or grades to report the outcome. The scores, marks or grades awarded to each assessment item are then aggregated. The aggregation is based on some sort of weighting scheme which usually reflects the course objectives. This aggregated grade represents a student's overall level of attainment in the course.

Assessors seek to ensure that the marks and grades achieved by students accurately reflect the attainment of the course objectives. Hence, the reliability and validity of the scores, marks and grades is an important consideration in the design and implementation of an assessment scheme [3]. Reliability can be described as the dependability of any measure of achievement, whereas validity encompasses the relationship between the assessment items and the objectives of the course. If the grading of assessment items is not reliable, misleading judgements about validity can be made. Reliability is not only a product of what is being examined but how it is examined, including how an aggregate grade is calculated. McLachlan and Whiten [1] suggest that 'scores should always be converted to grades before aggregation', because different methods of assessment give different distributions of scores. However, Wood [4] suggests that 'an overall grade based upon combinations of marks will, in general, be a more reliable measure of achievement than an overall grade based upon combinations of grades'.

Whether to use a mark or a grade may depend upon the type of assessment being administered. In a mathematics course students may be required to demonstrate content knowledge and understanding, apply this knowledge to solve open-ended questions and demonstrate the ability to communicate mathematical concepts and arguments. As a result, assessment items within the course may vary in their focus, e.g. skills-based questions testing content knowledge, open-ended problem solving questions testing application, written reports testing communication skills. These different types of student attainment may be more reliably assessed using different methods. Smith and Wood [2] suggest that 'marking schemes for extended and open-ended questions should also be developed, as this is not an area familiar to many mathematicians'. Open-ended problem solving questions lend themselves well to a criterion-based method of assessment. In this form of assessment the weights assigned to the components used to derive a total score (or grade) have important reliability and validity implications [5]. The existence of criteria should ensure consistency [6], but marker bias and variability may still pervade [7]. The criteria selected for grading should validly reflect what a grade means: level of student achievement [3, 8].

In certain circumstances criterion-based grading, in contrast to criterion-based marking, has advantages. Use of a grade scale often leads to a greater spread of results, provided that the grade scale is not calibrated to a percentage scale [9]. Grading scales may provide a justifiable alternative to the unwarranted precision identified with percentage scales, as it is questionable whether a marker could maintain consistency with the degree of precision required by a percentage scale [10, 11]. Grade scales may appear to overcome particular difficulties associated with percentage scales but Bridges et al. [11] believe they may cause other problems when some grades are underutilised. This may occur when individual grades are not clearly defined in terms of the quality of the performance which they reflect.

This paper provides an overview of the types of assessment and the methods used to produce an overall grade in a first year university core mathematics course and investigates the reliability in calculating the final examination result using grades. In particular, it discusses the implications of grading each examination question and then applying a 15 point equivalence scale to aggregate these grades using a weighted average.

2. About Foundation Mathematics

Foundation Mathematics is a core first year course within Engineering and Surveying, Sciences and Information Technology award programs at the University of Southern Queensland, Australia (USQ) and enrols approximately 600 students each year (250 on-campus and 350 distance education). Students come from a diversity of mathematical backgrounds with more than a quarter of the students having no experience of high school mathematics with calculus [12]. In order to cater for this diversity, students can complete the course by one of three pathways – Entry Point 1 students who have the least mathematical experience may take two semesters to complete the course and work through three study books starting with basic arithmetic; Entry Point 2 students who have a bit more mathematical experience may take two semesters to complete the course and work through two study books; and Entry Point 3 students who have generally had some experience of calculus take one semester to complete the course and work through one study book [13]. The examination for the course is based on the content of this final study book.

3. Assessment Items

Assessment for this course includes a preliminary test, three assignments, quizzes, a workshop submission and a final examination. Students initially complete a preliminary test to determine their entry point. This is followed by Assignment 1 which is a compulsory planning assignment. It does not contribute to the final grade and is designed to encourage early engagement in the course. During the first semester students are progressively assessed on mathematical and problem solving skills and mathematical communication in a group work environment (entitled 'workshop submission'), in tutorials for on-campus students and in a WebCT Discussion Group environment for distance students (4% of the final grade). Online quizzes related to the six examinable modules of content in the course are taken at the end of each module (6% of the final grade). These quizzes are a mixture of multiple-choice and short-answer questions which test the skills-based content and give immediate feedback.

Apart from these developmental assessments, students are assessed on their problem solving skills and mathematical communication through two problem solving assignments. The first

(Assignment 2 – 10% of the final grade) is delivered in week 8 of the first semester and the second (Assignment 3 – 20% of the final grade) at the end of the first semester for Entry Point 3 students and at the end of the second semester for Entry Point 1 and 2 students. These assignments are open-ended and designed to be solved effectively using a range of mathematical skills and strategies. The assignments are graded (HD, A, B, C, or F), rather than allocated numerical marks. A marking scheme allocates percentage weightings to each component of the reports (Aim = 5%, Method = 35%, Results = 30%, Conclusion = 10%, Communication = 20%). Because of the descriptive nature of the assignments, criterion-based grading with generic but fairly prescriptive marking criteria is appropriate for these assessment items. A grade is allocated to each of these five aspects of the assignments. These grades are converted to a point equivalent and then aggregated according to the weightings allocated to each of these aspects of the assignment.

Finally, students are formally assessed on their mathematical and problem solving skills in a 3 hour end of semester examination (60% of the final grade).

4. Marking and Grading

According to USQ's Academic Regulations [14], examiners may choose to either mark all assessment items or grade all assessment items in a course. A student's final grade and corresponding grade point for a course is determined by combining the student's grades or marks for each assessment item in accordance with the course specification which lists the relative weightings of each of the summative assessment items. A table (see Table 1) of point equivalence for each of the grades available for an individual assessment item is used by all examiners who grade assessment items. For the purpose of the determination of the student's final grade, calculated point equivalences are not rounded off.

Table 1: Relationship between Grade, Point Equivalence, Final Grade, Grade Point and Final Mark

Grade for Assessment Item	Point Equivalence	Final Grade	Grade Point	Final Mark
HD+, HD, HD-	15, 14, 13	HD – At least 13	7	At least 85%
A+, A, A-	12, 11, 10	A- At least 10 but less than 13	6	At least 75% but less than 85%
B+, B, B-	9, 8, 7	B – At least 7 but less than 10	5	At least 65% but less than 75%
C+, C, C-	6, 5, 4	C – At least 4 but less than 7	4	At least 50% but less than 65%
F+, F, F-	3, 2, 1	F - Less than 4	1.5	Less than 50%

To comply with these academic regulations, assessment items that contribute to the overall grade for Foundation Mathematics are either assessed using a grade or need to be converted to a grade as represented by the point equivalence before they are aggregated. The quizzes are marked out of 100, an arithmetic mean across the six summative quizzes is calculated and this is then scaled to the 15 point equivalence scale to reflect the 15 possible grades from HD+ down to F-. Assignments 2 and 3 are graded for each of the components (aim, method, working, conclusion and communication) with a HD, A, B, C or F. These grades are converted to the middle value of the point equivalence scale (see Table 1), e.g. A = 11, and then aggregated according to the weights for each of the five components of the assignment. The students are then informed of their grade and its point equivalence. The workshop submission is given a point equivalence of 15, 8, 4, or 2 based upon the student's participation. Use of the whole 15 point scale was not deemed necessary as the emphasis for this assessment was on participation reflecting its formative focus. A point equivalence of 0 indicates that the student did not participate.

Calculation of the point equivalence for the examination is a two-stage process. The examination consists of ten questions – six questions testing basic mathematical skills and four problem solving questions. Each question is graded and then each grade is recorded by its point equivalence. The point equivalence values are aggregated according to the percentage weights of each question.

The point equivalence for quizzes, assignments, workshop submission and examination are aggregated according to the weights described in Section 3. This point equivalence is converted back to a grade as described in Table 1.

5. Potential Problems with Aggregating Grades in Foundation Mathematics

A number of potential problems have surfaced during the operation of Foundation Mathematics in relation to grading the examination, in particular the use of the 15 point equivalence scale. This scale is not an interval scale and this may have an impact on aggregation of grades [1]. With a traditional passing percentage of 50%, there are in fact only 3 point equivalence values to represent the bottom 50% of student achievement, while there are 12 point equivalence values to represent the top 50% (see Table 1). Further, during the moderation of all results, it was noted that some papers which were on the boundary between a pass and a fail had a considerable number of questions with a point equivalence of three (i.e. failing questions) and very few questions demonstrating a reasonable degree of proficiency. The reliability of the grading process and the influence of the 15 point equivalence scale were thus questioned.

To investigate factors involved in aggregating grades three alternatives were compared using a random sample of examination scripts. It should be noted that the original sample was found to not include papers at the upper end of the distribution of grades so it was decided to include three extra papers from the upper end in this preliminary investigation. All papers in the sample were photocopied prior to grading/marking to allow for independent assessment.

5.1. Grade each examination question then combine grades

Strict adherence to university assessment policy meant that each examination question was graded and then each grade was recorded by its point equivalence. The original papers were graded by a group of three, each person grading all papers for a particular question for consistency. The point equivalence values were aggregated according to the percentage weights of each question.

5.2. Mark each examination question, combine marks and then allocate a grade

Because of concerns about the impact of aggregating so many grades the photocopied examination scripts were marked using percentages instead of grades by a person not involved in the original marking process. An overall percentage for each paper was converted to a point equivalence based on values from Table 1 with linear interpolation.

5.3. Use a 22 point scale rather than a 15 point scale for converting to grade equivalences

Because of concerns about the uneven distribution of grades – 12 passing grades and only 3 failing grades in the 15 point scale, the implications of a more even distribution of passing/failing grades using a 22 point scale was investigated. The original grades from each question were converted to the 22 point equivalence scale and then aggregated using the appropriate question weightings as before. This meant that the passing grade of a C- would be equivalent to an 11 which gave a broader number of point equivalence values to the fails and a more intuitive interpretation of 11 out of 22 or the more familiar 50% pass mark for students to consider. The aggregated grade was then rescaled back to a grade on the 15 point equivalence scale for comparison with the originals.

5.4. Comparisons

A comparison of 'Original grade – 15 point scale' with 'Marked then graded – 15 point scale' gives a measure of the effect on the overall grade of aggregating question grades compared with aggregating question percentages. If these two methods of grading were equivalent a random scatter of points about the line depicting a one to one relationship between the two variables would be expected. This was not the case (Figure 1), as a cluster of points in the lower part of the scale lie above this line. This indicates that aggregating grades based on the 15 point scale appears to inflate grades in the range of 1 to 7 point equivalence and supports the concern expressed by the examiner and moderator when the original papers were moderated.

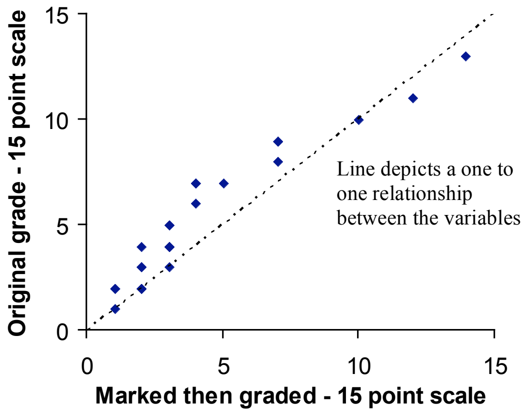

Figure 1: Relationship between 'Original grade' (15 point scale) and 'Marked the graded' (15 point scale) marking protocols

A comparison of 'Original grade – 22 point scale' with 'Marked then graded – 15 point scale' gives a measure of the effect on overall grade of aggregating question grades based on a wider more evenly distributed scale compared with aggregating question percentages. Again, if these two methods of grading were equivalent, a random scatter of points about the line depicting a one to one relationship between the two variables would be expected. Figure 2 shows that this is the case, meaning that there appears to be no difference between aggregating grades on the 22 point scale and aggregating percentages then grading. (It should be noted that this is a preliminary investigation and is limited by the size of the sample and the lack of data at the upper end of the scale.)

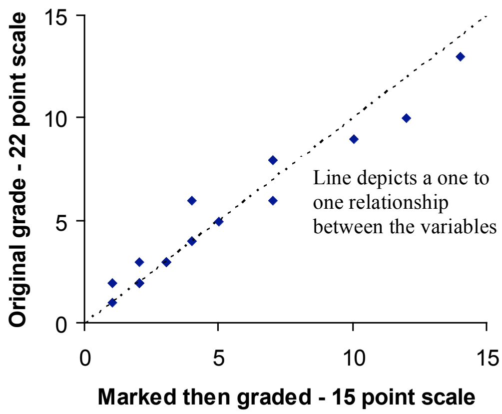

Figure 2: Relationship between 'Original grade' (22 point scale) and 'Marked then graded' (15 point scale) marking protocols

6. Conclusions

Whether to aggregate marks or grades can influence the outcomes of student assessment. The point equivalence scaling using the 15 point scale appears to adversely influence the outcome – inflating results in some of the lower grades in particular. This is not unexpected as examination papers receiving a lower grade will have more failing questions. These failing questions are graded in the restricted point equivalence range of 1 to 3 and may lack precision, whereas the passing questions are graded in a point equivalence range of 4 to 15. When the grades were rescaled to a 22 point scale, the results were not significantly different from the grade using an overall percent. The problem may not be with aggregating grades per se, but with the point equivalence scale used. Are there any advantages in using grading on a 22 point equivalence scale compared with just marking using percentages? The answer to this question may ultimately depend on the type of assessment.

It appears that a more reliable examination result can be obtained by aggregating marks using percentages and then converting to a grade. In the light of these conclusions, the impact of aggregating grades across the five assessment items – assignment 2, assignment 3, quizzes, workshops and the examination - to give an overall course result should be investigated.

If the grading of assessment items is not reliable, misleading judgements about validity can be made. Since reliability is not only a product of what is being examined but how it is examined, the process of marking or grading needs to be questioned and revised in line with the changing mixture of types of assessment used in mathematics courses.

The changing nature of mathematics education at universities has necessitated changes to the nature and style of assessments used. However, changing assessment types, for example from the traditional mathematics problems to open communicative questions, requires changes in marking practices which can lead to further unforeseen difficulties. In this course the use of grades rather than marks was believed desirable within some assessment types, yet the aggregation of such grades compared with the aggregation of traditional marked assessments has revealed inequities in the process.

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Do student reflections increase undergraduate mathematics learning?

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The students taking the theme based first year mathematics course (MATHS 101) at the University of Auckland are required to write a 'reflection' after each theme has been taught. The reflections written by the students in the first semester of 2004, based on the theme 'Figure it Out' were investigated and their use, for both the lecturer and the student, considered as a way to increase undergraduate mathematics learning. The student comments suggested increased learning occurred and it was decided to continue teaching this theme within the course and continue requiring the students to write reflections. For 2006, a higher level first year entry course (MATHS 102) will also require their students to similarly use reflections as part of the assessment practice. The implications for the use of reflections in higher-level mathematics learning and the theory behind writing in mathematics will be discussed within this paper.

1. Background

Some educators (e.g. [1], [6]) have recognised the value of writing in the teaching, learning and assessment of mathematics. They also recognise many of the factors that influence students' achievement when doing mathematics, as there is no subject quite like mathematics to generate such strong feelings amongst its students. Students' emotional responses to mathematics vary from those who are fascinated by the logic, the patterns, the investigations, the use of technology and more, to those who feel 'mathematical anxiety', 'confused', 'dumb' [6], 'panic' [5], 'shame' [2], or a lack of confidence (e.g. [7], [11], [9]). Phrases such as: 'Maths is hard', 'Maths is difficult', 'I hate Maths', 'Maths is boring', 'I cannot do Maths', are commonly heard; these are emotional responses displaying feelings of anxiety that can interfere with the students' ability to learn mathematics [5]. Encouraging students to express these feelings in writing, and hence be able to reflect on their reactions to the mathematics being studied, can assist both the students and their teachers to understand and overcome the reasons for the anxiety. Gay and Thomas [9] pointed out that valuable information could be obtained from both verbal and written interaction, since "just because they got it right does not mean they know it" (p.130), but students often lacked the vocabulary with which to express themselves in mathematics ([2], [12]).

1.1. Basic Skills

From its conception, the communication of mathematical ideas has been a major part of our first year undergraduate mathematics course (MATHS 101) that is offered as an optional course at the University of Auckland. The students are encouraged to think about their difficulties, their successes, their feelings and their beliefs [8] and communicate both in writing and orally as part of the learning process (e.g. [1], [3], [12]) and as an important part of the assessment process. The 120 students who enroll in MATHS 101 each semester often have very little mathematics background or are adults coming back into education and needing an introductory mathematics course. While teaching this course in 2002, it was found that many students lacked a basic understanding of the number skills needed to handle much of the mathematics offered and were fearful of mathematics as a whole. In order to meet the need for a review of basic number skills for these students, the extra mathematical theme 'Figure it out' was intro-

A personal reflection on the current lecture theme is to be done individually and is to consist of brief answers to your choice of **three** of the following questions or statements

i. Provide a list of mathematics content learnt for the first time, or covered previously and then forgotten.
ii. Note anything that is still unclear, that you are worried about, or that you would like further work on.
iii. How could this week's theme relate to the multi-cultural aspect of today's society?
iv. An explanation of any different strategies learnt during this topic.
v. An explanation of any applications or relevance of the mathematics taught in this topic.
vi. A discussion on any connections between different representations and different terminology that you have discovered within this topic.
vii. How did you feel this week as a learner of mathematics? Explain.
viii. How did the topic relate to the primary school curriculum (mathematics and other subjects)?
ix. What is one teaching idea that you have developed from this week's work? (Give details of activity and approximate age level.)

Figure 1:

duced into the course in 2003. This was then investigated to find the reactions of the students towards the content and the method of presentation of this new theme.

1.2. Reflection

In the second semester of 2002, to encourage students to think more about what they were learning, and to help with an essay that was part of their final assessment, the MATHS 101 students were asked to write a 'reflection' at the end of each theme. The six reflections required throughout the course made up less than 1% of the total marks, but having some value led to most students attempting to write something. No real guidelines were supplied and it became obvious students did not fully understand what was required of them [14], so in 2003 more direction was given. Students were asked to select, and write a paragraph on, 3 items from a given list that had been adapted from those used by FitzSimons [6]. It was found that providing some structure helped the students to respond appropriately (Figure 1.).

In 2004, the 'reflections' written by the students gave an opportunity to investigate whether this new theme 'Figure it Out', was meeting the needs of the students and to assist in decisions about future changes within MATHS 101. Further thought was given to other uses of these reflections as a research tool, while questions relating to their value for the student, the lecturer and any possible increase in mathematical learning were considered. This paper describes the teaching techniques used in the theme 'Figure it Out', how the student reflections were used to gather information on the students' responses to this theme, and the use of reflections as a way to increase their mathematical learning.

2. Research Study

The study involved 28 students from MATHS 101 who volunteered to have their reflections from MATHS 101 (in semester one of 2004) used for research purposes. These students include international students, younger students and adults returning to education, all with very little mathematical background. As the researcher was one of the lecturers of the course, no analysis was started until grades had been finalised to ensure no possible bias. It was hoped the nature of the comments would assist in giving some indication of the value of the theme under investigation.

2.1. Figure it Out

The theme 'Figure It Out' was taught in two one-hour lectures followed by a one-hour collaborative tutorial. The focus of the teaching was:

- doing mathematics can be fun [10] and everyone can achieve success;
- the many different strategies needed to solve problems involving addition, subtraction, place value, large numbers, fractions, and percentages.

The students were encouraged to think about not just the 'doing' of mathematics, but also the 'knowing' of mathematics [3]. FitzSimons [6] maintained that if a student can find enjoyment and success when doing mathematics, then a positive feeling towards mathematics is developed. It was hoped that the students enrolled in MATHS 101 would have a similar experience with the theme 'Figure it Out'.

2.2. The Introduction

At the beginning of the first lecture the theme was introduced with a warm up activity 'Mad Minute' in which all students could participate. It went like this:

> I wrote two numbers on the board, one under the other:
> 5
> $\underline{3}$
>
> I explained we were going to do an activity called 'Mad Minute' and that the two numbers were to be added together and their sum was to be added to the number above. We would continue in this way for 60 seconds then mark and count up the number of correct additions achieved.
> 5
> $\underline{3}$
> 8
> 11 etc.
>
> I put two new numbers on the board, and said 'go'. The students copied them down and proceeded to do the calculations while I timed them. After 60 seconds, I called "Stop, everyone stand up". I explained that they were to mark their additions as I called out the answers and to sit down when they made a mistake or had finished marking.

This procedure was designed to minimise the exposure of those students who sat down early and enabled the lecturer to stop calling the answers once there was only one student left standing. The activity was repeated with another pair of numbers and the majority of the students improved on their first result. Many 'emotional' comments were made in the reflections regarding this form of introduction. These included

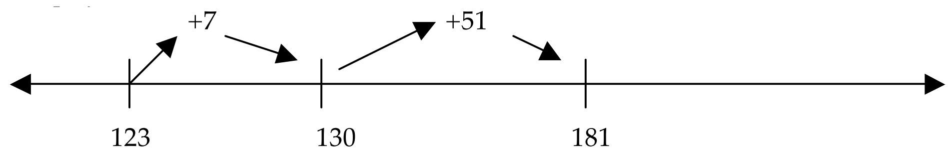
> "In the exercise 'mad minute' I was not confident on the first try because I only got 7 levels without mistakes, but I am pleased I made it up to 19 levels without mistakes on the second trial showing that I improved a lot." "It has given me confidence in my ability to succeed in MATHS 101. It has been fun and I enjoyed the warm-up of 'mad minute'." "I went to my first lecture feeling a bit tense about starting new work, but found the 'mad minute' was a great ice-breaker and left me feeling at ease."

Other comments made about this type of introduction used words such as feeling relaxed and having a sense of achievement right at the beginning of the first lecture; having fun for the first time when doing mathematics; gaining confidence etc. For such a basic activity involving addition some negative reactions were expected, however there were only positive comments expressed in the reflections analysed.

2.3. The Content

The content of 'Figure it Out' included the basic number skills of addition, subtraction, decimals, fractions, ratios and proportion, place-value, percentages and scientific notation. The introduction was followed by a brainstorming session on the strategies used when adding and subtracting:

> The students were invited to explain the different strategies they used when using the number skills of addition and subtraction. This 'brain storming' session produced many different strategies, although the younger students seemed to primarily use the standard algorithm. The use of alternative methods was encouraged and to communicate these different strategies resulted in the use of an 'empty number line'. For example one strategy that was used by a student to add:
>
> 123 + 58
>
> was displayed as:



> The older students supplied many strategies that they had devised in their working life that were more practical than the standard algorithm.

Many students saw the different strategies as 'new methods' for adding and subtracting with comments such as:

> "another reason I felt excited was due to learning previously unknown strategies for ways to figure out things more simply." "new methods opened my mind to more possibilities. I developed a faster adding skill." "Not only have I understood more points on various concepts in 'FIO', but I have also learnt new strategies in things that are familiar to me. For example the number line can give a clear glimpse of how a result was obtained and from this I have learnt how to add and subtract quicker."

Many other students commented on the fact that a problem can be done in many different ways and how clearly the open number line can be used to display them. They were unaware that a method other than the standard algorithm was possible or acceptable. Similarly the different methods that can be used when working with percentages were commented upon.

> When teaching percentages, we discussed different strategies they could use to get a possible answer. Once again many followed the traditional methods taught at school by calculating the percentage first then adding or subtracting it from the original as required. We looked at an alternative strategy where in a 20% discount sale, we need to pay 80% of the price. Therefore just one calculation is required for us to know what we have to pay.

Student comments included:

> "..percentages was one of my weakest parts of maths years agonow I understand it really well after the lecture." " I never thought about working out first what percentage we actually want it makes a lot more sense and is easier"

The reflection comments indicated that students were starting to think about their learning and how this theme with its relaxed approach and alternative strategies was having a positive effect on students' attitudes towards mathematics. Similarly skills involving fractions and large numbers were discussed within the lectures, using investigations involving diagrams and calculators. The existence of a 'fraction' button on a calculator and the ability to use it was a new experience for many. However, there were two comments relating to the common difficulty students have with fractions:

> "still unclear are improper fractions." " Fractions make me feel overwhelmed and I still feel uncertain after leaving the lectures". "I could never understand the measurements given in the form of a fraction in the cookbook but this week I learnt and understand those fractions and I'll never forget."

These comments showed that while some students gained a clearer picture of many of the basic mathematical skills they had previously met at school, not all were able to overcome their feelings about fractions. The more general comments by the students about their feelings relating to this theme as a whole were mainly positive. Words they used to express their feelings included excited; enthusiastic; motivated; challenged; fun; stimulated; a sense of anticipation; refreshing.

> "I thought you had to be gifted with the ability to do math, where now I feel that teaching and learning styles play a big part in my way of learning." "This week as a learner of mathematics, I felt both challenged and excited. Challenged because working on 'Figure it Out' showed me the huge amount that I had forgotten in the last 24 years (since I was last at school). Excited because after going over it once I was able to pick it up again fairly quickly. I felt challenged at trying to use my calculator." "I did feel I'm going back to the juniors, but this time look more carefully, and try to understand them and not just memorise the formula."

The researcher had expected more comments like those made by the last statement above, about going back to the juniors and was pleasantly surprised at the overwhelmingly positive response to the content and the teaching methods used in this theme.

2.4. The Resources

Both lectures included write-on-notes and a resource pack. The write-on-notes gave an outline of the lectures with spaces for the student to record their own notes. The resource pack included some activities done in lectures, some 'fun' activities for the students to enjoy in their own time, and worked examples with exercises related to each topic covered within the theme. Students commented in their reflections about the usefulness of these resources with statements such as:

> " Those booklets are really good because they have the answers, which makes it easier to find mistakes which I can circle and then ask for help." " The activities provided in the resource pack allows students not to rely on their calculators. Instead, working it out in their heads using exercises like: 'Will you do it in your head?', 'Adding compatibles' and 'Doubling and halving' is a very healthy way to teach."

3. Results and Discussion

This investigation was started to see whether analysing student reflections could assist in a decision to be made regarding the continuing inclusion of the theme 'Figure it Out' in MATHS 101 in 2005. The continuing question of the use of writing and the use of reflections as a means of increasing students learning in mathematics became part of this investigation, as the future use of reflections within higher level mathematics courses was under consideration.

3.1. Using reflections for decision making.

The emotional comments, the comments on the content, and the comments on the learning style made by the students of MATHS 101 have given an indication of how students feel about this theme. They commented about the content when discussing the different strategies that they had not previously experienced and about their difficulty with fractions. The method of teaching this theme was recognised by many as positive with its unusual introduction, its explicit references to real life situations and the resource booklets provided.

The comments in the reflections indicated that the 'Figure it Out' theme' successfully demonstrates an example of the change of emphasis within the teaching of mathematics as expressed by Burton ([4], p.314).

Emphasis on	competition	changed to	collaboration
	individual work	group work	
	knowing	enquiring	
Emphasis on	answers	changed to	questions
	formalisation	informality	
	substantiation	conjecturing	
	replication	creation	

The change of emphasis above was used in the theme 'Figure it Out' to review the basic number skills in a different way to that which the students have previously experienced. The reflection comments indicated an increase in students' confidence and understanding when learning mathematics within this theme, had occurred.

The use of reflections about a theme within MATHS 101 seemed to be a good indicator in determining its value. Even though the reflections were part of the course assessment and could be coloured by a desire to get the best marks, the positive feelings revealed about the theme 'Figure it Out' indicated the students believed that it helped them understand and enjoy doing mathematics at this level. As a result, it has been decided to continue teaching the theme 'Figure it Out' as part of MATHS 101, and continue requiring the students to write reflections.

3.2. Reflections as a tool to improve mathematics learning.

Although all assignments are marked by a 'marker', the reflections for MATHS 101 are marked by the lecturer and written comments can be made, which give feedback to the student. As it is important to build on the prior knowledge of the student [1], it is important for the lecturer to be able to understand what that prior knowledge is. When the students selected 3 statements from the given list (Figure 1.) it was found that the three most popular choices for their reflections on the theme 'Figure it Out' were:

i. Provide a list of mathematics content learnt for the first time, or covered previously and then forgotten.
ii. Note anything that is still unclear, that you are worried about, or that you would like further work on.
vii How did you feel this week as a learner of mathematics? Explain.

These gave the lecturer an indication of the students' mathematics knowledge and any difficulties they were having. The lecturer often addressed these by making comments and suggestions on the reflection sheet, or, when there was a common problem, explaining during lectures. Reflections, where some kind of structure is in place [14], give excellent feedback for both the lecturer and the student and give an opportunity to acknowledge the needs of the students and hence increase their mathematics learning.

The writing of these reflections encourages the students to express their feelings about mathematics while preparing them for the essay that is part of the assessment for MATHS 101. Although most mathematics courses do not involve an essay, many students have little communication with their lecturer, and the lecturer has little idea of the mathematical knowledge of each individual student. These reflections helped to fill that gap and as a lecturer, although the time involved has to be recognized as an issue, I found it worth the effort.

3.3. Would reflections be effective at a higher level of mathematics?

As writing reflections is a form of communication between the lecturer and the student, as well as focusing the student on what they are learning, it must be seen as being effective at any level of mathematics. At the University of Auckland, the mathematics modeling course, MATHS 102, is a higher level mathematics entry course and the course coordinator is currently planning to include reflections as part of the assessment process for 2006. The mathematics tutoring course MATHS 202 uses reflections in the form of a journal and the students' reflections were supportive evidence for the research done by Sundaram [13].

4. Conclusion

While the use of reflections is important to assist the students to think about the learning of mathematics, their use as a means of communication between the student and the lecturer could be recognized as even more important. In an environment where there are a large number of students and few lecturers some form of direct communication is necessary if increased learning is to take place. It is hoped that reflections used in higher level mathematics courses could also be used as a means of communication between the student and the lecturer to encourage an increase in mathematics learning. Their use in assisting in the making of decisions about changes within the course, MATHS 101, was found to be very successful in relation to the theme 'Figure it Out'.

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Using Mathematics to open up windows in teachers' minds: Encouraging teacher talk about learning and teaching

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Early results from a longitudinal investigative study of teachers' reactions to the stimulation of mathematical input during professional development opportunities are reported. The study is situated within a larger project coordinated by The University of Auckland involving mathematics teachers of senior classes in eight low socio-economic schools. It investigates the effectiveness of utilizing tertiary level mathematics and statistics to create a situation in which the teachers are encouraged to see themselves as learners and consequently to discuss learning and teaching within the community of practice that is developing within the larger project. Initial analysis indicates that the stimulation and interactions are proving successful in encouraging teachers to connect with their mathematical learner selves. The model being used to describe this process is discussed in detail in the paper. In particular evidence is presented that the teachers see that their own learning experiences can be used to re-view student learning.

1. Introduction

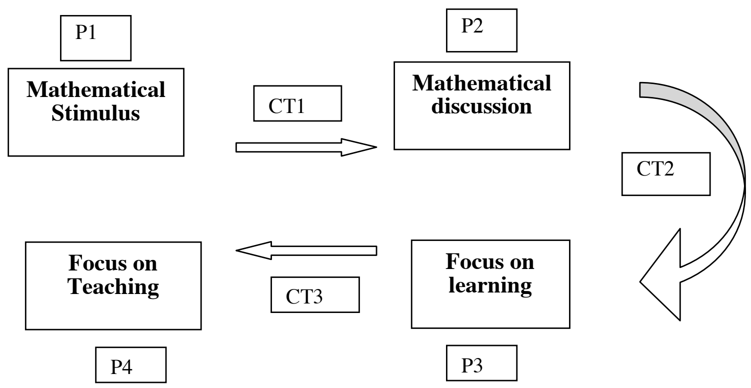
The study reported on in this paper focuses on reconnecting secondary mathematics teachers with their mathematical learner selves in order to prompt them into discussing and re-examining their teaching practice. The teacher in the following excerpt sees this as opening up windows.

> A: We get frustrated, I think, with every year doing the same type of thing with students. And in between we open up windows like this to see other worlds. Like we have not seen, I have not, for 30 years. .and then getting an opportunity like this to look at ourselves in another way.

It has been reported that while they do a lot of teaching, teachers do not, by and large, talk meaningfully about their teaching and reflect critically on their practice [1,2]. In addition, many of the teachers involved in this study have displayed reluctance to consider change [3], and some fall into the group of teachers whose attitudes to change must at least move from 'cynical' to 'skeptical' for any change in practice to occur [4]. The opened up windows allow the teachers to look both at themselves and their students in a new light. A model of the shift in focus of teacher discussion during professional development workshops from mathematics to pedagogy is developed in this paper. It is the progressive movement of individuals towards considering change through stimulation and discussion that is the focus of the investigation.

2. The Model

The basic model is shown in Figure 1. The model is based on the hypothesis that secondary mathematics teachers enjoy doing mathematics and that this disposition can be exploited to encourage them to discuss learning, both their own and their students', in an open, reflective manner. Further it is hypothesized that this discussion could lead to the productive examination of teaching practice and the possible creation of an openness to growth and change. The stages form part of a series that will be investigated over a period of two years, one of which has been completed. At each stage input from previous workshops will form part of a feedback loop to the teachers and to the researcher. The investigation aims to describe the teachers' responses to the stimulus and intervention and to enhance and adapt the existing theoretical model on the basis of collected data.



P = Phase CT = Connections and Transitions to be explored in 2005

Figure 1: One stage of the model

In P1 tertiary level mathematical content is being used to engage the participants in a learning situation. A number of researchers [5, 6] consider it important for professional development to have a content base. Whether 'content based' is taken to mean learning mathematics, learning mathematics in a new manner, or learning about the content pedagogy of mathematics, the teachers are engaging with the structures and connections that contribute to understanding mathematics. Garet, Porter, Desimone, Birman and Suk Yoon in large-scale empirical comparison of effects of different characteristics of professional development on teachers list content focus as one of the three core features of effective professional development activities [7]. In this study the role of the content is to promote the teachers re-viewing of learning (CT2) from a new perspective. It is recognized [5, 8] that examination of beliefs and existing behaviour patterns often precedes development and change in behaviour.

The model draws on the view that problematising existing ideas and perturbing learners' knowledge structures play a vital role in stimulating learning [9]. Studies convincingly show that success in any change effort always hinges on what happens at the smallest unit of the organization [10]. Guskey [11] argues that what this 'says to educators is that success in improvement efforts will always hinge on what happens at the classroom level' (p8). My contention is that improvement efforts hinge on how the individual teachers view their own growth and classroom change. The discussion in small groups affords the teachers opportunities to reflect on their practice (CT3) and to be part of a supportive community; these are important components of professional development programmes that support teacher growth [2,5,6,12].

3. The Study

This study forms a part of a bigger project, the Mathematics Enhancement Project described in [13] and [14]. The study's place in a larger project allows the researcher to gather data from multiple sources e.g. interviews conducted by co-researchers on the impact of the project. At the same time it needs to be recognized that completely separating out the effects of, and attributing impact to, the varied initiatives within the Mathematics Enhancement Project is clearly not possible.

In 2004 teachers of senior mathematics classes from the 11 low socio-economic schools involved, attended seven professional development workshops. At four of these a lecturer from the University of Auckland presented a talk (P1) the brief for which was, 'to find something which teachers probably haven't seen before but which should be interesting and accessible to them, and take about forty five minutes.'

In this situation the teachers needed to engage with new ideas in mathematics or statistics. The teachers were in the position of being learners of mathematics. They were introduced to the mathematics of traffic flow, tournaments, yachting, packing, Borovian knots and encryption, to the problems of rat re-infestation and the modeling of clam populations in estuaries and an analysis of the degree-diameter problem. Of particular interest to the Delta community is the opportunity afforded researchers to discuss applications from their field of interest with teachers and to forge connections, through the teacher's re-kindled mathematical interest, to tomorrow's university students. While the focus of this study is on secondary teacher development it can be argued that enhancing university-school connections, possibly through in-service post-graduate courses for teachers, which intertwine content and pedagogy, is an exciting prospect for Mathematics and Education departments.

In a pilot study Paterson [15] found that teachers found similar talks stimulating and that this stimulation fostered potentially productive discussion of both content and pedagogy amongst classroom teachers.

After each talk participants were encouraged to examine their own responses to the talks and their feelings about learning and to use these insights to inform discussions of teaching practice. The teachers sat in small groups and after the talk continued discussing and working on the mathematical ideas (P2) for about ten minutes, asking the presenter for input if they wished to. The researcher then asked them to turn the focus of their thinking to learning, their own and their students' (P3). They were given prompts to respond to both individually in writing and as groups. These prompts encouraged the shift of focus. Finally, in the light of the previous discussions, they were asked to think about teaching, both of the current topic in particular and in general (P4). This structure enabled the teachers to communicate with colleagues with whom they have become increasingly open and relaxed, as the project has continued [3]. In addition they could listen to themselves, an action that has been established as more effective than listening to others [Bem 1972, in 16].

In 2004 a total of twenty-six teachers attended one or more workshops. Half of them attended at least three. The unit of analysis for this particular study within the project is the teacher as an individual with a mathematical and teaching history and as a member of the group. At each workshop individual teacher responses were gathered in written form and all small group conversations were recorded. The teachers' written responses were categorized. The initial analysis of oral and written data led to the organization of responses and identification of recurring themes.

4. Data, Analysis and Discussion

In this paper written data from the first workshop in 2004 and conversational data from all four workshops is discussed. The data show that at the outset the teachers viewed their own and their students learning needs as being very different. The conversational data provides evidence that for a number of teachers the mathematical learning experience has stimulated reconsideration of the learning situation.

4.1. Analysis of written data from a workshop early in the year

After the talk and discussion at the first workshop in 2004 the teachers responded to the following two prompts: A: "I generally learn (mathematics) more easily when.." B. "I think students learn new ideas in mathematics when".

These data show that the 14 teachers at this workshop did not consider their own needs in learning situations to be the same as their students, however later conversational data show a much higher degree of identification with the student as a learner. A few responses did not fall into any of these categories and have been ignored.

4.2. Teacher response to the mathematical input in P1 and P2

The responses enabled the teachers' reactions to be categorized into three clusters: teachers reacting mostly with interest and stimulation; those reacting mostly as if intimidated by the

Table 1: Categorisation of teachers' responses to prompts A and B

Teacher Response	Prompt A %	Prompt B %
Actively Involved	23	24
Ideas based on prior knowledge	0	29
Stimulating and relevant	18	0
Confident, motivated to succeed	0	24
Improves my mathematics knowledge	23	18
Visualise the "big picture"	9	0
Clarity of explanation	18	0

mathematics and/or the language demands of the situation; and those exhibiting a variety or reactions, depending on a number of variables, both extrinsic and intrinsic. Those teachers exhibiting intimidation were, unsurprisingly, unwilling to engage in further pedagogical discussion. Consequently the evidence presented below does not usually come from this grouping. Future work will focus on attempting to minimize the factors that appear to intimidate these teachers.

Approximately half the teachers involved were generally stimulated; they shared insights into their own learning and could see similarities between their experiences and those of their students. A further third exhibited a variety of reactions.

Teachers whose responses fell mainly into the first category engaged willingly in the discussions after the talks, often leading the discussion and problem solving in their group. They were able to work around unfamiliar vocabulary and behaved as if they followed the thread of the ideas presented in the talk.

Those stimulated may so engage, apparently in proportion to their mathematical or linguistic confidence. In terms of the effectiveness of the talk in "opening up" the participants the big question seems to be, 'Are they as an individual generally stimulated or intimidated (by the mathematics, the presenter or their small group)?' It is suggested that this depends upon their own confidence in their ability to follow the mathematical ideas being presented in English. The importance of confidence in learning has been identified by Graven [17] adding to the work of Lave and Wenger [18] in their work on communities of practice. This aspect will be further explored in a new cycle of investigations in 2005.

A number of themes emerged in the initial analysis of the conversational data from the teachers for whom the intervention appears to have been generally or partially effective. These data support the hypothesis that the initial engagement with the mathematics can lead to purposeful open discussion of learning and teaching. Examples and discussion of those themes that relate to the teachers becoming aware of themselves as learners and the connections they have made to student classroom experience are presented below. The teachers spoke of: the importance of being energized in relation to teaching, the admission of fear of exposure and recognition that this impacts on their own and student participation in class discussions, and the recognition of different learner preferences.

4.3. Themes identified in responses from teachers who were stimulated by the mathematics

Opening up windows and being energized. The teacher quoted at the beginning of the paper continued, reflecting on what his insight has meant in terms of his classroom practice.

> A: So if somebody comes up with a story or something which may be exactly not related to . the formula once they start talking about that we can see the difference in their face and the classroom's changing. It doesn't happen every class, but once in a while we tend to support that. Before we said, "We have to get on with this one and finish this lesson but now we give other things a bit of time let them talk.

It is clear that his experience of learning has influenced how he sees the students' role in the learning process.

For a number of teachers the opening up of windows onto mathematics is energizing. Energy, enthusiasm, inspiration and excitement about teaching is seen by the teachers as very important and often missing in comparison with the time when they saw themselves as learners and viewed themselves as mathematicians.

> D: One of the things I found this year about these talks and stuff is that I think its kind of enthused me a bit. Into areas of mathematics which I have done but I've lost I've gone home and looked at things in my old notes. E: Because I was just thinking for me it's good to have stuff that you have to think about to turn your brain on again.

The effect of "running out of fuel" is highlighted in the next teacher's remarks.

> F: You go away really inspired and say OK I'm going to try that with my class. So it gives you all these ideas and you feel inspired to inspire your students Yeh. It's neat otherwise you teach out of like an empty fuel tank. You come here and you get re-fueled and get rejuvenated and refreshed.

The teachers said that their increased enthusiasm would make a difference to how they teach. The notion of refueling is interesting and raises the question of what it is that teachers need to keep going. In the debate about content or pedagogy based professional development [19] a third component may hold a key: teacher energy and motivation to engage in and implement professional development.

> H: So it's mainly your own interest that's been re-ignited. I: But that impacts on your teaching doesn't it? Because you're going to get more enthusiastic.

Sharing their personal learner preferences, recognizing that they are not all the same. This theme has been returned to throughout the year. The following excerpt is from the last meeting of the year and arose during a general conversation, not in response to a direct prompt.

> B I'm the opposite, I like to listen thoroughly I don't like to be interrupted. I don't like people asking questions I like to listen; I like to think as I am going. I just like to listen, I just like to listen all through and if I've got a question I'll try and shelve it and then at the end I like to think about it and I don't normally have questions at the end so I sort of think about it for the next couple of days. K: You're a thinker, I'm a talker.

A number of teachers shared insights into their own learning needs. One has realized that she cannot take notes and think at the same time and she reports that this has impacted on her classroom teaching. Another has found that it is often when he disagrees with the presenter that he engages most effectively.

> L: Well, sometimes I learn big time. If I have a negative attitude toward something that makes me query re what the speakers talking aboutSee I still don't understand.

Empathy with the students they teach. In a number of cases the teachers articulate the connection to how their students feel in the same circumstances and how this understanding affects their pedagogy.

> M: I came to one talk and it was a really good talk. But because it was in the afternoon, I was just really tired. And I just thought, you know there are kids in our classes that are the same.

Many of the teachers in the MEP have been in charge of their classrooms for a long time and the experience of being not in control has opened them up to the fears many students experience in class.

> D: Coming to these sessions in groups its made me look at what my students feel when they're in the classroom. And I think initially- the feeling is that when somebody comes up with something- you're taken aback a bit. And I think initially the feeling is fear. I think it's, I think that for me learning here and for them learning from me as well

It also appears that they had the confidence within their small groups to expose their mathematical thinking, their feelings about learning and to discuss how they think their feelings may be the same as the feelings of their students, for example anxiety about being made to feel foolish. Voicing this fear in a public forum appears to indicate that the lectures and discussions have been effective in providing an environment in which the participants could, as confidence in the community grew, use their learning experiences to discuss their teaching. Exposure of ideas and feelings may be components of openness to considering change.

> G: You're quite right I didn't ask a question because I was overawed by her confidence. Even though I reason, I know I'm an intelligent adult but I was too scared of saying what was going through my mind. G: But you suddenly understand where the kids are you know! I really think that's a good idea but she might shoot me down by pointing out that there is a logical reason behind why it's not done. H: So if you feel threatened by someone like that? D: I'm overawed by her competency. You can see that and I wonder if we do that? I'm suddenly thinking, "Hell am I doing that to the kids?" Even when I think I am right down at their level? Am I actually? Because I actually think I am.

5. Summary

For a number of the teachers involved the stimulation and focused discussion was effective in enabling and encouraging them to re-view themselves as learners of mathematics. In the process they voiced valuable insights into the students' mathematical learning in their classes and turned their thoughts on the role played by learning and learners in teaching. In addition veteran teachers shared thoughts often held private, for example anxiety about being asked a question in a large group. The implied question generated by the research is whether there is an impact, other than self-reported, of these insights on their classroom practice.

The evidence presented supports the claim that using the mathematical stimulation and focused small group discussion encouraged teachers to reconnect with their mathematical learner selves and that this has in a number of cases led them to re-examine learning and to consider their teaching practice. The "open windows" have encouraged the teachers to look at both their own and consequently their students' learning in a new light, often with more energy. Encouraging teachers to make their professional knowledge public is a step towards enabling them to live in Popper's World 3, allowing them to 'treat ideas for teaching as objects that can be shared and examined publicly' [20 p 7]. The public voicing of fears and anxieties is seen as evidence of being prepared to be vulnerable. It is suggested that being prepared to be vulnerable may be a component of the growth process. The interactions will be adapted to accommodate factors identified as impeding teacher engagement and will be trialed in 2005 within the same project. In particular the use of prompts and facilitators to enhance the movement of discussion along the path from discussing mathematics to discussing pedagogy will be explored.

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Using history to teach functions within an APOS approach to learning

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Our primary research objective was to determine the extent to which studying the history of the concept of function facilitates a deepening of student understanding. A secondary objective was to explore the notion that the cognitive steps required of students learning a mathematical concept frequently parallel the historical steps in the development of the concept. Four fourth-year pre-service secondary math teachers in a History of Math class completed worksheets and read original sources about the history of functions and graphs. Their level of understanding was determined both before and after through questionnaries. Further, 10 first-year statistics students read an original source about graphing of functions and completed a worksheet. We performed qualitative analyses of student responses based on APOS theory. The history of math students showed a growth in understanding of functions and both groups considered early graphical techniques an intuitive way of introducing graphing to middle-school students.

1. APOS Theory

APOS theory is a constructivist approach to learning mathematics at the post-secondary level. APOS theory, like constructivism, is a theory of knowing. The acronym APOS stands for Action, Process, Object, and Schema mental constructions made by students as they understand mathematics. Actions lead to processes, which come before understanding a concept as an object, which can be manipulated as part of a schema. The developers of APOS see their work as 'the result of reconstruction of our understanding of Piaget's theory leading to extension in its applicability to post-secondary mathematics' [1, p.41].

The action construction is similar to Piaget's action schemes. A student who has an action understanding of functions sees an algebraic expression as a command to calculate. This conception is like a recipe, and the action student thinks it must be applied to a number(s) before it will produce anything. The action construction, which is a necessary beginning, is considered the lowest level of abstraction. A reason so many students have trouble understanding, for example, piece-wise functions and composition and inverses of functions is that they do not go beyond an action understanding of functions [1].

A student with a process understanding of functions imagines a transformation as internal, totally under her control. She no longer needs to actually evaluate an expression to think of its result. She can reflect on and describe the linking of processes to construct a composition and the reversing of steps of a transformation to consider an inverse without actually performing those steps. When a student moves from an action understanding to a process understanding, APOS theorists say the student has interiorized an action to form a process.

Once students have practice working with functions as processes, APOS theorists believe the students are ready to begin to reason more formally about functions they can encapsulate the processes to form objects. A process construction corresponds to Piaget's operations, and the next level object construction corresponds to Piaget's objects. An object understanding of a concept sees it as 'something to which actions and processes may be applied' [2, p.19]. Dubinsky [1] claims that reaching this level of abstraction is incredibly difficult for students and has found few pedagogical strategies to be effective in achieving this end.

The schema construct is the highest level of abstraction and closely mirrors Piaget's schemata construct. The schema is the least understood of all constructs in APOS theory. It may be thought of as a collection of processes and objects that are organized in a structured manner. Asiala et al [1] suggest that the difference between schema and other constructs may be understood by comparing it to the distinction between an organ and a cell in biology. The organ (schema) provides the organization required for the cells (objects) to function for the benefit of the organ (schema). When a person has reached the schema level with respect to a concept, then that person's schema construct for that concept is the totality of her knowledge about the concept.

APOS theorists have developed a theory of instruction. They claim that students do NOT learn course material in a logical, organized and sequential order. Though students need to experience the levels of abstraction in the order described, they do not understand the material in a neat, organized fashion. Students gain partial knowledge, repeatedly return to the same knowledge, periodically summarize and tie related ideas together [1]. Students who have developed advanced levels of understanding often move from one level of abstraction to another, depending upon the demands of a given problem.

The instructional approach developed from APOS theory takes into account this learning pattern by using what Asiala et al. call a holistic spray. In this approach, teachers give students group activities that deliberately cause disequilibrium in their thinking while exposing them to as much about the topic as possible. The manner is holistic not sequential. Such perturbations cause students to question their current constructs and force intellectual growth as they try to figure out 'what's going on'. Different students may pick up different pieces and parts of the picture hence, the benefits of collaborative work. Each student shares her partial understandings with the group, thus adding to other group members' understandings. As a result, more pieces and parts are available to each student than would be available to any student working alone. This approach is certainly consistent with the constructivist viewpoint that knowledge is obtained by constructing and reconstructing understandings in a social setting.

In the current study, students are reading and analyzing primary sources in the historical development of the concept of function. Knowledge is obtained by reflecting on problems and by constructing and reconstructing actions, processes and objects in a social setting. It has specific implications for instructional strategies that are based on inquiry learning.

2. Teaching and the History of Mathematics

When a student of mathematics tries to read an original mathematics paper which was written before 1900, she often finds it surprising that the material which she is trying to read is related to the subject which she is currently studying. Thus, at first, it may seem surprising to use the historical development of mathematics as a guide for teaching mathematics today. However, when one reads the literature about the potential value that the history of mathematics can bring to teaching, one realizes that history of mathematics may be able to play an important role in enriching current teaching practices. For example, according to Grattan-Guinness, 'Without deep, penetrating motivations, all education is lost, and it is in providing these that I'm sure that the history of mathematics can be of great service in mathematical education' [3, p.122].

Commonly given reasons for studying the history of mathematics include the following.

- History can help increase student motivation and develop a positive attitude toward learning mathematics [4], [5], [6].
- Past obstacles in the development of mathematics can help explain why a certain topic is difficult for students [4], [6], [7], increasing teachers' understanding of barriers to student understanding [6], and helping them understand the stages of learning [4]. It may thus lead teachers to change the way they think about their students' errors and develop a more constructive attitude toward them.
- Learning the historical development may affect teachers' perceptions of the time students need to spend while struggling to understand a concept [4], [7], [8]: 'If it took several centuries for mathematicians to be able to make explicit our current concept of limit, for example, it is going to take a considerable time for our students as well' [4, p.65].
- It can provide an opportunity for developing an idea of what mathematics is; i.e., it helps students understand the reasons behind the development of algorithms and concepts and which problems math helps us to solve [4]. It leads to the idea that math is not a sequence of 'discrete chapters' but is 'an activity of moving between different ways of thinking about mathematical concepts and tools' [4, p.65].
- It shows that mathematical ideas evolve over a period of time, are struggled with and are subject to change [7], [8], and that grappling with mathematics is a common human experience [7].
- Since understanding the history of mathematics may change teachers' perceptions and understanding of mathematics, it may influence the way they teach mathematics and thus affect the way students perceive it and understand it [4].

There are, however, few, if any, research studies documenting the effect that studying the history of mathematics has on students' learning of mathematics. Many claims are anecdotal and most concern changes in attitudes, not learning. Po-Hung Lui [6] claims that there exist no empirical studies indicating that learning history helps students perform better on standardized exams. 'Although studying the history of mathematics may improve students' attitudes toward mathematics, the linkage between attitude and achievement is neither linear nor straightforward' [p. 420].

We understand that measuring the impact of historical studies on learning concepts is difficult. However, we hope, at least, to demonstrate that some students are able to move from one level of learning to another in part due to their exposure to historical readings. Such assessment may need to be of a qualitative nature. In the recently published ICMI study, E. Barbin et al [4] echo this sentiment. 'The question of judging the effectiveness of integrating historical resources into mathematics teaching may not be susceptible to the research techniques of the quantitative experimental scientist. It is better handled through qualitative research paradigms such as those developed by anthropologists' [p. 63].

Keeping in mind these possible constraints, the purpose of this paper is to begin investigating the following questions.

- How does a student's understanding of the concept of function change as she studies the history of the concept?
- Does studying the history of the concept of function deepen a student's understanding of the concept in any way and, if so, in what ways?
- Does a student's studying the history of the concept of function facilitate her move from an action level understanding to a process level understanding?

3. Methods

The students in our study were from two groups. One group of 10 students came from a lower level statistics course. These students were either beginning calculus students or had no prior experience at the calculus level. The other group consisted of 4 students who were enrolled in an upper level history of mathematics course. Their previous mathematics experience was beyond the calculus level.

In both groups, we assessed levels of student understanding of the concept of function with a pretest, which included open-ended questions and also several situations which might be described with functions. Students were asked to find a function in each of the situations. We then gave them a worksheet based on readings from the works of Nicole Oresme, (1320-1382), a French cleric and mathematician who first introduced two-dimensional graphs of varying quantities. In particular, students read about his representation of velocity changing with respect to time. We tested student understanding of the readings with focused questions that made connections to modern day function graphs, and also, we asked students to explain their understanding in journal entries.

Below is an excerpt from the handout we gave our students. Written by Oresme (c.1350), it appears in Clagett [10] and explains why Oresme thinks line segments are appropriate representations for measures of intensity of a characteristic or quality.

Again, intensity is that according to which something is said to be 'more such and such', as 'more white' or 'more swift'. Since intensity, or rather the intensity of a point, is infinitely divisible in the manner of a continuum in only one way, therefore there is no more fitting way for it to be imagined than by that species of a continuum which is initially divisible and only in one way, namely by a line. And since the quantity or ratio of lines is better known and is more readily conceived by us nay the line is in the first species of continua, therefore such intensity ought to be imagined by lines and most fittingly by those lines which are erected perpendicular to the subject. The consideration of these lines naturally helps and leads to the knowledge of any intensity. Therefore, equal intensities are designated by equal lines, a double intensity by a double line, and always in the same way if one proceeds proportionally.

The worksheet contained the explanation and graph below.

From these straight lines, Oresme constructed what he called configurations [9], a geometrical figure consisting of all perpendicular lines drawn over the base line. If the dependent variable was velocity, the base line represented time and the perpendiculars represented the velocities at each instant. A configuration might look something like Figure 1.

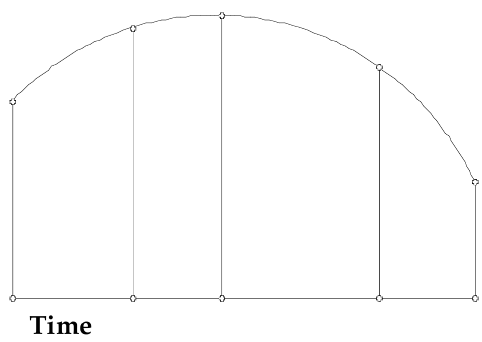

Figure 1: Oresme's construction an example

We then gave students tables of data and asked them to sketch graphs using both Oresme's system and the usual Cartesian coordinate system.

Three of the students in the upper level group also completed a series of 6 other worksheets over a 4 week period of time, based on the history of the concept of function. These worksheets focused on the works of Fermat, Descartes, Leibniz, Euler, and Fourier, and on the changing definition of function over the years. Three of these worksheets were from the Mathematical Association of America's CD, Historical Modules for the Teaching and Learning of Mathematics. The others were our original design. The fourth student in this upper level group chose to write a term paper. Her term paper encompassed the history of the concept of function through the time of Euler. Because of her term paper work, she did not complete any of the worksheets, including the one about Oresme.

Three of these four students completed a questionnaire at the end of the semester to discern their growth of understanding of the function concept. This posttest was similar in design to the pretest, with open ended questions and function situations. Further, on the final exam we asked the entire upper level class of 25 students to comment on the benefits of studying the history of mathematics in general, not necessarily relating to the concept of function.

4. Students' Responses and Discussion

Table 1 contains entries from the students' journals as they reflected on what they had learned from the Oresme worksheet and accompanying exercises. These entries show that they overwhelmingly believe Oresme's graphing method is appropriate and natural for young people and for people just learning to graph. It is interesting to note that Clement [11] reports about 10 year old children who have had little or no training in graphing, yet can represent known quantities fairly readily using vertical lines, suggesting that such graphs are a 'fairly natural and intuitive symbolization device ...and a useful starting point in training students to draw graphs' [p. 85].

Of the four students in the upper level group, only one of them correctly sketched, on the pretest, a graph of the height of water in a bulb-shaped bottle as a function of the amount of water in the bottle. On the posttest, all three who completed the questionnaire correctly sketched the shape of a bottle when given a graph of the height of water in the bottle versus its volume.

Other responses on the pretest and posttest showed growth in understanding, as well. While answering the pretest question 'What is a function?' student A made the common mistake of confounding the function definition with the definition of a one-to-one function. Her posttest answer to the same question, however, was concisely correct. Student B gave what appeared to be a correct textbook definition on the pretest, but he implied the necessity of an algebraic formula. On the posttest he emphasized the fact that 'functions are not necessarily analytic expressions', and quoted Euler in his definition. As this student was working through an activity critiquing different definitions of functions, he struggled with the idea that a function need not require a formula. He was impressed, too, with 'Euler's evolution of the definition of function'. He had a real epiphany, not only recognizing the generality of the function concept but also its dynamic nature. In fact, we had a very stimulating conversation about the dynamic nature of functions. Student C had provided a correct definition on both instruments though claimed that her greatest insight was 'how much better I understand the ideas after looking at them from a different perspective'.

Another interesting development was the students' ability to see functions in real-world scenarios. On the pretest, none of the three students correctly recognized a function in a list of student names paired with club dues owed. Student A claimed 'there is no correlation between the data'. Student B claimed that it 'Can be a function, just don't know how to write it'. These comments suggest a conception of function as needing some definable formula of correspondence. Student C left this question blank on the pretest. The responses on the posttests were, however, much improved. On a question listing students' names with a test scores, all three recognized the function. Student C, who left 10 of 16 function existence questions blank on the pretest, left only 1 of 8 such questions unanswered on the posttest. There was no class discussion per se concerning the pretest and posttest since only four students worked on this project. Thus, one can reasonably conclude that her working through the historical worksheets provided her the insight and perhaps extra confidence to work more correctly and completely with functions.

Also noteworthy are student comments when asked the benefits of studying the history of mathematics. The entire upper level class answered this question; a representative sample of comments follows.

Knowing and understanding the history is very critical. This gives a better understanding to problems we use every day and not even think of why or how it works. I think knowing the history of mathematics gives you a foundation of knowledge to derive from your thinking of math. I feel I know a lot more about every subject of math because of this [history of mathematics] class. One helpful thing is that it gives you a basis of the knowledge that you have acquired. For me the main thing that I gained is an overall better comprehension of the topics we covered. I found that I understood the concepts a great deal better when we went over the history of them.

Table 1: Summary of Student Comments on Oresme's Worksheet on Graphs of Functions

Upper Level Students (n = 3)		
Question	***Response (Y/N)***	***Journal Comments***
Do you think Oresme's approach is more or less intuitive than our modern-day technique of graphing? Consider how a very young student might represent such data if given absolutely no prompts.	Yes - 3	• 'I think Oresme's approach is how a little kid with no prompts would probably think of it. It seems a little easier at first…' • 'I think that Oresme's approach is more intuitive than our modern-day technique of graphing to a young student…. It seems a good way to first introduce it would be to use building blocks'.
Has Oresme's approach enriched your understanding of graphs in any way? Explain.	No – 3 (already had good understanding)	• '…my view on the nature of graphs has changed. I think that Oresme's methods for graphing are an excellent way to introduce graphing to students'. • 'I think his way gave me a new way to look at them'.

Lower Level Students (n = 10)		
Question	***Response***	***Journal Comments***
Do you think Oresme's approach is more or less intuitive than our modern-day technique of graphing? Consider how a very young student might represent such data if given absolutely no prompts.	More – 7 Less – 3 More, for younger students - 3	• 'In my opinion, Oresme's idea would be the more intuitive one for a younger student, who has had no prior experience with graphing data'. • 'Oresme's approach is more intuitive than our modern-day technique because he described it as a measure of intensity that can be measured and the magnitude from the graph'. • 'I like that it makes it easier to see which subject has more intensity, but it does not show any real relation of different subjects. I think that Oresme's method is close to what a young student would do if asked to graph without any instruction'.
Has Oresme's approach enriched your understanding of graphs in any way? Explain.	Yes – 6 No - 4	• 'I now understand the importance of graphs and what they actually mean more. I can see how they are used to determine the magnitude of certain quantities'. • 'Yes, Oresme's approach has helped in my understanding of graphs. It had given me a simpler way of looking at a graph'. • 'Yes, Oresme's approach showed me a different way to visualize graphs'. • 'The definition that he uses for defining the essence of a line using perpendicular lines greatly enriches my idea of the slope'.

5. Conclusions

We are trying to establish that having students study the historical development of a concept often improves their understanding of the concept. We are basing our notions of learning and understanding on APOS theory.

Based on a qualitative study of students' understanding of function and function graphs, the student responses overwhelming supported the opinion that Oresme's graphing methods are a natural way to teach beginners to graph functions. Further, our students seemed to move from action - early process levels of understanding functions to a solid process level understanding, and we believe the studying of the history of functions promoted this process level development. When one studies and writes about, for example, the historical development of functions, one cannot think simply in terms of individual function inputs and outputs; one must think on a more abstract level.

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Hardly Hardy: vulnerability and undergraduate mathematics students' identities

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As part of a longitudinal study, 'Student Experiences of University Mathematics', we followed a cohort of undergraduate mathematics students at two traditional UK universities with high ratings for research and teaching. This paper reports on identity issues for students pursuing an undergraduate mathematics degree. Specifically, student interview data has been interpreted from two perspectives: that of 'communities of practice' (following Wenger, [1]) and of narratives of the 'defended self' (following Hollway & Jefferson, [2]). At a methodological level, we claim that identity as a multi-theoretical construct is useful in explaining some student behaviours. Furthermore, we observe certain vulnerabilities students develop that are related to their burgeoning identity as they study mathematics.

1. Introduction

To undergraduate students, their identity, as a person and in relation to their subject is a matter of importance, for example:

> I'm not an applied mathematician, I'm not sure. I was, when I got here, I was a statistics person, but my first semester module was somewhat, put me off it, I'm somewhere pure or statistics but I'm not applied. (Rebecca, after one semester at university.)

'Students' Experiences of Undergraduate Mathematics', (SEUM), was a longitudinal study funded by the UK Economic and Social Science Research Council (Number: R000238564) of the 2000-2003 cohort of undergraduate mathematics students from two comparable and relatively high status English universities. Data in the form of questionnaires, examination results, observations and over 100 interviews were collected and analysed by the research team. The main aim of the project was to understand better the reasons why students experience undergraduate mathematics programmes in different ways. This paper uses alternative ideas about 'identity' to explain some aspects of student difference.

2. Why 'identity'?

Originally, some of our research questions concerning students' experience were worded in terms of seeking difference in attitude. For example, we were interested in how some students "maintain or develop more positive attitudes than others to the subject". By the time the data collection was completed we were questioning the attitude construct as a useful tool. This was because expressions of attitude, like "I enjoy maths", could be read in different ways for example: at face value, as an automatic response, or as a defensive self-protective response. There are two distinct threads to the concept of attitude. One is the quite informal, even popular-culture term used as in 'good attitude' or 'she's just got the wrong attitude'. The other is the refined construct developed in social science research (e.g. [3]). We acknowledge that 'attitude', loosely interpreted, is important in a student's career, but it is not clear, especially in contemporary cultures where questionnaires are so frequent and careless responses inevitable, how much the informal concept is reflected in data obtained from methods dependent on the technically-driven formal construct. Ruffell, et al. [4] also share disquiet about studying 'attitude to mathematics'. They make the point that the construct 'attitude' is more to do with the context, the questioner and the mood of the respondent than any ontologically prior disposition that can be extracted.

We found that the overwhelming reasons for choosing mathematics at university were that the student "enjoyed" it or judged that "it's my best subject". Such utterances sound like attitudinal stances, although the latter expression is admittedly ambivalent and in some cases may have reflected little more that a comparison of examination marks. So, informally at least, attitude and identity are in a symbiotic relationship: a student's attitude towards their studies depends on their commitment, including elements of personal risk, investment and pleasure seeking, and informs and is informed by their identity in the form of their conception of themselves as a learner/participator in mathematics. Their transition to university mathematics is stamped to varying degrees with either "pleasure" and/or "success"; this exposes the student to emotional risk. For inevitably, not all students experience success in and/or pleasure with mathematics at university. In particular it is inevitable that students who have regarded themselves as successful in mathematics in relation to former peers at school will be lower achieving in relation to the new specialist reference group. So such students, having taken responsibility and made this first (in many cases) major adult choice may be confronted with relative failure and thus 'identity-crisis'.

In the Higher Education system in England where the universities in our study were located, a student's mathematics-related identity emerges particularly early. This is because these students choose their degree discipline while still at school and as students on 'single honours' mathematics courses rarely study anything but maths. While some students did change their course, or their university, or 'dropped out', 70% of the approximately 200 students in our cohort who started on the programme leading to a degree in mathematics as their single specialist subject stayed with this choice for the three years necessary. Hence they have opted, at some level, to have mathematics as part of their young adult identity as they make the transition from home to university. (We had no mature-age students in the cohort, almost all were 18 or 19 when they started the course although a few were in their early twenties).

'Identity' is then a broad notion that encompasses some of the complexity of intellectual and affective sides of experience, and is therefore a more satisfactory explanatory concept than attitude. Nevertheless, in order that 'identity' can be used to explain and structure data from our research in ways that 'attitude' did not we need to employ some theoretical ideas.

2.1. Theoretical views of 'identity'

There are several different, potentially complementary, notions of identity that have been used in mathematics education research. For example, there is the socio-constructivist conception, from Op 't Ende and colleagues in the Netherlands [5] in which 'identity' emerges dynamically from complex affective responses. A different take on identity is employed by Boaler and Greeno [6] who use theoretical feminist epistemologies developed by Belenky et al [7] to explain how identity is central to how and what we are able to know. From an explicitly post-structural perspective, Mendick [8] explains the importance of imaginative projections for young people when they choose whether to opt in or opt out of mathematics and the subsequent 'identity work' they need to do to maintain a positive projection of them selves. Part of this identity work involved entwining their gender-identity with their mathematics-related identity. Bartholomew [9] also found that gender was significant in forging a 'maths-identity', for a masculinity seemed to be easier to meld with mathematical success than femininity, when such success involved competitiveness. In this paper we have used two further perspectives that seemed particularly suitable for addressing social and psychological issues: 'communities of practice' [1] and the 'defended subject' [2] which are outlined below.

3. A brief sketch of these interpretative perspectives

3.1. Communities of practice

Wenger's 'social theory of learning and becoming' is an explanation of how learning involves both practice and identity [1:14]. That is, knowledge develops for an individual as they engage with the world in a particular fashion: they participate in a practice. As they participate, they become part of the community of practice, taking as real (reifying) the tokens of that group: this beginning to belong helps fashion their identity, [1:151]. As reported elsewhere [10, 11], being a mathematics undergraduate is itself a practice. The experiences of attending mathematics lectures and sitting exams, giving pet names for modules or courses and being familiar with staff, contribute to a way of being. Participants in the community of mathematics undergraduates reify objects of mathematics as well as the mathematics department's curriculum, exams and assessment procedures.

The communities of practice framework has been used to analyse another group of mathematics undergraduates [11]. In this research, Solomon distilled a key feature of her undergraduates' identities as 'not belonging'. Her interpretation was that they did not participate in mathematics dialogues such that concepts were negotiated, as Wenger's model requires for participation. However, another interpretation is that what the 'negotiation' is within the undergraduate mathematics student community of practice is not the same as that in a community of 'grown up' mathematicians [10]. For example, to 'negotiate' you don't need to develop novel mathematics, but you do need to read your lecturer's worksheet, possibly say something like "what's this about" to another student and turn up at tutorial having 'had a go' at some questions. There you discuss mathematical meanings with the tutor in order that you are able to participate in answering curriculum-relevant questions.

3.2. Narratives of defended subjects Hollway and Jefferson draw on ideas from psychoanalysis as well as social science. Their principle is that all people who are the subjects in social science research both need to make their own sense of their experiences (are meaning-making) and they need to defend themselves psychologically. Consequentially, the interviewees:

- may not hear the question through the same meaning-frame as that of the interviewer;
- are invested in particular positions in discourses to protect vulnerable aspects of self;
- may not know why they experience or feel things the way they do;
- are motivated, largely unconsciously, to disguise the meaning of at least some of their feelings and actions. [2:26]

Hollway and Jefferson outline a "narrative" approach to interviewing and data analysis which aims to elicit stories which reveal something of an individual's underlying psychological take on situations they find themselves in. This approach was taken to analyse some gender issues in the SEUM project, [12,13] where we interpreted some female students' identities as 'invisible and special' because they were able to succeed and see themselves as mathematical, while not drawing attention to themselves as high performers.

4. Data and interpretations

In order to illustrate the two interpretative approaches, a selection of interview material from two quite different yet representative students are presented. In each student's case, after the quotations, analyses from each perspective are given.

A selection of interview material with Liz

> 'boring lecturers killed any chance I had with the maths' (Liz, final semester)

Liz tells of losing "interest in the subject" and her "down hill" journey. Having started off quite well, at least in terms of results and lecture attendance, she reports her "bad exam experience" that "knocks your confidence and like you're dreading it in a way you never knew existed". Her appraisal of school exams were that they "were fun" but by her final university year she is "really quite scared". While she's concerned her parents might "hate" her, as "I kinda hated myself", she acknowledges their support when she does not gain good results "they were quite cool with it actually".

By the time she is interviewed towards the end of her final year she says of mathematics that she "dislikes [it] with a passion I didn't know possible". Having come to university saying maths was "enjoyable" at the end she says she wants "to get through it for me".

For Liz, mathematics is "boring", "abstract" and "totally meaningless". She finds it impossible to "maintain interest" for an hour when lecturers "talk to the board" and "throw stuff" to the students. When prompted as to what could be improved she suggests that lectures be delivered so that she is "compelled to listen" (rather than "bored"), as her school teachers managed to do (and where she was successful).

Despite the severe discomfort that she is enduring, she sees her future using numbers, possibly in the financial industry "I would like to be able to apply mathematics". In fact when she talks about the research she did for her history of maths project which she selected to be part of her course, she tells of enjoying the independence and refers to it as "her own work", in juxtaposition to the standard maths fare which is described as "kind of da da da da da da".

When she is speaking about her part-time bar job she relates how some customers would get excited about maths, when she said that was what she was doing, but she herself could not talk about maths yet would have liked to. When asked what she'd do if she could start again she replies that she'd do classics as she'd "enjoyed it at A level". This seems ironic given "enjoyment at A level" was her rationale for choosing mathematics!

Interpretation using a community of practice perspective: Liz has come from a supportive home and well-managed school background and expects to enjoy mathematics in a similar fashion to her school experience where doing maths exams was "fun". Having been brought up within a culture of participation legitimated by testing, her bad exam results produce visceral emotion and question her identity as a maths undergraduate. She reacts as an outraged consumer blaming her alienation on "dull and boring" lectures; this also rationalises her lack of insider participation although she does manage to stay the course. Nevertheless, a middle class social identity propels her to keep face, remain peripherally participating, and to "get through it for me".

Interpretation from the defended subject narrative perspective: This "get through it for me" utterance signals Liz's protecting herself. To survive, her strategy is that of distancing herself from the subject and its proponents, so she may protect her own identity as a successful student. It is other peoples' fault when she does not understand. When she says that the lecturers (who were boring) "killed [her] chance with maths" she positions herself as a victim. The crime is to deny life to her potential mathematics future. And as she related that she'd have liked to talk about and be excited about maths and that she still sees a possible career in the financial sector, this 'killing' is unfair as well as denying her life. Her "da da da da da da" utterance represents her thwarted quest for meaning in mathematics.

A selection of interview material with Ian

> 'I thought with a first class maths degree I'd be able to do whatever I wanted to do and its not working out like that.' (Ian, final semester)

Ian has "knowncertainly since the last few years of primary school" that he was going to study maths. He says that has chosen to do "the things that would point me towards a maths degree". He applied to just two universities (most apply to six to increase their chances of getting in): to Cambridge (where his mother had studied science) and to Waverley (where his father is an academic in a science department.) He "didn't get in [to Cambridge] in the end". Even though Cambridge was his "first choice" he says "I don't mind now. Not at all. I'm glad

I came here now". He tells of coming to see his dad at work when he was younger and is "familiar with the campus layout".

Speaking about lecturers Ian says that "everyone sees a stereotypical statistics lecturer as someone who has a monotone voice shut up in an office with piles of paper and lots of numbers and a calculator and lots of boring notes". Some lecturers were like "the walking dead" though at the beginning of his second semester he judges the statistics lecturer as "we've quite a good one it seems". And the lecturer makes a difference: "it shouldn't but it does". He praises the lecturer who "makes you think instead of copying notes", "holding your interest, that's what they need to do". He is self aware in the way he does mathematics: "I tend to be very concise when working out answersI don't like writing lots, but you can't get away from it when you're doing proofs so that's something I'm going to have to work on".

Outside of the formal studies, he lives in a student hall "I can't cook but I'm determined to cook for myself next year. It's not a very good combination is it, going self-catering and not being able to cater? I'll have to learn". He doesn't go out in an evening if he has a lecture first thing "I'd just never get up so I've been quite careful really". "I don't do riotous stuff like students are supposed to do", but he plays snooker, is in frisbee club and maths society. He works "pretty much" on his own but checks with another lad in his tutorial group. He likes the group project work that's just started.

Catching up with Ian in his final year, he relates that his pursuit of an actuarial career has not been successful. The firm that he worked for for two summers does not want to give him a permanent job: "they said that I hadn't mentioned enough team working activities in my interview which, well is possible but it's a little bit vague stuff". A further rejection "knocked me back, but yeah I thought with a first class maths degree I'd be able to do whatever I wanted to do and its not working out like that", But he has a fall-back plan: "if other things fall through then I think a PhD might be interesting and maybe I could go on to be a lecturer from thatI think my maths degree will carry a lot more weight when applying for a PhD as opposed to applying for all these jobs for actuarial firmsbecause I think academic ability is to a greater extent what you need when you do a PhD".

Ian has "enjoyed" his course which has "not been terribly hard". His identity as a mathematician has changed, however: "one thing I've been surprised about is that I have changed from someone who thought they were more an applied mathematician to a more pure person. I'm not entirely sure why that is...the last couple of applied have been very, very hard and the pure ones have all seemed easy even the ones like analysis which most people seem to find hard". However, later he claims that it is because of his applied marks having "10%" lower average than pure that he has moved away from applied.

By the third year he has an insight into proof: "proving everything seems pointless at first, but when you get into more complicated things then it's useful knowing that everything you do has been proved before.. applied modules I think they build up more and more on a base you're not quite sure of". Ian is also doing a project module in his final year which involved a "big written report, and then do a presentation and it's been very useful for talking about at job interviews".

When asked how "stretched" he's felt, Ian replies that "I've never in any doubt that I was going to get a first overall, so it could have been harder". When prompted as to whether he sees himself as a mathematician or someone who's done a maths degree Ian says "to an extent I see myself as a mathematician, but if I get a job and then in a couple of years I might think differently about it, I think at the moment this is what I do most of the time, so yes" and he'd still do a maths degree either at Waverley or "I might apply to Cambridge again, cos I think I know everything I needed to get through their exams".

In terms of the curriculum, he says that "more problem solving would be nice but too much of it sometimes you get people who don't understand the subject at all and they wouldn't be able to go very far with problem solving because they don't have the foundations" when asked why some succeed and some don't Ian says "there's some who don't do the work, I think especially in first year it's vital to get a good groundingany uncertainties you have in the first year mean that you do less well in the second year.."

He has several things going on in his social life: his housemates are Spanish and he "gets invited to the Spanish parties and things like that" he goes to cinema a lot and to productions that university students put on.

Interpretation using a communities of practice perspective: Ian is a model participant in the community of undergraduate maths students at his university. He succeeds in the assessments, feels comfortable criticising his lecturers and understands aspects of the curriculum's rationale. He has prepared for this participation for most of his life and recognises that investment of his time is significant in honing his identity. He shows how he has aligned himself within the community over the three years by his explanation of the purpose of proofs. His second choice career as a university lecturer a reliable option fitting with his family background and experience indicates the depth of his social identification with mathematical or scientific academic communities.

Interpretation using a defended subject narrative perspective: Ian positions himself as a "first" and distances himself from those "who don't do the work". The issue of his rejection from Cambridge university continues to be an issue from which he needs to defend himself particularly in the light of his mathematician's identity. He chooses modules that are directly relevant to his goals, thus defending himself from being unprepared. His identity as one with a first class mathematician passport to the job of their dreams has been confronted with a reality of rejection from which he defends himself by continuing to apply "to get to the world of jobs" . Though his initial career ambitions have been thwarted, his story for this is because of his lack of team work experience, he defends himself by resorting - at least in his imagination to the PhD route to university teaching where he should be valued (protected) because of his "academic ability".

5. Vulnerability and successful adaptation

Both of these students exhibit vulnerabilities because of their identity as it relates to mathematics. Liz is clearly threatened and positions herself to survive. Her profound expressions of strong emotion signal her stress. Liz is also vulnerable because she is lost: she is frustrated that she cannot make sense of the mathematics. Ian's success and self-confidence as a mathematics student also produce vulnerabilities: he finds it difficult to understand how he could not be welcomed into mathematically-based work. The status that he thought his top maths degree would bring him has not materialised. He anticipates retreating to familiar academic environments. Nevertheless, in both these cases, to some extent they have overcome setbacks: Liz did get her degree (although with a low rank) and Ian has turned towards another acceptable career.

Not all students who experienced blows to their identity ended up with some success however. Another student, Raj, who entered university with very good entry qualifications had readily apparent problems with his identity, and, although staying with the course until the end, failed his final degree. Raj came from a working class Asian (Punjabi speaking) background. He was an only child and, at least in first year, phoned his parents daily. He was the only student in his inner city school to have achieved top university entry grades which he achieved mainly by self-study with teacher encouragement. Observers at lectures and tutorials saw Raj in the centre of the lecture theatre, surrounded by what seemed friends. In contrast, in interviews he off-loaded his distress in his failure to get good marks and his feeling of isolation.

> 'It was probably the worst start I could have hoped for. But, to be honest, I couldn't realise what it was going to be like because I really hadn't a clue what was involved in going to university. Hadn't a clue when I applied and then I came here and it was like the biggest eye opener I could imagine, just being here. Nobody told me, nobody knew and I never knew what to ask, kind of thing because I never really thought much about it. And nobody I knew knew much because people at my school don't go to university. My mum and dad's

> never been so nobody could tell me.' (Raj, second semester.)

This short extract indicates some of Raj's vulnerabilities. In terms of the two interpretative theories used: initially he was able to participate, at least socially, but when the bad grades kept coming, he did not turn up to tutorials and hand in homework putting his participation and undergraduate maths identity at risk. From other extracts, he said his parents were proud to have a son doing maths at university, but this aspect of his identity is vulnerable, given his poor performance, and he constructs defences that involve considerable emotional stress.

5.1. Vulnerability, identity and mathematics assessment

In our students' mathematics-learning environments, their assessment results were a key factor in their maintaining and producing their identities. The university mathematics majors' curriculum at both institutions in the study, had the majority of courses designed to 'deliver' a body of knowledge and only a small minority of courses orientated around students solving original problems. This type of curriculum militates against a 'mastery orientation', where the goal is leaning, [14] and trains a performance-goal orientation. A mastery orientation involves incremental processes towards understanding and produces learners who are more robust in the face of setbacks indeed they do not seek straight forward problems; their identities as learners are more as problem-solvers. Whereas a performance orientation to learning associates authority-verified results with self-worth and so creates vulnerabilities in an environment where the gate to participation is through assessment. This orientation requires a defensive stance to protect the self when there is failure as the learner identities are marked by assessment achievement. These vulnerabilities produce defences that often involve 'blame'; sometimes others are blamed (Raj's "nobody told me") or, as we frequently heard, students blame themselves: "I must work harder".

One student Leah was quite explicit in how her performance orientation contributed to her developing identity, telling us in first year, "we'll see if maths is for me when the results come in". This is a vulnerable identity with respect to mathematics, yet follows from the authority validation which she had, like many students, relied on to get to university. Over and over again, students told us that they had chosen maths because it was their "best subject" in the sense of exam performance. And at the end of three years their mathematical self was still being validated, or undermined, by their exam results.

6. Conclusion

There are myths about maths that contribute to the vulnerable, performance-orientation conception of self that associates externally verified success with self-worth. These include maths being 'useful', 'hard', 'abstract', 'high status', and that you have to be 'clever' to do it. When struggling, performance orientated students who were 'good at school' have their identity threatened by bad results, how do they manage to get through? While overall, academically buoyant students were able to get support from lecturers, marginal students did not participate in this way. In some cases, peripheral students survive due to the social support they find, for example, Liz did have "cool with it" parents and we found other middle class students who were able to get emotional and, sometimes, academic support from their families. In this regard we can see the classed nature of students' identities. Indeed, in Raj's case, he seemed to have two parallel lives and his social support from parents and peers did not help him to ultimately succeed in his maths degree; he was blocked from seeking help from lecturers by "not know[ing] what to ask" and his apparent need to keep face made him even more vulnerable.

One of the truths of research such as this is that there are no grand truths; students' identities, defences and patterns of participation are influenced by many factors and hence are at root unpredictable. The affective aspects of our psyches are sensitive to nuance that would be hard for even intimates to observe, let alone a team of professional (emotionally distant) researchers. What we have presented here is a snapshot of two different takes on the same text, one social and one psychological, with 'identity' as the common theme. These methodologies can be used by others for their own purposes. The issue of identity will be on-going but the theories we have used underpin the idea that: what you practice does make you and how you defend this does maintain you, whoever you are, in whatever communities you find yourself and whatever stories you need to tell.

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Mathematics communication for graduates

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How do mathematics graduates make the transition to the professional workforce? This study reports results from in-depth interviews with 18 graduates who have moved into a range of industries. The study is from the perspective of graduates who have graduated within the past five years from five universities. We investigate how they perceive that their university study has helped them move into the workforce using communication as an example. Language is powerful and is mediated with workplace interactions. Whilst there is an argument for other university services, such as careers programs, to help with some of the transitional processes, we argue that most career preparation should be done within the teaching and learning of the technical subjects themselves due to the discipline specific nature of the language and communication required.

1. Introduction

The transition from university to the workforce is one of considerable adjustment. Let us consider the following quotes from two recent mathematics graduates.

> Evan: You can, you can um, not talk to anyone for 3 years, and youll be fine. Do a presentation now and again, you dont really need to talk to anyone. Or be nice to anyone. Or be tactful. Or know how to teach someone something. Im just calling it the way I see it. As a student, you dont need that.
>
> At work, youd be out the door quicker than anything, doesnt matter how good you are, if you are not tactful, or dont know how to talk to someone, you cant have a, provide a client-style consultative relationship with, with people you work with, doesnt matter how good you are. Its, its secondary, far secondary. Ive learnt that firsthand, you know, that you need to be able to communicate and have other skills beyond the technical skills. Its so much more important.
>
> William: I ended up leaving [my job] after a year, so when I first entered an office and worked in an office and Id expected there to be much more When I went into an office I think I had the same sort of attitude that, you know, I could be a bit of a young cocky whatever and that my superiors were there to, you know, they would care about me and, you know, guide me and frame me and build me. That wasnt the case at all. It was they had the attitude, you do your work, you get serious or we were really going to and they did they gave me a cold shoulder and I didnt quite, at the time, understand what was going on but it was more like I still worked and I wanted to work hard and learn but, I think, I wasnt prepared for the environment.

Adjusting to the workforce can be problematic for students as they discover that what they have learned at university needs to be contextualized for work and that there is as much to learn at work as there was at university. Part of our job as educators is to help students prepare for this adjustment through the development of pedagogies that support students in their ability to transfer their content knowledge and their understanding about their own learning to the workforce. Both of the graduates quoted above had done well in their university studies and went into the workforce with excellent technical skills. They went into their new jobs with high expectations which were appropriate to their technical skills. One survived in the job but the other left after one year and has retrained. Does it have to be like this? With some training in interpersonal skills and office dynamics, the story of William could have been very different. Instead of Evan finding out, on the job that communication skills were important, he should have known this before he started work.

As mathematics educators it is important that we address this issue of transition to the workplace because, at a time of high employment and skill shortages, mathematics graduates are not doing well. Australian government graduate destination surveys of 2004 show that of those graduates who are seeking full time employment, 64.4% of mathematics graduates are in full time employment and 35.6% are seeking full time employment. In all the graduate fields surveyed, it is the second lowest full time rate for students graduating in 2004 [1]. Previous surveys show similar results.

The study reported here investigates recent mathematics graduates transition to the workforce and their use of communication skills as the indicator for the ease of transition. We broaden the work of Burton and Morgan [2], who investigated the discourse practices of academic mathematicians, to examine the discourse used by new mathematics graduates in industry and their perceptions of how they acquired these skills. We share the beliefs of Burton and Morgan in relation to the power of language choices. We have also investigated graduates perceptions of how they use this discourse and how they learnt the discourse. We use advanced mathematical discourse to refer to the uses of language in university mathematics learning and teaching as well as in professional life [3, p. i]. Through this paper we use discourse, communication and language to mean mathematical discourse. Because of the interactive nature of discourse, participants in the context of communication commonly mention interpersonal skills.

1.1. Background

Students experience mathematical communication in a variety of ways; through the provision of mathematical texts and the discussion surrounding the representation of mathematical ideas. Morgan [4] describes how differences in mathematical texts often reflect the differences between the experience of students and professional mathematicians: "On the whole, students have worked only on relatively short, routine problems for which little elaboration or explanation is required. Their writing has been addressed only to their teacher In contrast real mathematicians tend to work on relatively substantial and often original problems. Their anticipated audiences are expected to be genuinely interested in knowing the results and to need to be persuaded of the correctness of the results." (p. 2). Evans experience confirms this view, as he was required to make sense of his mathematical work for others in their language rather than reverting to purely mathematical descriptions. There has been insufficient attention paid to this divergence within curriculum design at university: "Most attention, however, has been paid to the difficulties children may have in making sense of mathematical language or the benefits that may arise from engaging in talking or writing in the classroom rather than addressing the nature of mathematical language itself." [4, p. 3] Whilst Morgan identifies the problems of mathematical language she situates it within the context of children learning and describing mathematics, but the point is made that it is the representation of the mathematics to others that is problematic.

This is particularly true of the group of French mathematics educators, exemplified by Raymond Duval [5, 6]. At university level, researchers influenced by Duval, such as Durand-Guerrier [7] have investigated how details in mathematical discourse, such as the way letters change status over the course of a proof, lead to deep misunderstanding in students. She argues that these questions have not been seriously addressed in mathematics education. We agree that these difficulties have not been addressed but believe, as implied by Morgan [4], that an enhanced course of action is to investigate how practicing mathematicians use mathematical discourse and to develop teaching and learning strategies to inculcate students into the discourse of the discipline. This will enable students to see a connection between mathematical discourse and the ways in which they may communicate mathematically at work. The work of Durand-Guerrier may change practices within the mathematical community, as their discourse practices are made explicit.

At post university level, the work of Burton and Morgan [2] examined the ways that academic mathematicians write. They argue that writing plays a critical role in many mathematical practices. "The stakes involved in producing mathematical texts that are seen as acceptable are often high, both for students who are likely to be assessed on the basis of their written work, and for professional mathematicians whose status within the community and even job security may depend on the quality (and publishability) of their writing" (p. 430). Burton and Morgan believe that "knowledge of the forms of language that are highly valued within mathematical discourses and the effects that may be achieved by various linguistic choices would empower them" (p. 431) to make choices, to break conventions and express their own personality. While they quote some published guidelines for the development of mathematical writing they conclude, "the training of mathematicians does not appear to include any systematic attention to the development of writing skills." (p. 448)

Despite the postulations of these eminent mathematicians, little is known about how graduates integrate their knowledge of mathematical discourse to the workplace [8]. The study reported in this paper investigates that problem.

1.2. Methodology

For this study, we used the approach of phenomenography which looks at how people experience, understand and ascribe meaning to a specific situation or phenomenon [9]. It is a qualitative methodology that is often used to describe the experience of learning and/or teaching. Phenomenography allows you to study the range of experiences not the numbers of people who had those experiences. The outcome of a phenomenographic study is a hierarchical set of logically related categories, from the narrowest and most limited to the broadest and most inclusive. This is referred to as the outcome space for the research. Phenomenography defines aspects that are critically different within a group involved in the same situation. In this study we are considering the ways that graduates experience the workplace from the viewpoint of communication. We investigate the different ways that they learnt those skills and have arranged these into a hierarchy. The questions posed are designed to encourage the participants to think about why they experience the phenomenon in certain ways and how they constitute meaning of the phenomenon. For the purposes of a phenomenographic study 1520 interviews are sufficient [10].

While it is reasonably easy to make contact with students at university for research purposes, reaching graduates has a range of complications. There is the difficulty, noted by others [12], of finding information about graduates. Once they have left the university there is no obligation for them to provide up-to-date contact details. There were issues of confidentiality until a graduate had decided to participate in the study. All initial contact with graduates was through an administrative assistant and the alumni associations of two universities. In all, 18 in-depth interviews were conducted. The interviews resulted in approximately 100 000 words which were analysed with the assistance of the software package NVivo [11].

The names of the graduates have been changed for confidentiality and their area of work included in brackets if necessary. The quotes in this paper are as they were spoken with some deletion of ums and other repetitions. Spoken English is full of grammatical errors and these have not been corrected. Quotes have been shortened and sections that are not relevant to the particular context have been omitted these are shown by the use of an ellipsis (). At all times care has been taken not to change the sense of what a participant is saying.

1.3. Attributes of the participants

The participants in the study had a range of employment situations, age, ethnicity and academic backgrounds. Some graduates had laboured through their degrees with many failures and others had high grades. Some had done well (by their own reckoning) in the workplace; others had struggled. Participants had graduated from five universities and had a range of degrees with mathematics majors. Most of the participants were in their early to mid twenties.

There were three people who had studied mathematics as their second degree and one had come to university as a mature-aged student. There were 10 males and 8 females and 8 spoke languages other than English as their home language. Their work situations ranged from police constable to cryptographer. The commonality between these participants is that they all regarded themselves as qualified mathematicians.

2. Results

All the graduates believed that they had been changed by their studies. Their studies gave them "the skill set where I could put my foot out in the door " (Nathan), "problem solving and just thinking about a problem logically is, you know, transferable to anything" (Melanie) and "It made me appreciate the big picture. It sort of opened up avenues of my career that I never would have imagined could have existed" (David).

No graduate suggested that they had studied mathematical communication systematically at university and there was some robust criticism of their preparedness for the workforce and the process of getting a job "I think it would be humane for somebody to have some kind of in-your-face train[ing] I dont know if training is the right word, just exposure to what happens when you go for a job" (Leah). There was also strong criticism of some of their teaching at university. Graduates also had received little or no career advice "You should have an accurate idea of what potential jobs are there" (Sally). Graduates were divided as to whether these functions were the role of the academic departments, the university or themselves.

Let us consider in detail the way graduates use mathematics in the workforce and how they learnt mathematical communication skills as these most impinge on the need for new classroom practices.

2.1. Use of mathematics

Patterns emerged through an initial summary of the participants responses. Those who used very little of their mathematical knowledge explicitly ("Bugger all!" was how one graduate described her use of mathematics in the workplace) said that they had taken a logical way of thinking from their degree or a way of looking at the world numerically. Those who used a medium level of mathematics found that their use of mathematics is very subtle; that they could often see patterns in numbers that others are not able to discern.

> David: Lets be honest in the sort of stuff we do which is money market focused, the math that we use in our day-to-day work would be more focused on, to be blunt, very simple arithmetic. And the pricing instruments, which are very vanilla in nature, so were simply pricing bank-bill type instruments and the formulas well published, well understood, well used across the whole market. It is, its very subtle, its often in the job we do you sometimes see examples of, its very difficult to explain, but people whove got it and people who dont. Theres a difference between being able to memorise a page of numbers, and being able to understand a page of numbers

Those who use high-level mathematics are required to make large adjustments in the ways that they use mathematics in the workplace as opposed to how it was presented at university and it has required them to change their view of mathematics:

> Roger: Oh yes, one problem for people with pure maths, well that I certainly had, is it is quite a shock to them in the real world to see how maths is used in this strange way where assumptions are made left and right, whereas in pure maths one dare not. But it is very important to accept that, if you are going to go and work in the real world so to speak, one has to be able to just allow oneself to make assumptions and so on, and even though it is completely against what one is trained to do, in pure maths, which is everything is to be verified, perfectly, so you must then get rid of that idea.

When they finished their degrees, graduates had little idea of how other areas of mathematics worked. Pure mathematicians, for example, were not exposed to modelling or statistical approaches. All the graduates expressed the idea that mathematics is pure and logical even if it is not used in that way. They also believe that they think differently to others: “the links that I make with things are perhaps a little bit different to other people, just the logical thought processes.” (Christine).

2.2. How did you learn these communication skills?

Graduates expressed three levels of understanding of how they learnt mathematical discourse. The key qualitative difference here is the level of control that the graduate perceives. Here we particularly consider mathematical communication, that is, when the graduate is trying to communicate their technical ideas. The first level is one where you go with the flow and use trial and error to work out the right communication for the situation. For the second level graduates learnt communication from outside themselves, from their bosses, or an outside agency such as a church group or sporting organization. On the third level, graduates stood back and systematically observed good and bad communication and actively reflected and modelled good practice. None of the graduates learned communication skills as part of their studies; in fact more than one graduate made a comment like: “Those sort of people skills I do not think, one certainly cannot learn them at the maths department!” (Roger). Here are some statements from graduates to exemplify the conceptions.

Level 1: Go with the flow, trial and error. Heloise is working in a medium sized company where she and her boss are the quants. She is mirroring the language used by those around her and feels comfortable enough to question if she doesnt understand. The interview showed that she is not in control; she is making it up as she proceeds in the job.

> Interviewer: So, how did you learn to talk differently to different people?
>
> Heloise: I dont know. I just kind of picked it up. Mainly Id respond by the way they spoke to me. If they speak analytically to me with my manager. So, I know that I can speak like that to him, type thing, and then again with the sales lady and with my director, hes a director, he just wants the bottom-line, as all directors want.
>
> Interviewer: But how did you figure that out?
>
> Heloise: Just kind of picked it up.

Evan works in teams to deliver IT solutions for the financial products of a large bank.

> Evan: Trial and error. Its usually that glaze-over effect! When they start glazing over its time to stop! Trial and error, very much so.
>
> Interviewer: So nobody taught you how to do this?
>
> Evan: Id have to say that its been a lot of, ah whats the word, internal hit and miss.

Level 2: Mediated by others and outside situations. People with these experiences have worked with a supervisor or an outside influence that has taught them how to communicate in an appropriate style.

> Paul. Experience. After, when you first come out of uni and you try to explain, youre really keen, oh Ive got this great idea and here, blah blah blah, and your boss just looks at you blankly and says thats too technical, I dont understand or if you get asked to write a report for someone and you run it past your boss, who says it is way too technical, take all the technical stuff out of it and just give them what they can understand, write for the audience. If you hear that and get told that often enough, then sooner or later, it changes the way that you deal with people at work so it just is trial and error, and going through the process often enough.
>
> Kay: Yeah, I think I guess, actually just learning about generic communication skills in the environment that were never related to my academic life, like (?), Im quite involved in leadership and things in areas outside uni like in a helping situation, also in Church and things

> like that and Ive done a few courses and Ive actually read a lot of leadership books and things like that.

Level 3. Active, detached observation. People who express these experiences have approached communication in the workforce in a scientific way and have closely observed the good and bad ways that interaction occurs and have modified their behaviour accordingly. They have not used trial and error but have observed rigorously and are making controlled, conscious decisions about how to communicate.

> Roger: Oh, well, the, from the people that, like there were two kinds that stood out. Firstly those that stood out in a positive way, and those that stood out in a negative way! So then the ones that stood out in a negative way, they were exactly the sorts of people you would expect to come out of the maths department almost, very dry and techy, [] But, so then, that you realise you must avoid, and then on the other hand the interesting people had a different style, you know, they spoke about interesting things, not just in the context, not just in more general human things but even in the context of their work, theyd always try to approach it from the more general perspective. Yeah, it was quite a learning curve I must say. So, because when I went there I didnt know, obviously it was the first time Id worked in a, in the so-called real world, which is not as real as people think.

3. Conclusion

This study has described graduates different levels of perceptions of their discourse needs and how they gained those skills. At the lowest level, graduates learn through trial and error, others are assisted by their manager or outside learning, whilst a small group view communication and interpersonal skills as a scientific process and stand back and use their mathematical observation skills to model their behaviour. Not one stated that they had systematically studied communication as part of their degree. These graduates were unaware of the power of language choices in the workplace. One of my colleagues who read this paper said "Why not?" At his university, students have a compulsory third year professional development course that incorporates communication skills and teamwork. There are several ideas to consider here. Firstly, few universities have a compulsory professional development subject for all students (though our data suggest that this would be useful). Secondly, a one-off subject is not sufficient to develop the discourse skills required to induct students into the language uses of their discipline. In the same way that it takes time to develop mathematical confidence and competence, it takes time to develop mathematical communication skills and confidence. Thirdly, this study examines the graduates experiences and their reflections on their learning. It is possible that they studied communication as part of their degree but they did not explicitly notice.

Many of these problems could be alleviated by changes in curriculum that would be cheap and effective to implement. Firstly, course designers and lecturers should give an overview of the different areas of mathematics and ways of mathematical thinking to their students. Group work in subjects needs thought but is relatively easy to implement and will not add cost to the course. Other teaching and learning suggestions should be seriously considered. A third year Transition to the Workforce subject could lead to the study of a project, consulting or peer teaching and develop many of the generic team and communication skills needed for the workforce as well as consolidating mathematical skills. This could be complemented by smaller real projects in previous subjects. Universities can coordinate opportunities for work experience in the university vacations. There is need for all agencies of the universities to work together to assist in the transition to the workforce.

Changing classroom practices in mathematics and other scientific areas can assist with the transition to the workforce and give our graduates the skills to succeed. Technical skills are not sufficient. Whilst there is an argument for other university services, such as careers programs, to help with some of the transitional processes, most should be done within the teaching and learning of the technical subjects themselves due to the discipline specific nature of the language and communication required. There is a crucial need for curriculum reform to assist with transition to the workforce in technical areas so that we do not squander people who have technical skills, but not the communication and interpersonal skills. The following quote sums it up.

> Nathan: Now, if youve only learnt half of the skills at university and the other half to do with, you know, communicating, whatever that you learn on the job, arent you going to be better off if youve got all the skills when you finish university? So I think it should be part of the degree.

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Short Papers

First year student experience with Maple laboratory classes

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The Computer Algebra System (CAS) Maple is being used to fundamentally change the teaching, learning and assessment paradigm for many of our mathematics subjects. For some advanced subjects, the traditional lecture mode has been completely replaced by teaching with Maple in a computing laboratory. However this paper discusses our experience with a weekly Maple class in the laboratory which supports an otherwise traditional (second semester) first year mathematics course. The content of the subject includes differential equations, surfaces, lines and planes, directional derivatives, Taylor series, matrix algebra, eigenvectors and eigenvalues. Maple is used to do numerical computation, plot graphs, perform animations and do exact symbolic manipulations and word processing. Initially, for this component of the course, pen and paper exercises were carefully integrated with the Maple exercises. However all student computation is performed now using Maple in small groups: this provides such a rich learning environment that no pen and paper work is done. We present our observations from tutor diaries and a student feedback questionnaire. Using the computing laboratory as an integral part of the course was new, educationally effective and enjoyable for the students and staff.

1. Introduction

Internationally, Computer Algebra Systems (CAS) are increasingly being used in tertiary (and secondary) mathematics, see [4, 8]. Our university has a site licence for Maple, which is widely used for undergraduate mathematics. For some advanced subjects, the traditional lecture mode has been completely replaced by teaching with Maple in a computing laboratory [1, 2]. In 2002, Maple was first used to support MA911: the second semester of a traditional first year mathematics course (MA910 in first semester, followed by MA911). For this initial implementation, the weekly computer laboratory sessions consisted of working through very carefully designed Maple worksheets which integrated exercises to be performed by "pen and paper" and Maple exercises. The Maple exercises had to be completed and printed out to be submitted as hardcopy and (for efficiency of marking) the "pen and paper" exercises had to be done using the template solution sheet.

This careful integration of "pen and paper" and Maple exercises was implemented in 2002 [7] since it had been well received in two other courses (for quite different second year classes, see [3, 6]). A detailed analysis of this approach and the pedagogical benefits is available in [7]. This approach is consistent with the study [5] which supports a balanced use of computer software and "pen and paper" work. However following some student feedback and requests, the worksheets for 2003 were rewritten (by the authors) without any "pen and paper" component.

The students (numbering about 100) of MA911-Mathematics for Scientists met five times a week with common lectures. For their weekly one-hour Maple lab classes, the lecture class was split into two groups (of similar size):

1. the maths group with some previous Maple experience having used Maple in the previous semester, and
2. the non-maths group with no Maple experience.

Over the course of the semester we spent eleven class sessions in the computer lab. The 11 Maple sessions in the lab were structured as follows: 1, Maple Intro; 2, First order Differential Equations; 3, Second order Differential Equations; 4, Modelling with Differential Equations; 5, Surfaces and Space Curves; 6, Vectors, lines and planes; 7, Directional Derivatives; 8, Taylor Series; 9, Solving Systems of Linear Equations; 10, Basic Matrix Algebra, Transposes and Inverses; and 11, Eigenvectors and Eigenvalues.

The maths group was already familiar with CAS: they had taken, in first semester, a non-linear maths course that used Maple. However even those who had no experience caught on quickly with the Maple Introduction worksheet.

From 2003, (semester two) this Maple component of MA911 has had no "pen and paper" work and from 2005 has been completely electronic - only submission by email was allowed. The worksheets include some elements of e-Learning which are not possible with traditional courses. The students rarely had any difficulties with the worksheets for the course. On lab day each week, the students would come to the lab, launch Blackboard that opened with the Maple worksheet page as the default homepage, and set up the day's assigned Maple sheet by downloading the Maple file and arranging the browser and Maple windows. After the first two classes in the lab, students never waited to be told what to do. They just got started on the day's worksheet. Their self-motivation was wonderful, and we were elated to see that they were taking responsibility for their own learning. Part of their motivation stemmed from wanting to make good use of their time. Our lab sessions were only 60 minutes long. The assignments were part of each worksheet and generally took them between 30 and 60 minutes to complete. As they worked through their worksheets, most students strongly preferred to consult with one another (which we supported and encouraged them to do - the assignments had to be submitted by groups of size 1,2 or 3).

Their enthusiasm was high. Those students who didn't have any other class scheduled in the following time period, often stayed back to finish their tasks immediately, rather than having to go back to them later. From 2005 assignments were submitted by email (hardcopies were no longer accepted). The weekly Maple worksheets were graded and contributed 10% to the final mark.

In many lab sessions during the course, using Maple for computations meant that it allowed the students to focus on the bigger picture rather than becoming lost and frustrated during the course of long calculations, such as finding the eigenvectors of a 4 by 4 matrix. As this enabled them to complete so many more computations at a much faster pace, they found they were left with time, to be able to look for patterns, in ways that would have been impossible if they were doing their computations by hand, confirming even more so, the efficiency of Maple. Students made strong gains in their conceptual understanding as they used Maple to complete the worksheets.

2. Our observations

Student feedback, along with our own observations, was recorded in tutor diaries in 2003 and 2004. (A student questionnaire was also used, see the next section.) We noticed that both groups were similar: this was supported by the responses to the questionnaire, see Figures 1 and 2. (Also the final exam marks gave an average of 67% for the maths group and 66% for the non-maths group.) The fact that the maths group had experience with Maple (unlike the non-maths group) didn't impact strongly on performance in the Maple labs. From our observations, some characteristics for both groups were evident. We noticed that there were three distinguishable behaviour types as discussed below.

High achievers (about 45% of the students). The students who moved into groups quickly, sat close together, used their first names and student numbers to title their assignments and did much better than the others by getting straight into their work. In general, common behaviour for the high achiever groups is to not engage in other discussions, but to encourage each other to participate in the team's solution of their tasks. They read aloud all the given information and instructions and they attentively listened to each other without interruptions.

Getting all of the facts into perspective helps ensure that everyone is aware of the assumptions and the expectations of the assignment. They respond to (or at least acknowledge) comments made or questions asked by other group members. They do not accept confusion passively. If they do not understand the information given in worksheets or tasks in assignments they openly and curiously discuss these issues and freely ask for assistance, which highlights their eagerness, their need to understand, to learn and to ensure they fully comprehend everything, rather than shrugging anything off. In addition, we found that they easily adapted and quickly became familiar with the terminology used.

It was also apparent that the high achievers choose mostly to work in teams (and those who choose to work individually need more assistance than those students who work in teams).

Middle achievers (about 45% of the students.) Having observed the middle achievers, we noticed that their main characteristic was to split up the work. They didn't all focus their attention on the one same problem at the same time. It was a problem sometimes convincing them that it is much easier to resolve a task when group members work together and to frequently check for agreements. It was hard to get them into the habit of explaining and reasoning or "thinking out loud", to ask and consult with each other and to convince others in the team to do the same. The process of constructing and refining explanations helps everyone to relate to the worksheet and the actual topic in it. They were aware of the time constraints against them, but they were convinced that the best strategy was to split the tasks between them. They realized that, by individually concentrating on different parts of the assignment, they couldn't monitor the groups progress. We tried to encourage them by emphasizing and explaining that it is an important and appropriate practice to ask each other, how they are doing, what they are doing and to "brainstorm" to help the group complete the assignment. If they got stuck, we reviewed and summarized what they had done so far. This process created new opportunities for group members to ask questions, and often it revealed important connections that had been overlooked. Asking questions is the engine that drives mathematical investigations, even though the information being presented often relates to what is already known.

Lowest achievers (about 10% of the students). Groups with the lowest achievements were mostly confused with the Maple commands. Some of them were passionate "Internet billiards" players: they were very competitive but only in billiards tournaments. We found it difficult to get their attention in any way and to help them overcome their difficulties with Maple.

3. Student feedback: a Maple Questionnaire

The histograms in Figure 1 and Figure 2 present data collected in 2004 from a questionnaire (see the Appendix) completed by both groups. The results indicate that the students were finding maths easier to learn with Maple. Most students found that the lab Maple sessions were useful in helping them understand mathematics. The results are generally positive. Most students enjoyed the Maple lab sessions and the quality graphics in Maple. The material presented in the lab sessions was well accepted and many students saw the Maple worksheets as not being too difficult. Results proved to be rewarding once we encouraged the students to get comfortable in communicating with each other in the lab and in creating a team work atmosphere. They found it was a great learning experience.

Some of the comments made in the final "open" question, were far beyond what we would have expected. For example:

- I thought the topic that we covered I understood a lot better.
- I found using Maple very helpful and had a lot of fun overall!
- I would like to have done more Maple as I think I would be able to understand the maths

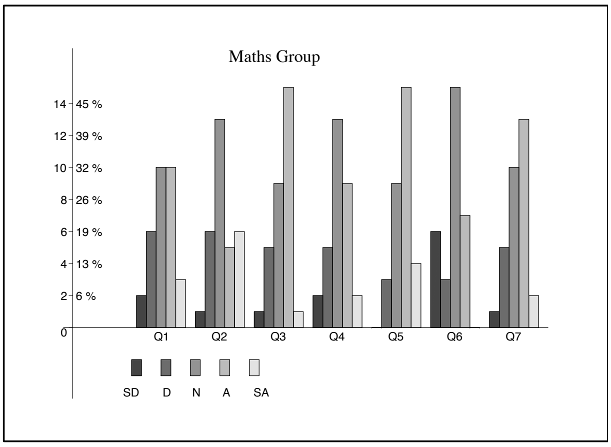

Figure 1: This histogram gives the responses by the maths group to the Maple questionnaire.

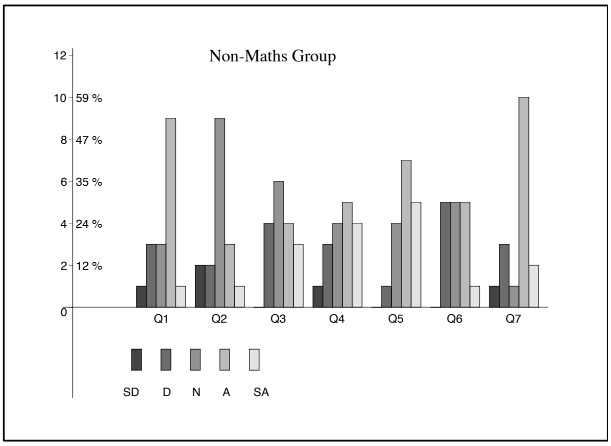

Figure 2: This histogram gives the responses by the non-maths group to the Maple questionnaire.

a lot better this way.
- Interesting and challenging.

In 2003 and 2004, only some of the assignments were submitted electronically, but many students reported that they enjoyed submitting their assignments as Maple files via email. It would save them the trouble of waiting in queues for hard copy printouts. They also enjoyed receiving marked assignments as annotated Maple files, which notably provided comments (using a font with a dark green colour) on what they did wrong. As a result of student feedback, only electronic submission was accepted in 2005.

Some educators express concern that the students will use a tool like Maple to compute the result without any mathematical understanding. However the our observations indicate increased conceptual understanding (as reported in the literature [8]). For example, many students would often ask questions like:

- What if we change this part to . . . ?
- Can I apply this on . . . ?

Such curiosity allows them to explore a number of possibilities to find meaningful patterns. What we have found remarkable is that with these types of questions students connect different topics and extend their logical thinking. Beyond asking questions, students have more opportunities to make connections and to discuss comparisons between different topics.

Plots, graphics and animations encourage the students to look at the material, and study with greater intent, rather than merely trying to memorize how the problems were solved in class. Those lab sessions do away with this traditional approach. Since students use the Maple software to help them step through the boring routines of traditional mathematical learning, they are free to achieve a better conceptual understanding of the material while still gaining good knowledge of the methods of problem-solving. For an e-Learning example, for a given matrix A, students ran animations of arrows for X and AX to illustrate the special cases of the eigenvectors. The end result is a student who really understands the material he or she is working on.

Some students who left feeling somewhat negative, were concerned that if they continued to take lab sessions, they would not actually learn how to do any of the maths and that Maple would simply be one more confusing factor in their learning process. We tried to point out to them, the importance and need, to firstly understand the lecture's topic, and that they would have another chance to do it in the worksheet with Maple, together with (and incorporating) all the features Maple has.

Some students found that one of the better features of Maple worksheets is that they are so interactive. Students were able to customize their Maple worksheets by typing additional text comments and notes on certain topics, which enabled and assisted them with far better understanding. Some of them modified the explanations in the worksheet, or tried to solve the same problems but with commands different from the commands given in the original worksheet. They knew they could always download a fresh copy of the original from BlackBoard if their experiments were unsuccessful.

4. Conclusion

The use of CAS with visualization of mathematical theories and concepts helps students to understand maths. By generating graphs that cannot be easily drawn by hand or which would take a lot of time for students to draw, the students from both groups responded with interest and understanding. We believe that CAS is an excellent tool to accomplish these objectives, for the majority of students. Students in the lab responded very positively to 3-dimensional graphs generated by Maple. Lab sessions in MA911-Maths for Scientists proved that with CAS it is easily possible to close the gap between students who had some experience in Maple and those who had none. As we received quite similar responses from both groups surveyed, it is evident that it doesn't take too long to learn the nuisances of the relevant software, in this case being Maple. We interacted more with the students in those lab sessions than in any other teaching situation and students certainly enjoyed this course using Maple. We believe that "deep learning" (see [8]) was achieved. Despite the argument for a "sensible balance between the use of software and the 'pen and paper' manipulation which is so essential to an understanding of fundamental concepts" (see p. 6 of [5]), we found that innovative use of Maple and cooperative small group work provided a sufficiently rich learning environment that "pen and paper" work could be dispensed with. Following the success of the weekly computer lab sessions described here, from 2005 a similar approach has been introduced in the course preceding MA911.

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Appendix – The Maple questionnaire

```
 1. Seeing how Maple solved problems helped
me to understand methods better.
                          SD     D     N     A     SA
2. The worksheets were too difficult.
                          SD     D     N     A     SA
3. The graphs produced in the Maple sessions were helpful to my
understanding.
                          SD     D     N     A     SA
4. I think I understand the topics we covered with Maple better
than other topics.
                          SD     D     N     A     SA
5. I like being able to sit down and get on with the worksheets
straight away.
                          SD     D     N     A     SA
6. I would have liked to have done more Maple work on other topic
areas.
                          SD     D     N     A     SA
7. Cooperative work helped me learn.
                          SD     D     N     A     SA
SD - Strongly Disagree
 D - Disagree
 N - Neutral
 A - Agree
SA - Strongly Agree

Overall, how did you find the experience of using Maple this
semester?
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Using Diagnostic Assessment For Mathematics In Distance Teaching

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Economies of scale usually suggest that admissions testing has no place within a distance learning context. However, at the University of South Africa (one of the worlds mega-universities) diagnostic assessment, particularly with respect to reading comprehension and quantitative literacy, was found to play an important role in student performance in mathematics[1]. The primary objective of the research described in this paper was to establish whether the introduction of diagnostic assessment for mathematics at UNISA was feasible, and whether students could be placed into meaningful categories according to potential risk. Assessment procedures used elsewhere were taken into account, and specific assessment tools were selected. The main outcomes of the first phase of diagnostic testing (2004) suggest that such assessment is possible within a distance-learning environment, that initial categorisation procedures were reasonably accurate, and that specific aspects of the questions assessing quantitative literacy appear to cluster well together. Further research into the predictive validity of different aspects of the test should yield information that could be useful in test design and in informing aspects of student support.

1. Background

Between 2000 and 2003 staff in the Departments of Mathematics and Linguistics investigated the reading difficulties of students registered for the Mathematics Access Module , and implemented a reading intervention programme. Phase I [1] involved initial investigations with a small group of volunteers, development of a reading test, and analysis of test results after it was administered to all students registered for the module. Phase II [2] saw the development of a 22-week face-to-face intervention programme, involving volunteer students. In order to attempt a meaningful intervention under the constraints of distance learning, Phase III [3] involved the production of a video and a supplementary video workbook. The video 'Read to Learn Maths' focused specifically on the reading skills relevant to mathematics discourse. The video script and the video workbook text were based on the content of the face-to-face.

In all phases various methods were used to obtain data pertaining to reading speed, comprehension levels, and particular reading skills, such as anaphoric referencing, logical relations, aspects of visual literacy, vocabulary (academic and technical) and sequencing.

2. Results obtained

2.1. Phases I and II

The most important result from Phase I was that students with overall reading scores of below 60% were unlikely to pass their mathematics examination. The results suggested that the stronger a student's reading ability, the better his/her chances of performing well in the examination. While high reading scores do not guarantee mathematical success, a low reading

[1]It has subsequently become clear that the results described here, based on research undertaken in 2004, are relevant to other mathematics courses or modules, at other universities, as well. However, for the purposes of this paper, references to 'the module', or to the study of mathematics, relate to the UNISA Mathematics Access Module.

score seems to be a barrier to effective mathematical performance.

In Phase II, pretests showed that students had low reading scores, were reading at well below recommended speeds, with poor comprehension levels. They had not been exposed to print, and did very little reading apart from reading for study purposes. In spite of the 10% increase in the group mean reading score after the intervention, the majority of students would still have been regarded as weak readers. Correlations of performance on a number of variables against examination scores yielded results in line with the findings from Phase I. The most important result was that the strongest correlation (highly significant) was between pretest reading scores and the mathematics examination mark.

2.2. Phase III

Phase II suggested that most of the students would have benefited from a more intensive and longer intervention programme, hence the development of the video. Feedback after the first attempt at using the video/video workbook suggested that a longer intervention programme had little impact, as the students did not, or could not, use the material effectively on their own. This gave impetus to the idea that a diagnostic process was required, which would identify students likely to benefit from additional support.

From the 2003 data, the total reading score correlated more strongly with the examination mark than did other aspects of reading ($r = 0{,}455$; $r^2 = 0{,}207$). This moderate correlation reinforced the notion of a relationship between academic performance in mathematics and the ability to read efficiently. The results cannot show the extent to which weak reading skills may have affected drop out.

3. Additional problems contributing to poor performance

3.1. The problems evident after the use of a project

During 2001, 2002 and 2003 a project[2] was used as an alternative approach to assessment. The project identified several factors that appeared to be undermining student success [4]. It is essential to gauge the extent of such problems as soon as possible. This further highlighted the need for pre-registration diagnostic assessment.

3.1.1. Reading/language problems

The project further demonstrated that limited academic literacy (reading and language proficiency) appeared to be a barrier to mathematical achievement.

3.1.2. Meta-cognitive problems

The project showed that students could not easily assess whether they understood concepts, whether their answers contradicted or supported concepts, and whether their answers made sense. Lack of critical reasoning skills undermines conceptual growth.

3.1.3. General knowledge problems

Poor levels of general knowledge, in particular aspects of quantitative literacy, impeded students' ability to learn from examples used to illustrate mathematical concepts.

4. Aspects of diagnostic assessment

4.1. Theory and principles

Internationally testing in various forms is practised at many insititions, in the form of aptitude and interest tests, proficiency tests, diagnostic tests, etc. (See [8] [9], [10], [11] and [12].) In South Africa, in 2001 a Council for Higher Education discussion document, *A New Academic Policy for Programmes and Qualifications in Higher Education*, proposed an outcomes-based education model in which commitment to learner-centredness was central [5]. In a learner-centred

[2]The project focused on the application of simple mathematical concepts to everyday situations.

approach learning activities should match students' levels of competence. This cannot occur when an institution has no clear sense of learner competence on admission. Students who attempt tertiary-level study with insufficient levels of skill may need more support than can be provided. Significant wastage of resources occurs as a result of incorrect placement in courses, either as a result of misguided student choice or lack of entry-level guidance [6].

The prediction regarding the massification of education in South Africa made by Lewin in 1997 [7] is also relevant: '... as higher education expands towards a mass system, the quality of students will change as a result of this fact alone. They will be a less select group with a wider range of learning needs and new and different support needs' (p. 157). Regarding science, engineering and technology (SET) education he also suggested that ' ... there will be a point at which recruiting further down the pool of applicants will become cost ineffective as a way of increasing the numbers enrolled at tertiary level in SET with any reasonable chance of success' (p. 162). Entry-level assessment can facilitate smoother articulation from school to tertiary study.

In South Africa the government accepted that institutions 'will continue to have the right to determine entry requirements as appropriate beyond the statutory minimum. However, ..., selection criteria should be sensitive to the educational backgrounds of potential students ....' (Government Gazette, 18 April 1997, no. 17944). The disparity of educational opportunities and resulting unreliability of matriculation[3] results as predictors of academic performance, particularly for so-called disadvantaged[4] students, has resulted in the development of various alternative admission routes, where the focus was on academic and quantitative literacy [13]. Examples are the Alternative Admissions Research Project (AARP) tests developed at the University of Cape Town (UCT) in the late 1980s, and the placement tests piloted in 1999 at the Nelson Mandela Metropolitan University (NMMU; formerly the University of Port Elizabeth). (See [8] and [13].)

The AARP tests stretch over most of one day, and consist of an English test and a mathematics comprehension test, which assess potential rather than knowledge; also a mathematics achievement test which based on content knowledge. Test results are analysed and adjustments made to smooth out differences in educational background. The AARP tests are now used in about 20 institutions across South Africa.

NMMU uses ACCUPLACER CPTs (Computerised Placement Tests) obtained from the Educational Training Service (ETS) in the USA. ACCUPLACER assesses English language and reading, and basic algebra and arithmetic. The ACCUPLACER algebra, arithmetic and reading tests are computerised adaptive tests. Written versions (the COMPANION, purchased in booklet form) are also available. There are many entry-level similarities between UNISA and NMMU, such as variable levels of schooling, diversity of language and culture, and registration several years after leaving school.

Assessment and placement testing appears to have been successful in reducing the number of high-risk students admitted to various degree programmes, and has the potential to provide institutions with critical information that can address increased access and throughput rates, and also offer students a learner-centred, developmental approach ([15]; [16]).

A significant factor for distance education is that economies of scale favour large enrolments. However, if the 'the "open door" [of distance learning] is also a "revolving door" through which ill-prepared students pass only to re-emerge as drop-outs' the university no longer serves the purpose for which it was designed [17].

[3] At the end of twelve years of schooling students write a matriculation exam.

[4] The term 'disadvantage' refers to students whose schooling had been inadequate due to repressive political structures. By definition it thus refers mostly to students from African, coloured and Indian communities. The term is normally used 'to describe students whose schooling has been negatively affected by (mis)education or other circumstances such as poor socio-economic and political conditions. Problems often manifest themselves in communication ..., cognitive ..., and subject-specific deficiencies' [14].

4.2. Introducing assessment at UNISA

4.2.1. Purpose

Diagnostic assessment, with the goal of accurately placing students into categories according to potential risk for mathematics, and of developing appropriate support, was approved at UNISA in 2003 and introduced in 2004. This paper focuses on two specific questions that needed to be addressed:

- If testing were to be introduced for mathematics students, what aspects would be tested?
- In particular, what test aspects would have predictive value for mathematical success?

Although language proficiency is critical, it seems to follow, and to be distinct from, reading comprehension [18]. In the UNISA distance-learning context, reading has been shown to play a fundamental role in the construction of meaning from mathematics texts [2]. It was thus important to include reading as one of the test components. From 3.1.2 and 3.1.3 it also seemed necessary to include aspects of quantitative literacy.

4.2.2. Assessment criteria

Several criteria were accepted. Assessment procedures should be based on internationally accepted standards, but could be adapted to suit UNISA requirements. Reliability and validity were important, as was the need for practical and cost effective measures. Tests needed to take into account educational, cultural, and language background, and recognise potential, rather than acquired knowledge and skills.

5. Assessment tools

5.1. Reading

It was decided that where possible, existing tests would be used. This has the advantage that even though norms and standards might differ validity and reliability are established. The location-specific nature, and length, of the AARP English test made it an unsuitable instrument. ACCUPLACER offered a reading comprehension and a language proficiency test. Given the link already established between reading and mathematical performance, t thus seemed that for mathematics the ACCUPLACER Reading Comprehension (ARC) was possibly more appropriate than the English Language Proficiency test.

Although computerised testing for all potential students at UNISA is not yet feasible this would be the most efficient and effective means of carrying out diagnostic assessment, particularly within an open and distance learning environment. This provided further motivation for using the ARC, even though initially the paper-and-pencil version was used.

At NMMU these tests were used in their American format, but empirical item bias analyses are continually undertaken. Initially NMMU developed their own standards on the tests, and did not use the USA norms or standards. UNISA also adapted the standards used in interpreting ARC scores. In spite of possible problems arising from ambiguities and culture-specific items (seven questions), it appeared that the possible bias would not detract from the items' potential to assess students' ability to construct meaning. In the interests of validity and reliability it was agreed to leave the items in their original format for the first round of testing, and delete or adapt items later if necessary.

Table 1 shows the different aspects of reading assessed in the ARC.

5.2. Developing a test for quantitative reasoning

The level of conceptual knowledge of the mathematics module initially linked to the diagnostic assessment process is below the level of knowledge tested in other available mathematical. The AARP Mathematics Comprehension Test does not draw on previously taught mathematics, but is a dynamic test that primarily assesses reasoning potential, using limited mathematics that students can realistically be expected to know, emphasising understanding

Table 1: Aspects of reading tested in the ARC

Aspect tested***	Number of occurrences*	Questions
Causal relations	2	2, 4
Contrastive relations	4	3, 7, 8, 23
Recognition of sequence	1	5
Interpretation of implied or inferred information	9	10, 14, 15, 17, 20, 27, 32, 33, 35
Comprehension/interpretation of factual information; detailed/ general.	5	9, 11, 23, 28, 29
Identification of main/subsidiary ideas	12	13, 16, 18, 19, 21, 22, 24, 25, 26, 30, 31, 34
Academic vocabulary	Many	All questions except 5, 26
Number sense	1	23
Recognition of contradiction, inconsistency	1	12
Substantiation	2	1, 6

*The ARC consists of 35 multiple-choice questions. It is clear that there is overlap between categories.

and reasoning above recall. Although appropriate, as in the case of the English test, the AARP tests were not an option for UNISA. It was thus necessary to design a quantitative reasoning test (called the Basic Arithmetic Test (BAT)) that would similarly determine whether sufficient understanding of rudimentary concepts has been achieved. Test design drew upon common misunderstandings that have been seen as significant barriers to understanding, and relevant aspects of similar tests.

Questions categories are shown in Table 2. Questions were classified as easy, moderately easy, moderately difficult, and difficult, and weighted accordingly.

Table 2: Aspects of quantitative reasoning in the BAT

Aspect tested	Number of occurrences*	Questions
Simple arithmetic operations (whole numbers) (SAW)	19	36, 37, 38, 39, 40, 46, 50, 51, 54, 57, 58, 59, 60, 63, 65, 66, 67, 68, 70
Simple arithmetic operations (fractions/decimals) (SAFD)	12	41, 42, 43, 44, 45, 48, 49, 52, 53, 62, 64, 69
Pattern recognition (Patt)	3	46, 54, 61
Number sense (NS)	5	46, 47, 48, 55, 62
Conversion of units (Units)	3	54, 60, 70
Academic vocabulary in relation to mathematics (e.g. satisfy) (Ac Voc)	3	46, 47, 55
Technical vocabulary (e.g. fraction, ratio)	4	47, 63, 64, 66
Comparison (e.g. bigger than, earlier, more than half, double) (Comp)	10	47, 48, 49, 50, 56, 57, 60, 61, 67, 68
Time (how long) (TL)	3	48, 49, 62
Time (when) (TW)	2	60, 70
‘Translation’ from words to mathematical statements (Trans)	17	48, 49, 50, 51, 52, 53, 54, 56, 57, 59, 60, 62, 65, 67, 68, 69, 70
Recognition of insufficient or redundant information (IRI)	4	56, 68, 69, 70
Learning from explanation (Exp)	2	64, 66
Spatial awareness (SA)	4	58, 61, 65
Insight	14	48, 49, 51, 52, 54, 56, 57, 58, 60, 61, 63, 67, 69, 70

* The BAT consists of 35 multiple-choice questions. A glance at the categories shows again that there is some overlap between categories.

5.3. Grading criteria

Institutions use many different scores (including school performance) and apply regression formulae, classification functions and other measures, based on research, to determine risk category. Criteria needed to be determined to categorise UNISA students considered likely to be successful with no additional assistance (Category 53)[5], likely to be successful provided they had support (Category 52), and those who were unlikely to be successful without assistance beyond that available at the university (Category 51). See Table 3. Initial criteria for classification were based on the ACCUPLACER guidelines and empirical results following Phases I, II and III of the reading intervention (for the ARC), and similar procedures for the BAT. An explanation regarding choice of category boundaries is available on request.

Table 3: Diagnostic assessment categories

Category	Reading Comprehension (weighted) (20 – 120)		Basic Arithmetic (weighted) (0 – 69)
51	score < 70	**OR**	score < 46
53	score ≥ 103	**AND**	score ≥ 59
52	All other scores		

Ongoing research will determine whether adjustments need to be made to question weightings or the categorisation process.

6. Results

6.1. ARC results

The ARC results showed good item-test correlations, with only five questions below 0,5, and with low item-test correlations in only three of the seven items that had been regarded as potentially problematic.

6.2. BAT results

Low (below 0,30) item-test correlations were obtained across all BAT question categories (i.e. easy to difficult), since several BAT items tested unrelated aspects: for example, there is no conceptual relation between being able to add two three-digit whole numbers, and being able to determine the number of dots in the 50th block in a set of nested blocks, given a picture showing the number of dots in the first three blocks.

6.3. Consolidation of results

Once the results were consolidated, students in the two groups (January and June) were distributed over the three different categories. Comparison of the January and June results[6] revealed that in spite of a few months exposure to the mathematics study material (and in many cases also to a preparatory English module), allocation of students to the different categories yielded similar results. See Table 4.

Table 4: Students according to risk categories

Category	**January** (n = 326)	% of total	**June** (n = 834)	% of total	**Total** (n = 1 160)	% of total
51	10	3	35	4	45	4
52	93	29	176	21	269	23
53	223	68	623	75	846	73

6.3.1. The impact of changing category boundaries

Some adjustments were made in the categorisation process to try to distribute the students more evenly in the different categories. However, the pattern (as in Table 4) remains the same. When the ARC was excluded and categorisation based on the BAT alone, there were surprising results (see Table 5). The table shows that whereas it had been thought that the ARC might

[5]Details of the categories (51, 52 and 53) and the implications of this classification were explained to students in the tutorial letter they received when they registered for the assessment.

[6]For a number of administrative reasons testing took place in two stages. Students who registered early were tested in January, the balance in June.

skew the categorisation adversely in relation to the high-risk students, in fact the converse is true, i.e. the reading score may have played a greater role in determining the categories into which students were placed.

Table 5: Exam results of students in the three risk categories *including* and *excluding* ARC

Category	Including ARC No. of students (n = 463) (% of total in cat.)	Mean exam score	Excluding ARC No. of students (n = 463) (% of total in cat.)	Mean exam score
51	332 (72%)	30%	136 (29%)	26%
52	106 (23%)	45%	189 (41%)	32%
53	25 (5%)	57%	138 (30%)	48%

6.3.2. Cluster analysis

Several clusters of questions emerge. See Figure 1. The clusters A, B, C, D E and F are indicated. Cluster D is the most interesting, particularly since it contains C and overlaps with most of B and E and half of F.

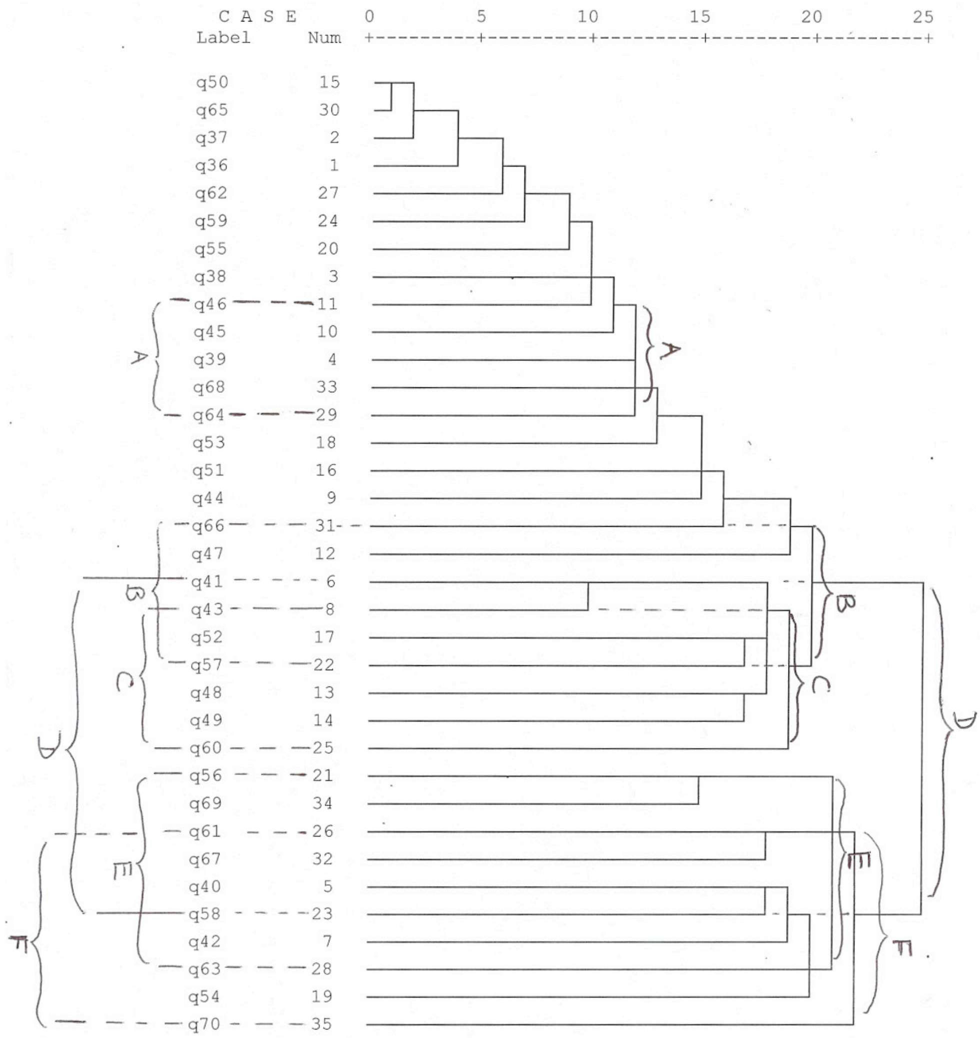

Figure 1: Cluster analysis of BAT questions

Details of the cluster D (obtained from Figure 1 and Table 2) are given in Table 6. Over time further analysis of the predictive validity of the items in Cluster D will yield useful information regarding specific items related to mathematical performance.

Table 6: Analysis of Custer D

Questions in Cluster D			**Item-test correlation** (to two dec. places)	**Number and nature of aspects assessed** Common aspects: SAFD (6), trans (8), insight (9), comp (7)	
Overlapping/ included clusters					
B (with 66, 47)		41	0,64	1	SAW
	C	43	0,63	1	SAW
		52	0,74	3	SAW, trans, insight
		57	0,66	4	SAW, comp, insight, trans
		48	0,56	6	SAW, NS, comp, insight, TL, trans
		49	0,50	5	SAFD, comp, insight, TL, trans
		60	0,61	6	Insight, units, SAW, comp, TW, trans
E (with 42, 64)		56	0,63	4	Insight, IRI, comp, trans
		69	0,47	4	SAFD, IRI, trans, insight
	F (with 42, 63, 54, 70)	61	0,24	4	Insight, patt, comp, SA
		67	0,22	4	Insight, SAW, comp, trans
		40	0,40	1	SAW
		58	0,73	3	Insight, SAW, exp

7. Conclusion

Low-risk students are considered likely to cope without additional support. High-risk students need careful counseling. Provision of adequate support (dependent on staff and resource allocation) is essential for medium and high-risk students.

Ongoing research, both quantitative (further statistical analysis into reliability and predictive validity of individual items and clusters of items, in the ARC and BAT) and qualitative (investigating student perception of diagnostic assessment, and potential problem areas relating to support) is critical. Further analysis of the relationship between specific clusters and mathematical performance will provide valuable information.

The costs, benefits, advantages and disadvantages of instituting diagnostic assessment for mathematics students need to be thoroughly investigated. In the interests of equity and access, throughput and success, it is crucial that students are given appropriate career guidance, and diagnostic assessment could serve as one of the tools in this process.

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[7] AL refers to reading in a second, or alternative, language. In this thesis the term L2 has been used in the same way.

Cooperative learning in Higher Education in South Africa

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Each student is unique and one of a kind. Each student has his own personality and has different ways of learning; progress differently and reach different degrees of success with their methods of study. Learning of Mathematics is a complex matter.

Cooperative learning provides for active involvement of the student in the learning process; it provides the opportunity for the student to accept responsibility for his own learning success as well as that of his fellow students and it provides the opportunity for mastering social skills which are a necessity for our modern, complex and integrated society.

Change starts in the classroom. In order to adapt to the new demands in Higher Education in South Africa, a new way of teaching and learning must come into play. This study proposes cooperative learning as the best suited methodology of teaching Mathematics in a multi-cultured environment. To address the needs of the modernizing economy of South Africa, Higher Education must produce graduates that can take responsibility for their own life-long learning, that is socially able to function in a multi-cultured environment and that can work productively and cooperatively towards a common goal. Cooperative learning can give the students these skills.

1. Introduction

Each student is unique and one of a kind. Each student has his own personality and has different ways of learning; progress differently and reach different degrees of success with their methods of study. Learning of Mathematics is a complex matter. No two answers will correspond if inquiring into the method in which Mathematics is mastered. The same is also true if inquiring into the teaching strategies to be followed in order to acquire success in teaching this subject.

There is a chronic mismatch between the output of higher education and the needs of the modernizing economy of South Africa (Geldenhuys et al., 2004:31). In particular, there is a shortage of highly trained graduates in fields such as science, engineering, technology and commerce (largely as a result of discriminatory practices that have limited the access of black and female students), and this has been detrimental to social and economic development. The responsiveness of the higher education system in South Africa to present and future social and economic needs must improve, including labor market trends and opportunities, the relations between education and work, and in particular, the curricular and methodological changes that flow from the information revolution, the implications for knowledge production and the types of skills and capabilities required to apply or develop the new technologies (Geldenhuys et al., 2004:31).

Change starts in the classroom. In order to adapt to the new demands in Higher Education in South Africa, a new way of teaching and learning must come into play. This study proposes cooperative learning as the best suited methodology of teaching Mathematics in a multi-cultured environment. To address the needs of the modernizing economy of South Africa, Higher Education must produce graduates that can take responsibility for their own life-long learning, that is socially able to function in a multi-cultured environment and that can work productively and cooperatively towards a common goal. Cooperative learning can give the students these skills.

Cooperative learning provides for active involvement of the student in the learning process; it provides the opportunity for the student to accept responsibility for his own learning success as well as that of his fellow students and it provides the opportunity for mastering social skills which are a necessity for our modern, complex and integrated society.

This study concentrates on co-operative learning, since the point of view is held that it provides a comprehensive framework within which effective learning can be achieved in a multi-cultured environment.

2. Higher Education in South Africa

The Minister of Education, Me Naledi Pandor, (Cape Town: 2004), at her first address in office, outlines the key challenge that is facing education in South Africa today:" Let us share a passion for quality education for all." Amongst others, she listed the teaching of Mathematics as one of the priorities in order to face the challenge of equipping students and pupils with necessary skills. Mathematics, Science and Technology have be promoted on all levels of education and no effort should be shared in order to teach the necessary skills in these subjects.

The South African profile of student distribution and characteristics changed dramatically as can be seen in the following table:

Shifts in enrolments, 1993–1999	%
Increase in African students	80%
Decrease in African students in historical black institutions	9%
Increase in African students in historical black technikons	138%
Increase in African students in historical white English-medium universities	100%
Increase in African students in historical white Afrikaans-medium universities	1120%
Increase in African students in historical white technikons	490%

According to Kapp (2004:1.2) these changes have the following consequences: it changed the social relations on campuses between staff and students; between academia and administrators; and between institutions and national government. The result is a new administrative and managerial architecture that altered the physical organization and social character of the institutions.

Du Prè (2004:22) states that in order to adapt to these changes, the core academic functions must be imbedded in the relevance in knowledge and applicability of skills. The focus shifts from faculty productivity to student productivity; from faculty disciplinary interests to what students need to learn (du Prè 2004:4). These functions must lead to greater learning opportunities, work-integrated learning and "Just-in-time" education. Re-skilling, up-skilling and multi-skilling must take place (upgrading of knowledge and skills to graduates where and when they need it.)

3. Cooperative learning

"One day, in a land far away, ruled a wise king over his people. Everything in this land was colorful and the rest of the world called this people the rainbow nation. However, there was one problem. The people of this rainbow nation were hungry for knowledge and skills. A delegation was sent to the king to inform him about their need. The wise king told the delegation that a new system will be developing that will address their thirst and hunger. He demanded the nation to gather in two weeks in front of his castle. On that day and the days that will follow, all will be fed at the castle's gates and no-one will be hungry anymore.

The word spread like wildfire and on the expected day, the whole rainbow nation was gathering in front of the castle. The king appeared and instructed his soldiers to open the gates and to bring out the knowledge and skills soup. Under loud applause, the soldiers brought large bowls of soup and put it down in front of the masses. All was given big spoons with long handles to dig deep into the bowls. "Enjoy!" said the king and went back into his castle.

The rainbow nation rushed forward and started digging their spoons into the big bowls of soups. It is just then when the problems started. Because of the long handles of the spoons, they could not reach their mouths. As soon as they brought the spoons to their mouths, all the soup spilled on the ground. The people became more and more frustrated and before long, chaos broke out. "We demand to see the king again! This is not helping us!" The king was called. The people confronted him:" How can we be fed? The system is not working! Here is all the skills and knowledge we will ever need, but everything is falling on the ground!" The king looked at them and replied: "You will only be fed if you work together. Feed one another..."

Cooperative learning is a structured, systematic instructional strategy in which small groups of students work together toward a common goal. (Cooper, Robinson & McKinney, 1990:1) This grouping is usually heterogeneous in terms of race, sex, prior achievement and other characteristics deemed appropriate by the instructor. This promotes attitudes towards persons of different backgrounds that are an issue of great concern in dealing with the student diversity that challenges the Higher Education environment in South Africa.

Astin (1992) completed a study of over 200 colleges and universities to assess the factors that could make a difference in undergraduate education. He examined nearly 200 environmental variables, including curriculum factors. He concluded that student-student interaction and student-teacher interaction were by far the best predictors of positive student cognitive and attitudinal changes in the undergraduate experiences (Astin, 1992:4). Based on this and related research, Astin has pressed for greater use of cooperative learning in college instruction.

Cooper and Robinson (2005:2, 3) published a status report on small group instruction in Science, Mathematics, Engineering and Technology (SMET) disciplines. They concluded that small-group instruction at the college level has a positive overall effect on learning outcomes. They also reported on the research base for small-group instruction in SMET classes. The field of research on small-group instruction in SMET disciplines in higher education can be characterized as promising, but relatively immature. The number of studies done on this is relatively small and the quality of these studies is often less than rigorous.

It is therefore clear that Cooperative learning can form the correct environment in which optimum learning can take place in a multi-cultural and rapid changing environment. In this study, cooperative learning was applied in the Mathematics 1 group of engineering students at a University of Technology; in specific team-assisted individualization (TAI) as a method of cooperative learning. This method of Cooperative Learning is the best applicable method for Mathematics, since it allows the student to work individually as well as in groups. Slavin and Maiden (Davidson, 1990:21) developed this method and emphasize the applicability of TAI in Mathematics:" TAI would provide a means of combining the motivational power and peer assistance of cooperative learning with an individual instructional program capable of giving all students material appropriate to their levels of skill in Mathematics and allowing them to proceed through these materials at their own rates."

Slavin and Maiden (Sharan, 1994:24) did seven field experiments to determine the effect of TAI on Mathematics-students with respect to their academic results, attitudes and behavior. Academic results were tested in six of the seven experiments. TAI groups did throughout better than the control groups; especially in concepts and applications. The effects of TAI were the same on the top achievers as on the lower achievers. Positive results were also obtained with respect to classroom behavior, self confidence in Mathematics and racial relationships. Buys (1998:167) conducted a research project on TAI by applying the method in the Mathematics class at a Technical College. Substantial improvement in the academic results was found. Positive feedback from the students as well as the lecturers showed that TAI can be successfully implemented with positive results.

4. Process

The students attend Mathematics 1 classes five hours per week, of which one hour is theory that covers all the theory for the week; three hours practical classes where the students do worksheets in groups with champions; and one hour tutor class where the students do tutorials based on all the worksheets.

A pre-test is written in the first practical class of the week. The result of the pre-test determines the forming of the groups and champions are identified based on the outcome of the pre-test. Each group must have a champion. The champions are the students with the best results in the pre-test. They form the "leader" position in the group and must ensure that the group stays productive and focused. Champions can be same person every week or new champions can be identified.

Students divide into groups of 4 or 5 with mixed results. Regrouping will take place regularly. The syllabus is divided into units. Pre-test is written at the beginning of a unit. This pre-test determine the level of knowledge and skill of the student with respect to the unit. Specific worksheets must be completed according to the outcome of the pre-test. A worksheet consists of the following:

- Theory of the sub-unit
- Examples
- Exercises

Steps in TAI:

- All students attend the theory class for the unit.
- All students write the pre-test.
- The result of the pre-test determines individually which worksheets must be worked out.
- Champions are identified and the groups are formed.
- Students work individually on their worksheets; assistance will be provided from within the group. The lecturer facilitates the process and helps if the group cannot solve a problem.
- An intermediate-test can be written in the group. The champion marks this test and provides the result to the lecturer. This test is not compulsory, but can be used for self- or peer-assessment.
- As soon as all worksheets are completed, the worksheet for the tutorial class will be given. This worksheet can be done in class or in the tutorial class. Worksheets not completed in class, must be completed at home.
- The tutorial class is scheduled at the end of each unit. A worksheet, consisting of all the work in the unit, must be completed individually and not in groups. The tutor and champions assist the students.
- A post-test is written in the tutorial class as soon as the student thinks that he is ready and all worksheets are completed.

5. Results

5.1. Questionnaire

The attached questionnaire was filled in by the students. The questionnaire was filled in anonymously and there was no opportunity given for the students to discuss the questions. This ensured that each student had the opportunity to express his/her individual view. The responses are qualitative of nature and cannot be measured precisely. The results are therefore given in percentages. 60 students completed the questionnaire. The following results were obtained:

	%	
Question	Yes	No
I learn more in the group than I would have learnt on my own.	75	25
I enjoy working in a group.	83	17
The group motivates me to do my share of the work.	82	18
The group helps me understand the studying material better.	83	17
It is fun working in a group.	67	33
In a group, I get the benefit of everyone's ideas.	82	18
When I have problems, I get help from the group.	90	10
Group work gives me the opportunity to talk and discuss course content.	95	5
I prefer to learn Mathematics in a group.	77	23
I experience group learning as successful.	80	20

5.2. Examination results

There was no significant difference in pass rate in the examination when compared with previous examinations.

6. Conclusion and recommendations

The students experienced group work overall as positive. They agreed that it gave them opportunity to discuss course content and in doing so, got the benefit of everyone's ideas. 77% prefer to learn Mathematics in a group and 80% experienced group learning as successful.

The fact that there was no significant difference in the pass rate of the students, must not take away the value from applying this method of instruction to Mathematics. The students were very actively involved in the learning process and valuable interaction skills were developed.

The following recommendations can be made for implementing cooperative learning in the classroom:

1. Each student must be fully aware of the method of teaching and they must be well informed about the assessment opportunities. In doing so, the students will be aware that they must take full responsibility of their own learning.

2. There must be strictly adhered to the work scheme as it was set at the beginning of the semester. Deviations from this can result in not having enough time to finish the syllabi.

3. Success within a group must be assessed on a regularly basis through peer- and self-assessment.

4. All course material must be available on time.

5. All the above can be summarized in one word: planning. The lecturer must know what the required outcomes must be, how the students will reach the outcomes and when the students reach these outcomes.

Further studies are now conducted to bring in technology into the whole process. Since most students have cell phones, it will be used to have quick responses when doing assessment. This will happen through blue-tooth technology.

7. Summary

Teaching is sometimes characterized by a high degree of teacher control and student passivity and powerlessness. This creates an environment of zero-involvement and zero-responsibility and zero interaction with other students. In cooperative learning, most of the responsibility for learning is explicitly placed on the student.

To address the needs of the modernizing economy of South Africa, Higher Education must produce graduates that can take responsibility for their own life-long learning, allowing them the ability to function in a multi-cultural environment where they can work productively and cooperatively towards a common goal. Cooperative learning can provide students with these skills.

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Mathematics curriculum changes in high technology environments

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This presentation reports on the work of the Belgian Ministry "Exploot" project [10]. In this project a large set of modules was developed for efficient teaching of mathematics using symbolic software.

1. Mathematics in a technology environment

Modern technology is influencing mathematics in many ways.

1. Technology has expanded the scope of mathematics;
2. It encourages experimenting and browsing by students;
3. It is well-adapted to the problem - experiment - conjecture - proof paradigm;
4. It executes exact symbolic computations and automated proofs;
5. It offers better approximate computations, not necessarily limited to numerical problems.

1. We see a widening gap between the volume of mathematics taught in schools, and the available knowledge. Our students do not seem yet to realize to what extent their learning environment and goals have changed. Part of this inertia can be attributed to the teachers: such changes take an entire generation to implement. To attain advanced goals in the limited time allocated for learning, one has to find the shortest path to teach powerful applications from as few prerequisites as possible.
2. Researchers, teachers and students can obtain free information and content over the internet from MathWorld [2], dedicated internet sites (e.g. [3]), functions databases [4], electronic publications and libraries. Understanding of this information requires a basic knowledge of mathematics.
3. Teaching of mathematics can exploit the eagerness for experimenting in students [10]. This is also the way mathematics has evolved over the centuries, while experimenting with paper and pencil used to take lots of time, computer experiments can be done in a few seconds, and a large number of experiments leads to reasonable conjectures.
4. The first technologies were limited to numeric computing. Modern software systems can now handle symbolic expressions in an exact way, and have control mechanisms that allow approximate computations to arbitrary precision. These are called convergence theorems in theoretical mathematics. Over the years, traditional mathematics and technology have become more similar.
5. There is a huge gap between the simplicity of models and the complexity of many real-world phenomena. While a classic paper and pencil approach is limited to simple applications, technology makes complex phenomena tractable. We also have to get rid of

the belief that everything is deterministic, since reality is more stochastic. Technology makes this uncertainty tractable through simulations, numerical analysis and interval arithmetic. This is used in fields where no exact mathematical model is yet available. The most powerful computer infrastructure in the world is used to model everyday needs such as weather forecasts.

2. Effects on the curriculum

Symbolic computer software (CAS) originated around 1980. A report commissioned by the European mathematical community on its history is in [1]. CAS and graphics calculators only became popular in the late eighties and nineties, and their introduction in education has been uneven and slow in some countries. In the meantime, innovation has occurred at an increasing speed. In spite of the rapid evolution of technology, mathematics curricula have little changed over the last twenty-five years. Most curricula have even been around with little changes since the seventies, after the dispute over introducing ”new” mathematics was decided. Today society is questioning the relevance of mathematics education and the popularity of mathematics is diminishing as technology is taking over.

To fill the gap it is not sufficient to teach a technology manual. A specific technology manual will be obsolete in a couple of years as new versions appear, or as new software is adopted.

Technology use should not be limited to generate drill questions for testing and grading. This way of using technology falls short of attainable objectives both in mathematics and in technology. If technology is used in teaching, it is natural to use it in testing as well. But it should not be used in testing if it is absent in the learning process. All parties concerned, teachers, students and the society, are calling for a new curriculum and typical content that is better adapted to the new technology.

Introduction of technology should start early after a playful introduction to counting, reasoning and geometry in primary schools. This is the mathematical content for everyone, even people that may never use any technology. Basic activities involve reasoning and some computing (exact, approximate or guessing) where a calculator or computer is not always welcome. Knowledge of plane geometry is a good basis for an introduction of technology since it fits on a computer screen and offers the composite structure of objects: point, segment, line, angle, broken line, polygon, circle, etc.. Excellent interactive material is available [5, 6]. It prepares learners for more advanced commands in current comprehensive symbolic softwares such as MatLab, Maple or Mathematica [7].

3. A technology driven curriculum

A more advanced mathematics curriculum can concentrate on features that are imbedded in every symbolic technology: e.g.

- representations in the plane (on a computer screen) instead of numeric tables,
- fast graphing and audiovisual capabilities (including web delivery),
- 3-dimensional viewing capabilities and animation,
- the usefulness of constructive proofs,
- fast simulations that mimic the ”for all” clause in mathematics,
- frequent switching between continuous and discrete phenomena,
- list or vector handling,

and so on ... For instance, a calculus curriculum can be organized using the following principles linked to technology capabilities:

- the ability to make computations with arbitrary precision,
  - e.g. exploit continuity at a point to locate zeroes or attain limits;

- the growth of functions (including big-O and little-o notations):
  - no growth as seen with horizontal tangents (as in Rolle's theorem),
  - the order of contact between graphs,
  - estimating growth by computing derivatives;
- fitting data by well-chosen functions, depending on the nature of the data:
  - polynomial approximation near a point,
  - Fourier approximations for nearly periodic data,
  - bell shapes or wavelets for hump-like data;
- the speed of approximations and the fastness of algorithms in general;
- modeling of real world problems discussing the quality of the model;
- trying to do analysis on data for which the prescribed function is not yet known;
- the ever-present interplay between continuous and discrete mathematics;
- uniform continuity on closed intervals and uniform limits on closed intervals [8], and even finer convergence criteria whenever needed.

Modules [13] for delivering mathematics courses following these principles were developed in the author's project [10]. These modules now cover the full undergraduate mathematics curriculum.

Teaching has to offer the fastest approach to learn new techniques, bypassing old habits or redundant material. For each topic we have to ask questions such as "is it still relevant?" and "what is the fastest way to introduce a topic from material the targeted student group already knows?". A new curriculum will be overloaded if it follows the traditional linear structure, delaying the introduction of new concepts until all more general concepts (or even all simpler concepts) are handled first. A mathematical curriculum looks like a directed graph mapping the prerequisite - objective order. In these graphs the fastest way to attain a goal is selected by the teacher. This requires from the teacher a deep understanding of mathematics and the logical dependency between topics. Example [13]: for a learner who is familiar with elementary geometry in the plane (e.g. from using a geometry package) it would take a short period (less than ten hours over a couple of weeks) of guided experiments to grasp the concept of discrete Fourier transform and its applications in probability and signal filtering [9]. Once this transform is understood, the learner has access to many more state of the art applications.

4. The role of experiments

Contemporary curricula and study material are all static, in spite of the inclusion of beautifully typeset formulas (LaTeX) or brilliant graphics. Delivery of course material over web pages does not change this fact since interactivity is limited for security reasons or copyright. Excellent content can be created on the web using Java programming or WebMathematica [7], but limits the user experiments to specific tasks by fencing off undesired inputs. In a flexible learning environment we should allow students to have an impact on the study material and to help them develop new simulations to convince themselves and their peers.

It is surprising that most students learn advanced technology commands faster than their teachers or parents. But experimenting should not be limited to predefined commands. We can offer templates to encourage structured use of commands. Some of these can be built in an interactive way with output constantly varying according to a change in parameters (e.g. using the GUIKit interface in Mathematica [7]). To allow further experimenting we should not try to hide content programming from students. They should have access to the full functionality of the technology. This is no longer a problem since most software manufacturers have full licenses available at a special price for students or for class use.

As an example, we present a module in Fourier theory. A selection of typical outputs is in the Appendix. Starting from elementary geometry and the Parker Spirograph toy we can give an introduction to Fourier analysis and its applications in a couple of hours. This uses little trigonometry or complex numbers [9] and all computing and graphics rendering is done in the background by the computer.

5. The role of history

The teacher has to concentrate on historical developments that are considered milestones in mathematics. The relevance of a subject in mathematics can be graded by its stability over a very long period. It turns out that essential content is linked to great names in the history of mathematics. Naming of concepts is less important, and the curriculum should refrain from trying to introduce new names for each slight difference between concepts.

Newton's notion of function is a good example of a concept that has been stable over time. Historically, approximations were always done by polynomials. Functions that do not allow polynomial approximations tend to exhibit pathologies that cannot be handled by calculus methods. Technology handles polynomial expressions as fast as it handles linear expressions. Fitting well-chosen functions to data is a basic operation in every technology [13] and the basic idea in modeling and simulations.

6. The role of programming

Efficient teaching may use some features or commands that are not built-in in the software package, and the teacher may have to add a few short program modules. This is easy in an expression-based software that is close to the language of mathematics [7]. Our documents [10,13] include programming templates for all visual outputs to help teachers.

A modern curriculum should link mathematical ideas to algorithms and programming. This is now less difficult than used to be the case with early procedural programming languages on typed data. Over the past decades we have seen a convergence of programming paradigms with the underlying mathematics in symbolic mathematical technology. Similarities can be used to integrate mathematics and basic computer science curricula. Mathematics is a rule-based system in which computing function values (lambda-calculus) is possible whenever the variable matches the required pattern. Each individual familiar with symbolic mathematics technology can use his knowledge as a basis for a programming course [11].

7. Control

A modern curriculum should teach the student a control attitude and warn for simplifications or shortcomings in technology. This is not different from learning control in classical mathematics teaching, as shown in [12]. This does not mean that a curriculum has to include a list of warnings against "errors" in the computations. Too many texts focus only on warnings of the kind "if you push this button after that one for such value, you may get a wrong answer". Since performance is better in advanced symbolic systems, it takes less effort to warn users.

Nevertheless, some of the technology side-effects are imbedded in a hard way because of the finite nature of all computing engines and users should permanently be aware of this [13]. It comes too late if control is only taught in a numerical methods course at the end of the curriculum.

8. Conclusions

A good mathematics curriculum prepares for a life long schooling and applicability of mathematics. No learner is ever going to use mathematics without technology, and therefore it is mandatory to teach mathematics using symbolic software. Since we cannot predict what kind of technology will be available for the next generation (in twenty-five years from now) we can only develop in our students an attitude of acceptance and eagerness for technology improvements.

Changes in the curriculum will take effect after a generation of teachers has been trained using this philosophy. This is where innovation has to start.

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[13] Sample Mathematica notebooks (special CD prepared for the Kingfisher meeting)

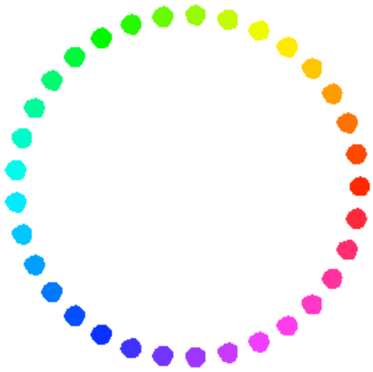
- Drawings.nb: repeated mapping of a matrix to a figure in the plane.
- FitPeriodic.nb: exhibits basic functions to be used for fitting given data sets.
- DFTBuildup.nb: introduction to DFT, including the proof of its inverse transform.
- SpiroFourier.nb: introducing Fourier series from Spirograph drawings (see Appendix).
- Growth.nb: studying order of contact between function plots, and limits at infinity.
- GraphicalInput.nb: inputting data by hand.
- LimitE.nb: controlling numerical instability in slow limits.

[copies of these can be obtained by mailing to the author]

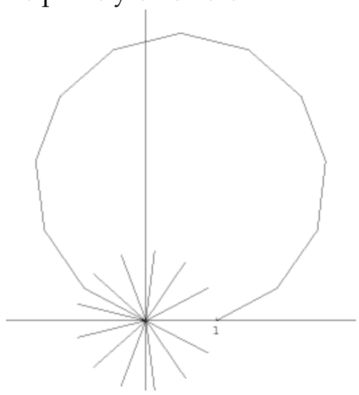
Appendix

Here we give a series of outputs generated by the spirograph function in the SpiroFourier document [see also 9]:

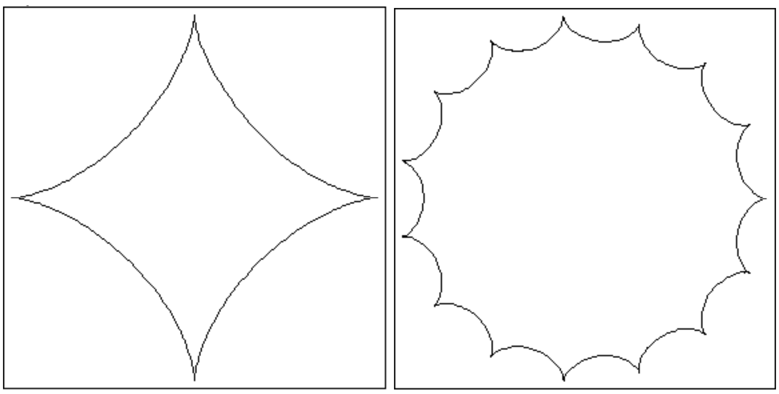
Directions in the plane, and roots of unity



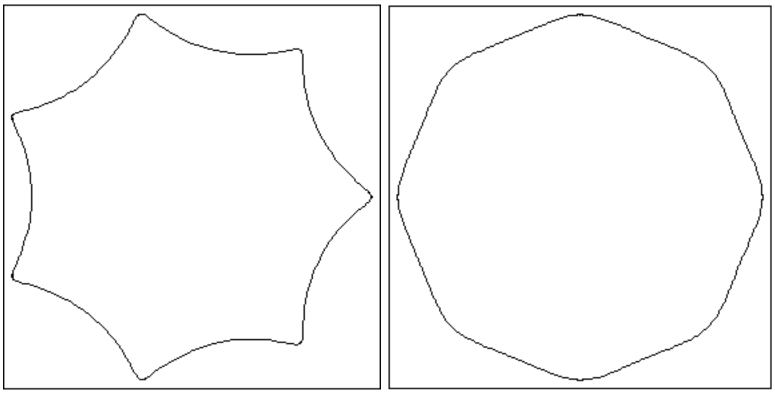
Proof that adding directions in the plane yields zero



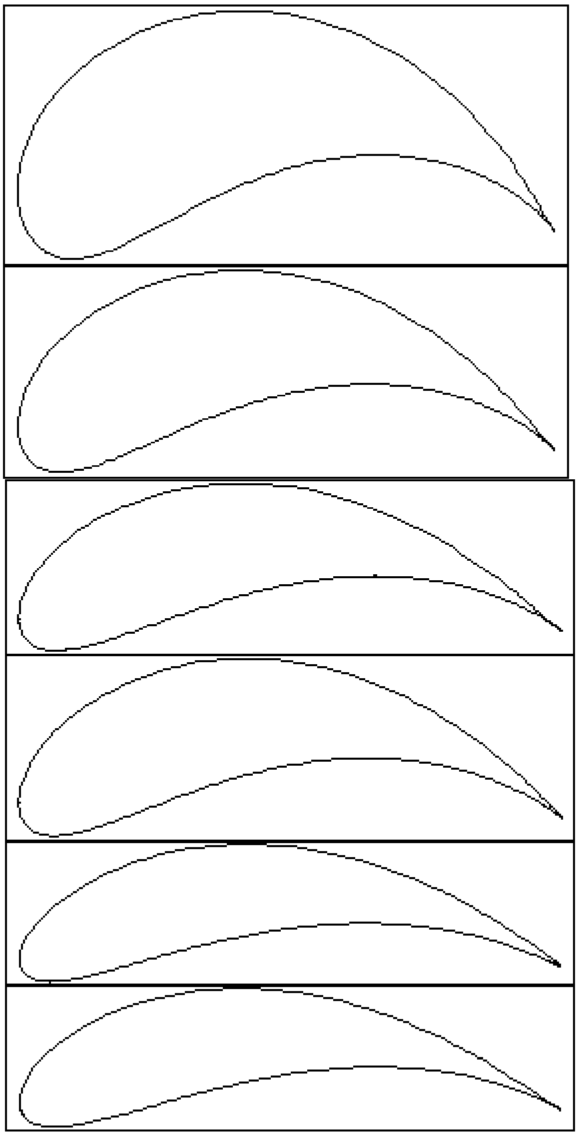
Cycloids



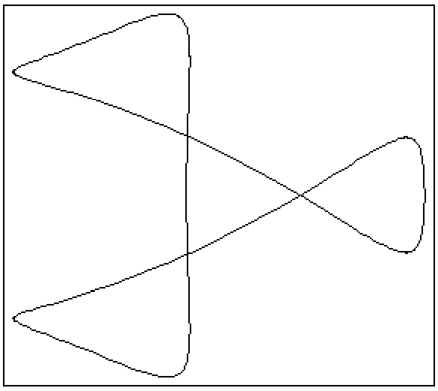
Hypocycloids



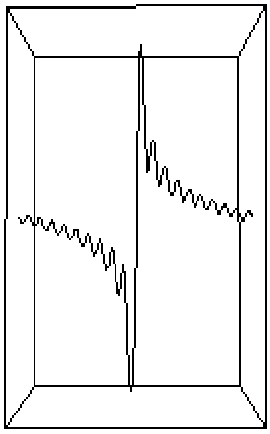
Airfoils (can be animated)



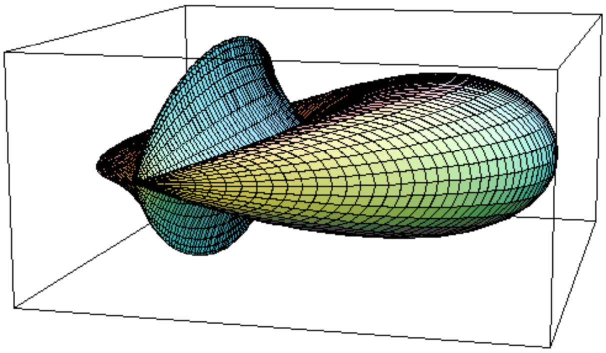
Higher order images of the circle



Fourier series expansions (a 3D image, can be animated)



Generalized spheres (a 3D image, can be animated)



Spicing up the classroom: how one subject turned the tide

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At the best of times the study of statistics for many students has had a reputation for being boring, irrelevant and just plain too difficult. Sometimes deservedly so. But it doesn't have to be this way since there are many exciting opportunities to demonstrate its power and application by using situations that are both challenging and stimulating. This paper looks at some of the techniques that were used to make the lecture theatre and classroom a more effective place of learning that led to an astonishing Australian success story. There have been questions raised among academics about the usefulness of the traditional lecture approach that has been used for many years at universities and other tertiary institutions. With increasing calls from some quarters to embrace modern technology and move to more interactive and flexible learning systems, the results obtained in this instance clearly demonstrate that such radical changes can actually work.

1. Introduction

The traditional face-to-face lecture has been the cornerstone of university teaching and learning until recent times. But cracks have recently appeared in this trusty method with many institutions reporting falling attendances [12] and a concerted effort is now in swing to consider other forms of imparting knowledge. To the forefront is the so-called technological revolution [6] with an astounding array of devices at the disposal of the modern lecturer. Common among these is the reliance on *Microsoft PowerPoint* overheads and of course the ready availability of the Internet. Those who refuse to yield completely to such revolutions are often accused of living in the stone age and retirement is sometimes suggested as an option.

Without question the Internet has caused a transformation in teaching and learning from many points of view. For the lecturer it has opened up so-called 'flexible learning' possibilities that now see entire degrees being offered without the student ever going near a classroom. As well as the convenience for the student, it also yields a darker side where there is now an abundance of reference material that can be downloaded at the press of a button. This of course also raises the hot topic issue of plagiarism, but that is a discussion for another day. Instead, this paper will explore the teaching culture at a leading undergraduate school for business located in one of Australia's most prominent universities, examining one of the teaching and learning strategies it has adopted.

Some lecturing staff take a minimal approach to the Internet, using the web simply as an adjunct to the main business of providing a teaching and learning experience for their students. Their efforts may be as simple as placing basic course material on the Web, including, for example, practice exercises, solutions, useful Web sites and other important information. It is a bold lecturer who displays their entire lecture notes on the Web since it may be viewed as an invitation to skip lectures altogether. It could be argued that if lectures consist only of simply note taking then it may well be a waste of student time. In many instances, lecture notes are rarely placed on the Web as an alternative to coming to lectures. To do so would see lecture attendance rates plummet in most cases and leave the students feeling that their three or four hours of contact time each week could have been much better spent as quality and productive time doing other things. It is also important that service quality assessment is performed at regular offerings in each course and appropriate heed taken of the feedback (see [3] and [8]).

There is something in between giving the students *all* the lecture material in printed or electronic form and giving them nothing so that they have to copy everything down as it is spoken or transcribed on the board. One common technique is to give the students in printed form *almost* all of the lecture notes but with vital pieces missing. To fill these in they must be present at the lecture, solving problems either individually or in a group. Extra material from the lecturer, as it arises from class discussion, is written on the board or overhead projector as desired. But copious note taking is in this way avoided. Such techniques have served many lecturers here very well over the years. The idea of having a fistful of *PowerPoint* slides from which there is little or no deviation can be quite one-dimensional and consequently a big turn-off for many students as a result of the lack of associated spontaneity. Indeed, none of the rated top five lecturers in the postgraduate school of management at Macquarie University have ever used PowerPoint as a lecturing tool except on rare occasions.

2. Teaching undergraduate business students

A particular challenge comes in an undergraduate offering at Macquarie University in Sydney, Australia, one of thirty-eight public universities in the country with an enrolment of about 25,000 students. There they cope with a massive course with the largest enrolment of any single subject at an Australian university. This course in introductory statistics caters for around some 2500 students each year and serves the statistical needs of all first year subject areas across the campus. One might be forgiven for thinking that the reason for such an achievement is that the material involved in statistics is so fascinating that students simply can't resist it. On the contrary, with very few exceptions, those enrolled are there only because it is a compulsory prerequisite to a great many popular fields of study such as business, economics, accounting, computing and behavioural sciences, to name a few.

The fact is that virtually no student comes to Macquarie to major in the field of statistics, and yet they have the largest statistics department in Australia and produce more than half of the country's statistics graduates. Excellence in teaching and learning techniques is the key and the statistics department has gone to extraordinary lengths to make the material relevant, exciting and challenging. With lecture theatres sometimes holding up to 500 students or more this is no easy task, particularly if many of the audience are somewhat hostile from the outset at having to be there in the first place. The lecturers are hand picked and the teaching material offered is first rate. It is virtually impossible to be all that 'interactive' with such a massive audience and staff find themselves lecturing on the stage that doubles as a movie theatre. There are two overhead projectors that are some twenty feet apart and an enormous cinemascope type screen behind. There is even a piano in the corner. Indeed, you feel like a performer and that is exactly what you have to do. Holding the attention of the audience so that the two hours seemingly flashes by before they realise it has gone is quite challenging.

The sheer volume of the course has a sizeable impact on how the material is presented. Honed over the forty years that it has been in operation, introductory statistics has survived to lead the way in effective methods of very large group teaching. There are a number of components for student assessment including computer aided testing, mid-semester examinations and a final examination (that itself must be passed in order to complete the course). Without going into full details, twenty percent of the final mark comes from twelve weekly short answer questions (on the previous lecture's material) that the students answer by logging with their unique username and ID onto a website constructed specially for that purpose. The questions are randomly generated and no two students will get exactly the same test. In fact, they may have up to two attempts at each quiz and count the better of their two marks. Feedback on their performance is immediate and they are told of the correct way to do any problem in which they have been wrong. This is an automated system that is tremendously resource intensive to set up, but that now requires only minimal maintenance. At the end of the semester the course administrators simple record a total score on all quizzes for each student.

A constant thorny question for the statistics staff at Macquarie was how to attract the students to do even more statistics subjects after their introductory course. What happened to these 2500 students after this first course? For the department to survive as a viable entity, it was essential to convince at least a portion of the captive audience from their first experience that the subject of statistics could be interesting, challenging and even relevant. Of course, this is no easy task. It was certainly true that some students continued in the subject because they had failed in or were disillusioned with their own major area (e.g. psychology, economics, accounting, actuarial studies etc) and were simply looking for something else to fill in the time. But these were not nearly enough in number and it fell upon the staff to come up with an idea that would revolutionise the way the department organised their offerings.

3. Innovative ideas

From time immemorial, traditional statistics departments were steeped in theoretical courses with such thrilling names as Mathematical Statistics I, II and III, Stochastic Processes and Theory of Probability. Student numbers were dwindling along with a nationwide (even worldwide) decline in students studying mathematics and by association statistics suffered along with it. There was a crisis point fast approaching as many university statistics departments simply disappeared or became absorbed into mathematics departments (also in decline) to retain some kind of viability. At Macquarie there was a three pronged attack on the problem that changed the face of statistics and made it the best in the country.

Even before it began its revolution, there had been repeated calls for the department to remove itself from its location in the undergraduate business school to a different school where mathematics was taught. The department repeatedly fought against this proposal and eventually won their battle, and in a gesture of seeming defiance dropping most of the mathematics prerequisites from their subjects. This naturally had an immediate negative effect on the number of students enrolled in mathematics courses but had the desired result of a modest rise in enrolments in other statistics courses. The downgrading of assumed mathematical knowledge had its predicted consequence without compromising the rigour of the material. But there was more to do.

The next step in the process was firstly to rename existing subjects that had such unappealing titles it wasn't surprising that enrolments in them were critically low. Whether we like it or not, the word 'statistics' simply does not sound very exciting to the majority of students and so the term was replaced where possible with another that might be more 'catchy', although the subject content was largely the same. Examples of these changes include replacing the course 'Applied Probability' with the name 'Risk and Chance' and 'Statistical Inference' with 'Modern Statistical Concepts'. The numbers in these subjects immediately increased markedly.

But these were for the most part still the same old subjects and it was now time to diversify and modernise the offerings in statistics where needed. Perhaps the most significant departure from the way in which the department offered its units was a radical change of direction in an attempt to lure some of the huge number of students enrolled in other business areas. To do this it was necessary to convince prospective customers that there were careers to be made in statistics and that employers were ready to embrace these skills. Although 'traditional' statistics had served reasonably well, the numbers of students had certainly levelled off and were in danger of falling off the radar altogether.

The situation was turned around by the introduction of a Bachelor of Commerce degree with a major in statistics that concentrated in the field of 'operations research'. This meant the introduction by the department of new offerings in statistics that covered new subjects, in addition to operations research itself. After much soul searching the following were introduced:

- logistics
- market research
- quality management
- e-commerce
- project management

The interest from students was immediate with the numbers enrolling at third year level operations research averaging over 200 per offering while some of the remaining theoretical statistics courses could only manage around 30 students. While some statistics staff lamented the 'downgrading' of these tried and trusted subjects, there is no doubt that the survival of the department was now largely dependent on the continued success of these operations research subjects that were firmly geared to the requirements of business and industry. This is a very important aspect as far as students are concerned since one of the most common questions asked is 'who will employ me if I get one of these degrees?'. There was now a much wider range of options in this respect and the department was sure it was on a winner. The path was now clear for the final assault on the program.

Having an attractive program in statistics at the higher levels was important enough, but it was even more essential that the students have their appetite whetted from a very early stage. The beginning introductory statistics unit covers the usual topics of descriptive techniques, hypothesis testing and linear regression. They became familiar with dedicated commercial statistical computer packages such as *Minitab* and demonstrated the virtues of the statistical attributes of *Microsoft Excel* since this was the program many of them would use after university life.

Although the compulsory first year course still had its predictable massive enrolments each year, around 90 percent of students were still choosing *not* to proceed further with the study of statistics – either because they weren't interested or they could not fit it into their program. In addition, there were simply many hundreds of students across the university who did not do any statistics subjects at all because they had no compulsion or inclination to do so. If they did enrol in a further offering, it had to be one with the unappealing title of 'Introduction to Statistical Practice'. In its early years the number of students enrolled in it had been relatively healthy but it too was suffering a decline due to a perception of its being somewhat boring in nature. If they didn't do this subject then they would not even get to the impressive new subjects offered at higher levels.

It was then that the final drastic change was made. With several other academics I conducted an informal survey of students to determine just what aspects of 'the real world' contained statistics that might be of interest to them. Perhaps of little surprise, the overwhelming favourite topic was gambling, followed closely by sport. No other application of statistics seemed to spark anywhere near as much interest at all, although there was some fascination in using statistics to study unusual phenomena such as the occult or the supernatural. The statistics staff decided that a third topic that might appeal to students was the application to medicine, particularly to crime scene investigation. The stage was now set for the end of 'Introduction to Statistical Practice' and the dawn of a new subject 'Gambling, Sport and Medicine'.

4. Teaching statistics using gambling, sport and medicine

Catering to the market, it fell upon me to design and teach this brand new offering and to introduce it at the first year level with no pre-requisites *alongside* the course in introductory statistics. Since it would emanate from the same department, there could be no mistake that it was indeed a statistics subject. It took over a year in the planning of a syllabus and another six months to select the key staff that I would involve in teaching it. The media became aware of what was happening and I happily did several radio and television interviews promoting this radical approach to the subject. It was a defining moment, especially since gambling in particular involves sensitive issues with associated social problems and, moreover, a number of the students had not yet reached the legal gambling age of 18 years. As well as teaching from a statistical point of view, it was also necessary to talk about the negative side of gambling and the devastating impact it can have on those who do not gamble wisely and the effect on their families and friends.

Finally all was in readiness. All that remained was to see if students would be as excited about the new subject as we were! The response was overwhelming to say the least. In the first year over 400 students enrolled from across twenty-six different study areas across the campus. This was a huge increase on its predecessor and it was now up to me and my team to deliver the goods. Firstly it had to be made clear that this was not simply a subject that taught impressionable minds how to win at gambling – on the contrary, its aim was to present statistical principles in a way that captured the imagination but at the same time being challenging and informative.

It was correctly felt that one of the best ways of achieving this goal was to employ visual aids or 'props', and there was no shortage of these when it comes to the topics under discussion. The department was fortunate enough to receive a donation from Penrith Panthers Club, the largest club in Australia, of two working poker machines valued at around $30,000. These machines turned out to be a highlight of the teaching of probability and expected returns and added a new dimension of excitement to the learning process. Other gambling devices that were purchased inexpensively included a two-up kip, coins, dice and packs of cards. A new roulette wheel was a more difficult issue since it costs around $25,000 for a new one and $3500 for one that was reconditioned. Since both were beyond the budget, an imitation plastic one costing several hundred dollars did the job nicely, although smaller in size than would have been liked in a large lecture theatre.

Gambling, the largest (and most popular) component of the subject was then under control and sport was the next topic to have much the same treatment. To this end, teaching staff supplied their own equipment including cricket bats and balls, tennis racquets, basketballs, soccer balls and ice hockey sticks to illustrate how these games were played. It was then shown how strategies could be developed using basic statistical principles, the very same ones that had been previously used in demonstrating their use by means of tedious agricultural experiments and the like. Even if a student wasn't particularly interested in sport they were still fascinated by the whole experience.

The final section on medicine, also being the shortest, did not require many aids but did provide enormous scope for a wide ranging choice of topics. Among these is a regular lecture by one of the country's leading handwriting experts on just how statistics can be used to detect forged signatures on documents. Another deals with the interpretation of DNA and other evidence in trials and just how easy it is to bamboozle a jury (or a judge).

These were all techniques that could be readily used in the lecture theatre and classroom and when required the poker machines were wheeled to the classes on specially designed trolleys. But there was still something missing. There was no suitable textbook on these subjects and so one had to be written. This had taken me eighteen months but finally it was finished, naming it *Gambling and Sport: a statistical approach* (see [4]). Students readily purchased it owing to the contents containing a great deal of relevant material on the gambling and sport sections that covered over 75 precent of the subject. Other reference books on gambling included the classics such as [7] and [11] while for the medicine section there were plenty of printed notes given to the students. There was one final step to round off the complete package of innovation.

5. Web based teaching and learning

The educational experience needed just one more enhancement that would require a great deal of time and, naturally, some money. Fortunately, Macquarie believed in the merits of the course to the extent that it awarded me one of the largest ever teaching grants of $45,000 to present the subject as one that effectively utilised the capabilities of the World Wide Web. This was a two year project that initially involved considerable time developing the specifications for a web site unlike any that had been devised before at the university. It was pioneer work that presented a real challenge to the programmers whose task it was to put my expectations into practice. There were also others who had written well-known references in the design goals in a learning environment and variations in the delivery of instruction (see [5], [10] and [13]). This project was going to use these principles to their fullest.

Space does not permit a full description of the harrowing time spent bringing the venture to its conclusion, but suffice to illustrate some of the features that it has to offer. The opening screen of the subject (also known as STAT175) website (http://online.mq.edu.au/pub/STAT175/) displays the course logo and a variety of sub-menus that can be reached by clicking on one of the three icons GAMBLING, SPORT or MEDICINE. The user is then led through a series of public pages that describe various aspects of the subject including detailed information on each topic along with the unit outline and due dates of assessment.

To access more in-depth material relevant to the unit, users can click on the LOGIN icon and must supply a username and password. Each student is given their own unique combination of these for the duration of the subject. This enables the teaching staff to communicate directly with each individual and they can also contact each other privately. The actual lecture notes are not be placed on the web since it is felt that this would be a discouragement to attending lectures. The specifications of the web site include the following requirements that have their individual icon on the screen.

- a discussion board (among themselves)
- a direct email to the lecturing staff
- access to unit information
- frequently asked questions
- practice exercises and solutions
- practice mid-term examinations and solutions
- practice final examinations and solutions
- files that may be downloaded
- assignments

After logging on to the private section of the site, the screen presented displays a dazzling array of icons with the most exciting feature being the introduction of an animated simulation component in which students can play games such as Two-up, Keno, Roulette, Craps, Keno Heads or Tails? and even slot (poker) machines. The simulations, written using the powerful authoring environment of Macromedia's Director Lingo, were the most expensive and complex part of the site but undoubtedly well worth the effort since, apart from being colourful and challenging, they can be used to test strategies and to verify the theoretical statistical results.

The Olympic Games in 2000 and 2004 provided a golden opportunity to demonstrate statistical techniques using sporting data. In particular, one lively topic of discussion was the times and distances that athletes and swimmers would have to achieve in order to win a gold medal. With an abundance of historical data, students were able to use their statistical skills to come up with various intervals within which they feel the winning performance would lie. Then they could compare their answers with what actually happened.

Apart from the Olympics, there are a number of popular sports including football, cricket and basketball that are all covered in some aspect. Some of these are along the lines of developing or testing of strategies for play, while for others it is considering the effect of rule changes. One of the beauties of teaching statistical principles using sport is that there is never any shortage of information to analyse and it is always topical. In all, after six weeks of learning about the statistics of gambling, the next four weeks is spent on the statistics of sport. Useful references here included books such as [1], [2] and [9].

Although the sports and medical statistics components of the course are well liked by the students, they do not enjoy the popularity of the gambling section. However, for those students who are keen to undertake further studies in the fascinating area of medical statistics, the department offers a complete unit at the higher level in epidemiology and a number of graduates with this interest have gone on to find employment in this area.

Undoubtedly the subject, which makes no secret of the fact that it features gambling, has been an outstanding success and has in a short space of time attracted thousands of students along with hundreds of thousands of dollars in funds to the department. The media has continued to be very supportive, as has the gaming industry itself in assisting a venture that provides an essential public education program in this controversial area. The university has even featured it on a promotional video and DVD and at least two other universities have tried to imitate the success by offering their own versions.

6. Remarks

There are many ways in which a lecturer can be effective and what techniques work for some will be a disaster for others. Curiously, Outstanding Teacher Awards at Macquarie University are much coveted and awarded by a panel who has never actually seen the recipient at work in the classroom. To be successful, superior student ratings over a sustained period of years are essential but not sufficient. It is also necessary for the applicant to write a ten page treatise on, among other things, their philosophy of teaching. The next step is to be selected, if the university deems it, as an entrant in the national Australian Awards for University Teaching scheme held each year.

It comes as no surprise that some classroom lecturers are pure magic with just a marker pen and a whiteboard at their disposal while others flounder no matter what modern teaching aid is at their disposal. That is why, with an increasing emphasis on good teaching skills, both the undergraduate and postgraduate business faculties at Macquarie have appointed their own Associate Dean/Director of Teaching and Learning to bring out the best in their staff. In doing so they have high expectations on a general raising of the level of teaching ability of their staff.

As far as the new subject involving gambling, sport and medicine goes, the outcomes for those involved in this fascinating project have been extremely rewarding. It has been very well received both within and outside the university and has spawned similar units at other tertiary institutions. The pressure is always on academics to be able to present their material in a way that is relevant, thought-provoking and stimulating. Although this may be easier to do in some areas of study than others, the results are well rewarded, as evidenced by the Macquarie experience in this unit. The hunt is now on to repeat this success by offering other types of courses that will also attract the attention of students who otherwise would not have ever envisaged studying a particular subject area.

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Recreating first mathematical impressions into lasting mathematical impressions

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Mathematics is a subject that evokes an assortment of emotions when mentioned at university or in the workplace. Often students welcome the first light of mathematics with admiration and wonderment but leave at twilight with apathy and disaffection. Can educators rewrite the script with the final act being one of mathematical joy for university students? Will the answer be found in the dialogue of mathematicians or in the performance of the learners? Can the solution to evaporate these feelings at university lie within primary pedagogy? This exploratory research provides an insight for teaching mathematics at university using recollections of schooling from preprimary to university.

1. Introduction

The positioning of service units in mathematics causes problems for both students and educators as first year study consists of other service units. The ability to relate to chosen fields is inhibited and for some students this adds extra emotions and attitudes that restrict learning rather than the actual mathematical content. A positive caring environment promotes an atmosphere where mathematical confidence and learning occur and grow simultaneously [1]. As the need for mathematical competency is recognised, educators have the task not only of teaching the mathematical content but changing the attitude to mathematics [2]. Students foresee hours of work with little knowledge gain therefore it is this quiet subset that must relinquish past prejudices in order to acquire a fresh approach. The engagement of university students in mathematics may be achieved by a more creative approach that will rekindle a long forgotten interest and appreciation [3]. Interest in mathematics may be revived by utilising devices such as puzzles to promote creative modes of thinking in students and thus resuscitate stagnated development of mathematical learning [4]. The intention of this scoping paper is to elicit attitudes and reactions at various stages of mathematical education from a diverse group. This provides the motivation at university to use primary teaching strategies in first year service mathematics courses. Mathematics education at primary school consists of games, discussion, small group activities in conjunction with whole class participation and will subsequently be referred to as 'primary teaching strategies'. These strategies will be used for all students during the usual contact hours regardless of ability and not as additional study. It is hoped that using these strategies may change students' self-perceptions and beliefs to study mathematics by evoking their creativity and curiosity. Qualitative responses will be sought to highlight the part played by emotions which obstruct/facilitate the learning of mathematics. What will be sought is not recollection of the memory but rather its emotional reside. This is the starting point for students because these emotions obstruct/facilitate learning. Prior research has focussed on various aspects of the mechanical processes of teaching mathematics [5–7]. I am exploring the imbrications of emotion on learning mathematics and from my findings will formulate a framework for understanding this issue.

2. Methodology

The survey was completed by current students studying in Health Sciences, Science, and Computing areas, and former students acting as volunteers in an extracurricular context. Most courses would include a first year mathematics/statistics unit.

Table 1: Career status of people who completed the survey

Status	Teacher	Student	Professor of Law	Recent Graduate	Office Workers	Nurse	Other Professions
Number	8	23	1	1	6	6	10

Table 2: Ages of the people who completed the survey

Age	20-29	30-39	40-49	50-59	60-69
	21	18	10	4	2

Table 3: Gender of the people who completed the survey

Sex	Male	Female
	17	38

3. Results

The following tables contain summaries of the responses at each stage of schooling. It is possible to respond to more than one category in any one section.

3.1. Kindergarten

Often there is no formal title given to mathematics time at kindergarten, activities including colouring and sorting shapes, completing dot-to-dot puzzles through to simple counting puzzles are linked to term topics. Presentation is highly valued through the use of bright colours and storylines so as to engage the child's thinking and prolong the attention span. Often play is interpreted as valuable but not real learning by many outside observers but a well-planned curriculum should incorporate an emphasis on play [8]. Play at this stage of learning is a process that captivates both mind and body with no particular goal or incentive; it has no other motivation for the child but enjoyment. The following quotes about kindergarten encapsulate this sentiment:

> 'Playing and arranging wooden blocks with numbers on to get the right answer to sums. Didn't really know it was mathematics we were doing but I remember it was fun' (Male entertainer)

> 'No we didn't realise it was maths, it was realistic we had physical things in front of us so the concept was easy. I think it was easy to learn cause in a way you didn't even realise you were learning' (female science student)

At kindergarten the term 'mathematics' may not be explicitly stated to the children during class time, more often words such as 'sums' or 'numbers' are substituted. The foundations of numbers are truly laid in this environment. This may be a reason why memories of mathematics are not recalled rather than just the passage of time. Language and gestures utilised in mathematics at kindergarten may be more important in shaping future learning than educators imagine [9].

3.2. Primary School

Primary school begins about five years old and spans around seven years. The environment alters to accommodate a change to more formal schooling though fun and games factor has not

Table 4: Responses concerning preprimary education

Question One Preprimary Preprimary	Little or no recollection	Unaware it is mathematics	Fun/play	Recalls teacher
Number of Responses	21	12	23	30

Table 5: Responses concerning primary education

Question Two Primary	Timetables /rote learning	Exercises from book	Fun	Recalls teacher	Feeling good	Hard work/difficult
Number of Responses	21	18	11	22	17	16

disappeared. Some of the changes involved are; individual desks replacing the communal mat, students are made more aware of a timetable with different teachers for different subjects, and students possess their own labelled equipment e.g. pens, pencils, and books. The community spirit and fun environment is still evident with working accomplished often together as a class or in small groups. As students progress through primary school more individual work is slowly introduced. Early primary schooling still revolves around and involves the teacher and that was evident from some of the responses. The role and image of the teacher had a major effect on some students as is evidenced by the fact that after many years, sometimes nearly fifty years, the name and demeanour of the teacher could vividly be remembered [10]. Recollection of timetables and rote learning varied and were not limited just to the classroom, some recalled posters at home. Timetables may be utilised to create many number patterns that may be recalled and used years later. This is often not fully shown to students and unhappy memories are created instead of number memories:

> 'Work focussed on operations and little else. Focus on the timetables. If you had trouble here you were in real trouble. Did enjoy this though...' (female early childhood teacher)

> 'We had to do our timetables out loud individually. I did not like this as I have a morbid fear of failure that I am only now overcoming' (female medical imaging third year student)

> 'In year three we started reciting the multiplication table... While this may have seemed like fun at the time it didn't seem more than a song.' (male physics PhD student)

> 'I remember learning timetables and chanting them out loud in class. We used to play games with them like timetable baseball. It was fun with the competitive element' (female media third year student)

These recollections have been placed in order of age, oldest to youngest, and thankfully there is a progression away from army style drill to a friendlier approach. A game-like approach often engages all students in mathematical skills and concepts indirectly with the fun component eliminating some of the stress, fear and boredom of the usual written exercises [11]. These simple games aid the pathway to discovering and enjoying mathematical concepts [12]. The

Table 6: Responses concerning secondary/high school education

Question Three Secondary/High	Hard Subject	Exercises from book	Easy and enjoyable	Losing interest	Recalls teacher	Recalls Topic
Number of Responses	38	19	19	27	17	11

Table 7: Responses concerning tertiary/university education

Question Four Tertiary/University	Maths part of degree	Losing interest	Recalls lecturer/lectures
Number of Responses	34	10	27

competitive edge is credited in motivating learning as well as making mathematics appealing. The following remarks illustrate this appeal:

> 'I remember as an elementary school student participating in competitions involving so-called 'flash cards' in which math problems were displayed on a card and students competed to identify the correct answer as quickly as possible. The game element motivated me and then I started to enjoy math' (male professor of law)

> 'We had mathematic competitions where we competed in class to see who got the most answers correct ... The competition was fun, as it kept me motivated.. I loved that kind of challenge.' (female graduate in cartography)

3.3. High School

The transition from primary to high school has seen much attention in recently with students' welfare a priority by providing parent information nights, tours and activities to alleviate the shock of a new environment. High school is often a daunting experience for students since the friendly atmosphere of early mathematical schooling is perceived to be lost and the reputation as a difficult subject is born. Mathematics is surrounded by a negative aura in many aspects of life from family and friends to press and film. Both mathematics and mathematicians are typically depicted in films and books as people who are tolerated rather than accepted in normal society due to either appearance or character. Many responses here reveal that interest in mathematics appears to wane in late primary and early high school.

> '...Maths was more enjoyable as it was challenging especially in the lower years. The teacher in the upper years was more traditional ... more monotonous at this stage.' (Female Nurse)

> 'High school maths was basically all from exercise books and not interesting at all. VERY BORING... We all lost interest pretty quickly' (Female School of Air Teacher)

> 'Yes I had to work for it ... Subject no longer stimulating- lost interest when I realised that I was memorising rather than understanding the subject' (Female Tax Office Worker)

An interesting quote from the mature-aged female fourth year education student after completing a classroom exercise incorrectly:

> 'I think I went off maths after that. I always heard my sister saying she was never good at maths and couldn't function with numbers. I used the same excuse and eventually thought it just ran in the family...'

Families help to promote the continuance of the negative cycle, therefore it is important to socially give the human touch to mathematics and make the subject more appealing and enjoyable to the next generation. Once mathematical confidence is present then learning can really begin, as the mind is now open to ideas and concepts [13–15]. This set of recollections is within the last five years, even as recent as a year ago, showing high school mathematics evokes memories of countless examples from books, and graphics calculator work. Discussion of mathematics appears not to be as memorable:

'Maths was not being taught at a practical level and it was more questions from books... I hated it.' (female science student)

'I started in the top year 8 and 9 classes then went down hill. It was all from a book. I lost interest very quickly. It was mainly taught on the board along with theory but it was always done in a rushed manner like there was no time to ask questions which eventually turned me away from mathematics and it ended up becoming one of my weakest subjects' (female graduate in cartography)

'We started doing maths from books now. Teachers would give whole chapters which either became too repetitive or challenging. That's when maths became boring.' (third year female media student)

Pressure on teachers to improve mathematical standards has been enormous, forcing some teachers to set more homework assignments. These stresses may be transferred to students and create long-term anxiety about mathematics thus reducing future enjoyment [16]. The play element is deemed impossible to accommodate within time constraints at high school. Discussion becomes an under-utilised tool being employed infrequently and thus loses its objective. By thoughtful questioning and carefully directing of students by teachers, discussion episodes have important implications for both teacher and student [17]. Students enjoy the chance to chat informally about strategies for solving problems where both the speaker and listener gain knowledge [18]. Play is an essential essence of all stages of life [19]. Sports and computer game fanatics discuss in great detail strategies, events etc... as it is not deemed learning but enjoyment [20]. Creation of this ambience in mathematics may well promote learning and re-establish the fun factor.

3.4. University

Study at university in mathematics usually consists of lectures and tutorials with assessments based on online quizzes, mid-semester tests, and final examinations. These types of assessment are all written-based ignoring any information that could be obtained by other means such as oral and body language. From the moment an individual is born, gestures are important being documented constantly by parents and teachers. The passage of time does not make gestures in relating information any less important. Research has shown that gestures may be the window to individual's thinking that lies beyond the lecturer's grasp in the written word [21]. Basic gestures and movements can make an impact on information moving to the working memory and consequently becoming memorable [22]. Simple gestures and body language can convey up to 80% of information and once recognised by the educator can be utilised to improve teaching at all levels [23–24]. Knowledge of different learning styles is a great asset at any juncture of education. One experiences many ways of acquiring and learning new information and along the pathway of life, students discover their own particular styles of learning [25]. The theory of multiple intelligences relates the different modes of obtaining information in the following categories: Linguistic, Logical-mathematical, Spatial, Bodily Kinaesthetic, Musical, Interpersonal, Intrapersonal, and Naturalist. Pictures, words, hands-on work, and music are modes employed at kindergarten; primary then extends introducing social, personal experiences and self-reflection modes. University tends to use principally linguistic and logical-mathematical modes in teaching thus missing the opportunity to relate to all of the learning styles and hence all students [26] [16].

The quality of teaching is heavily monitored throughout all levels of schooling and professional development courses are often compulsory. Research is often rated higher than teaching when considering the employment of new staff in departments at many universities. Teaching at university becomes less regulated now with only some lecturers possessing teaching qualifications. This may not necessarily be a major concern as induction and other associated teaching courses are offered to staff during the year. The next comments though show that some students feel that some lecturers may be struggling in this area:

'Throughout secondary or high school, I found math to be enjoyable. I didn't find it difficult until I got to college, where a poor calculus teacher definitely impeded my enjoyment. I blamed the instructor who was very poor in communications (and seemed to delight in confusing students)' (Male Professor of law)

'Most maths lecturers are very single minded and are of the opinion that maths is the most important subject to learn ... Not have the attitude 'What's wrong with you? Why don't you get it? (Female Education Student)

Hopefully the last comment about lecturers of mathematics is not echoed too loudly. The transition from school to university has had its profile raised recently after much research; help is readily available for students in all schools and departments. Emotions about mathematics can easily be changed by the lecturer's style and manner. The last quote here shows that even long held beliefs may be transformed:

'Although I'm sure their desire was for us to understand I did not like the style of teaching, could not understand. I would like to see your style of teaching Anne, introduced to the lectures. They switched on lights and made you understand... I need to understand anything in order to learn and/or remember. (Medical Imaging Student)

'I really disliked maths from yr 3 onwards and found it boring ... maths was rote learning, exercise books and copy of the board. My opinion of maths changed and I now confidently teach maths... It is all falling into place.' (Female Primary school teacher)

4. Implementation

A user-friendly learning atmosphere of cooperation is established with students recognising that their input is respected. Students are encouraged to ask questions during lectures, tutorials, or via e-mails. Colour, imagination, and variety form the ingredients of my lectures and tutorials. Students quickly adapt to the new style and appreciate that ideas can be discussed informally at any time in an interactive forum. Photocopied lecture notes are provided, so students may listen and are encouraged to personalised notes in picture, symbol or language. Activities at tutorials vary from passing questions on cards to utilising equipment in the room for concrete explanations. For example: numbered cards have different forms of algebra expressions such as linear functions, quadratics, etc... The aim is to collect all cards that can be manipulated to the expression of card 1. Mathematical skills involving negative numbers, expanding brackets, work with indices and many other basic arithmetical operations can be incorporated easily. Students are often surprised by the different forms of the expressions. This activity is enjoyed by students which replaces exercise questions about different types of factorisation. Students are encouraged to share ideas and solutions with others either at the board or in small groups. Competitions between small groups create discussion of different strategies. Simple rules are repeated aloud during class in an effort to make the rule memorable [22]. Incorporating book exercises intermingled with oral and game-orientated exercises will cater for all forms of learning styles [27]. Rewards are always welcome in any form particularly the edible variety. The aim is to enlighten students to the idea that mathematics can be fun and creative just like other subjects.

5. Conclusion

Market research listens and acts upon information received to generate a product that is desired by the consumer adding modifications with the passage of time to meet society's needs. Educators should acknowledge that education has similar marketing traits [28]. The students' responses to this exploratory survey give an insight into a non-mathematician's perspective of both teaching and learning of mathematics.

A renovation is warranted in the approach to mathematical pedagogy for university students with difficulties relating to mathematics. Play is not just the province of children. Play activities promote ideas and solutions that may not happen. Reinventing play can impart complex mathematical concepts to older students, where ideas can flow without prejudice [29]. Exciting, congenial games and puzzles are easily incorporated into various mathematical areas replacing what is perceived by students to be boring exercises. In tertiary education, peer support raises confidence and dulls previously held conceptions of mathematical ability. Small group work in primary school may be reintroduced to provide the support structure for many students. The knowledge that others have similar problems produces an atmosphere to learn with and from each other. Mathematical barriers may be reversed and loathing may be replaced with enjoyment again. An appropriate teaching strategy combined with a willingness to learn creates the perfect recipe for mathematical inspiration. It would appear that residual emotion from childhood memories affects contemporary mathematical learning. This indicates that learning is historically path dependent. Past emotion needs to be dealt with before future learning can begin. Current research is continuing with two groups studying the same material, one with and one without 'primary teaching strategies' during tutorial time with the aim to compare attitudes towards mathematics before and after the course. Future research may include a longitudinal study of students' images of mathematics through primary school to high school. The general perception of mathematics, even with change of pedagogical fashions, has remained the same. The following saying may hold the key to unlocking the mathematical puzzle:

> 'We don't stop playing because we grow old;
> we grow old because we stop playing.' George Bernard Shaw

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Cultivating Mathematical Understanding by using Innovative Technologies in the Lecture Theatre

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The teaching of mathematics today focuses on providing more meaning for students, encouraging them to think logically, developing their number sense and cultivating mathematical understanding. This paper seeks to explore issues in the delivery of mathematical course content using a Tablet PC to aid students' understanding. Clearly an understanding of the principles of adult learning is integral to this study in order to critique their application within this educational context. A preliminary evaluation indicates that the use of a Tablet PC in teaching Mathematics allows the teacher reach their audience more effectively.

1. Introduction

Mathematical knowledge has grown substantially in the last fifty years and the tools available to aid mathematics students have changed dramatically. The teaching of mathematics today focuses on providing more meaning for students, encouraging them to think logically, developing their number sense and cultivating a true mathematical understanding. This paper seeks to explore issues inherent in the delivery of mathematical course content using a tablet PC to aid students' understanding. Specifically this paper explores the opportunities, possibilities and constraints with regard to implementing Tablet PCs for mathematics education. It also includes a number of guiding principles for the design and development of mathematics education materials for use on a Tablet PC.

This study explores this linkage and examines a range of teaching and learning strategies for the effective use of tablet PCs appropriate for the students' study demands and real-life application of the tool within the wider society. In particular, meaningful teaching/learning contexts, in which students are able to effectively use a tablet PC, engaging in a range of activities appropriate to their learning styles, will be examined.

2. What is a tablet PC

A tablet PC is a portable computing tool. It is as powerful as a modern PC, but it does not require a keyboard. Instead, you can add information by writing on the screen of the tablet PC with a digital pen (this pen can also replace the mouse). Many aspects of mathematics education are being revolutionised by innovative technologies. The Internet, graphics calculators and course management systems (eg Blackboard) are gradually making their way into teaching and learning practices. Tablet PCs, which have just become available in the last couple of years, are another example of new technology that has expanding utility in course delivery.

There are two main types of tablet PC: a slate Tablet PC, which is a tablet with no attached keyboard; and a convertible Tablet PC, which is basically a laptop computer with a screen that can swivel and fold onto the keyboard to create the tablet.

The Tablet PC can be connected to a data projector and fulfil the functions of both whiteboard and overhead projector without the disadvantages of either. Unlike a whiteboard, all classroom board work can be recorded digitally and made available to students in a suitable electronic file format. Students can then repeat lessons, exercises and activities at a later time. Like an overhead projector, a tablet PC can be used to expand, amend, or highlight prepared learning materials, and it is a much more portable teaching tool. The Tablet PC use in this study was the Toshiba Portege 3500 and further details can be found at www.toshiba.com.au. This Tablet PC has been used in lectures since 2004 and streaming video of lectures in 2005 using the Tablet PC are available at http://streaming.cqu.edu.au/stat11048/2005_t1/.

3. Rationale for Using Tablet PC

Individuals now live in a complex and competitive computerised society, which requires many individuals to be "technologically" literate. As society becomes more technological and demands new and changing skills and understandings of its members, the study of technology becomes increasingly more necessary. Consequently there is an impetus upon the education system to provide access to knowledge and tools and heighten interest in and awareness about technology.

Taylor [1] suggests: "in the past twenty years, there has been a significant expansion in the availability of a wide range of technologies with the potential to improve the quality of teaching and learning". These advances have been accompanied by job specialisation, a multi-faceted approach to curriculum design and development and "opportunities for interactivity and access to instructional resources provided by the computer communications networks" [1].

Technology may be viewed as a process of human action and interaction. "People use it as a mode of action and thought, the praxis by which change can be affected in the physical environment and most importantly, the social and ideational environment" [2]. This conceptualisation of technology provides a framework for an explanation of the term "educational technology". When technology is guided by educational principles and practices, it essentially has been applied to the process of human learning. Mager [3] views educational technology as the application of scientific knowledge about learning, about the conditions of learning, to improve the effectiveness and efficiency of teaching and training. This educational technology provides for the design and development of learning contexts, which promote effective teaching and learning. The teacher assumes the role of technologist by ascertaining student needs, planning a program which caters for the students' learning styles and structuring the social and physical elements of the environment to support the students' endeavours. "The educational technologist is easily recognised by his/her application of the means of presentation (hardware) to a well designed program (software) which evaluates the total process of teaching and learning" [2].

Table 1: Comparison of Tablet PC, Notebook and Whiteboard

Topic area	Tablet PC	Notebook	Whiteboard
Pre-prepared material	Material can be pre-prepared using Microsoft PowerPoint. PowerPoints presentations can be reused.	Material can be pre-prepared using Microsoft PowerPoint. PowerPoints presentations can be reused.	Pre-prepared Material has to be written on whiteboard (errors are made more often)
Write on Pre-prepared material	Yes, using the Tablet PC pen for writing and highlighting.	No	Yes, using highlighting pens.
Make material available before and after the lecture	Yes, comments added to materials are included.	Yes, however comments added to materials are not included.	Yes, however comments added to materials are not included.
Videos and pictures	Yes. Useful for off campus students	Yes. Useful for off campus students	No
Cost	Tablet PC - $3000 to $5000. Data Projector - $1500 - $10000	Notebook - $3000 to $5000. Data Projector - $1500 - $10000	Whiteboard and pens are cheap

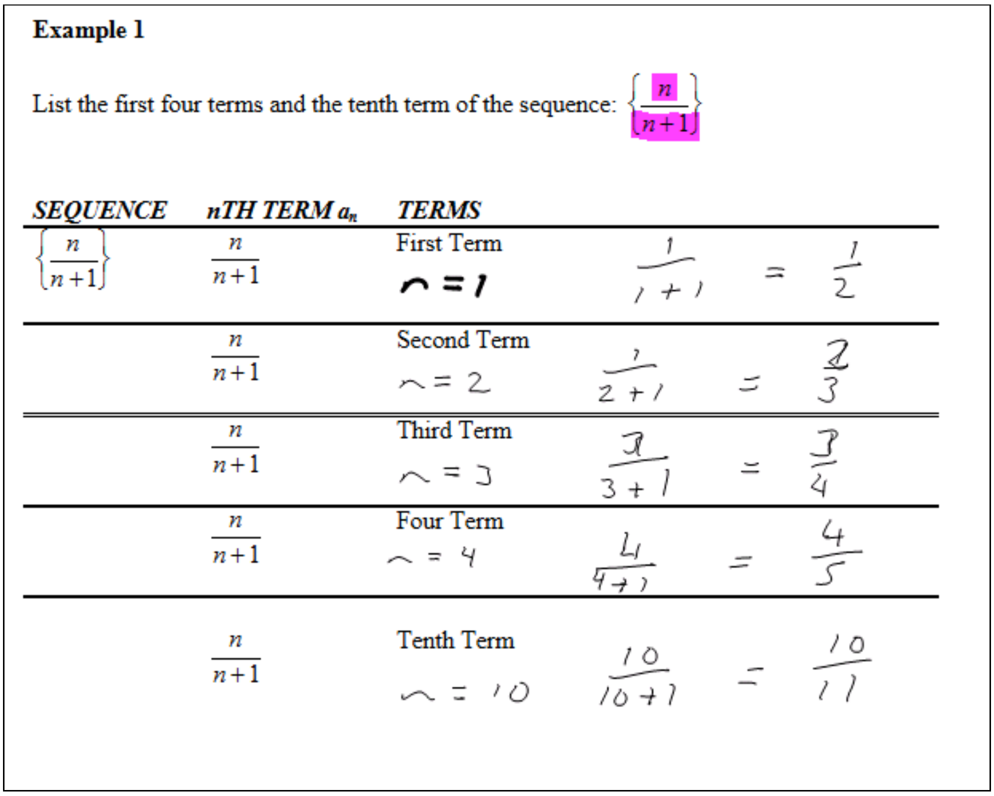
Table 1 indicates that a tablet PC has the advantages of a notebook and whiteboard in lecture theatre.

As noted above, there is an increasing need for individuals to have a high level of technological literacy and competency. At Central Queensland University, this need has been identified within the area of mathematics. In order to develop students' mathematical understandings and competencies, the Tablet PC has been introduced as a teaching and learning instrument. The use of a Tablet PC is an innovative approach to teaching and learning mathematics and allows lecturers to cover more work more quickly. Subsequently it is imperative that various instructional approaches are developed to foster effective teaching and learning strategies for the use of the Tablet PC for both academic and societal application. Coupled with this is a perceived need to document a range of innovative strategies to introduce and implement tablet PCs within the tertiary mathematics classroom.

4. Adult learning principles and teaching with a Tablet PC

These principles are mainly based on the adult learning work of Knowles, Holton and Swanson [4] and Entwistle and Ramsden [5] and the principles of the use of computer-based materials [6]. These aspects will now be briefly considered.

- Adult learners like to be involved in the learning process facilitated in a climate of respect. In order to accommodate these student characteristics it is necessary to provide a range of learning experiences to cater for differing learning styles, which include visual, auditory, tactile and kinaesthetic learners. Subsequently careful planning is required to support the students as they develop their mathematical competencies. For example, Figure 1 shows the real time interaction which is possible with a Tablet PC using with prepared material.



Example 1

List the first four terms and the tenth term of the sequence: $\left\{\frac{n}{n+1}\right\}$

SEQUENCE	*nTH TERM a_n*	*TERMS*			
$\left\{\frac{n}{n+1}\right\}$	$\frac{n}{n+1}$	First Term $n = 1$	$\frac{1}{1+1}$	$=$	$\frac{1}{2}$
	$\frac{n}{n+1}$	Second Term $n = 2$	$\frac{2}{2+1}$	$=$	$\frac{2}{3}$
	$\frac{n}{n+1}$	Third Term $n = 3$	$\frac{3}{3+1}$	$=$	$\frac{3}{4}$
	$\frac{n}{n+1}$	Four Term $n = 4$	$\frac{4}{4+1}$	$=$	$\frac{4}{5}$
	$\frac{n}{n+1}$	Tenth Term $n = 10$	$\frac{10}{10+1}$	$=$	$\frac{10}{11}$

Figure 1: Interacting with prepared content

- As individuals grow and develop, they accumulate a reservoir of experience that becomes an increasingly rich resource for learning. Accordingly, as a student's confidence increases, so does their ability to take risks with their learning and try new approaches to problem solving. During tutorials the importance of listening to adults is practised, both what they are saying and how they are saying it. This is the basis for responsive communication and enables students to express their concerns freely and openly. The Tablet PC provides a learning aid to support students' visualisation of problems, their experimentation with mathematical theorems, the validation of their answers and the means to explore other problems in differing contexts.

- "Research has shown that stepping though examples can improve classroom dynamics, boost students' confidence levels, and promote the understanding of mathematical concepts and function, and advance problem-solving ability" [7]. For example, figure 2 shows how the Tablet PC allows the process of interacting with students when stepping through examples.

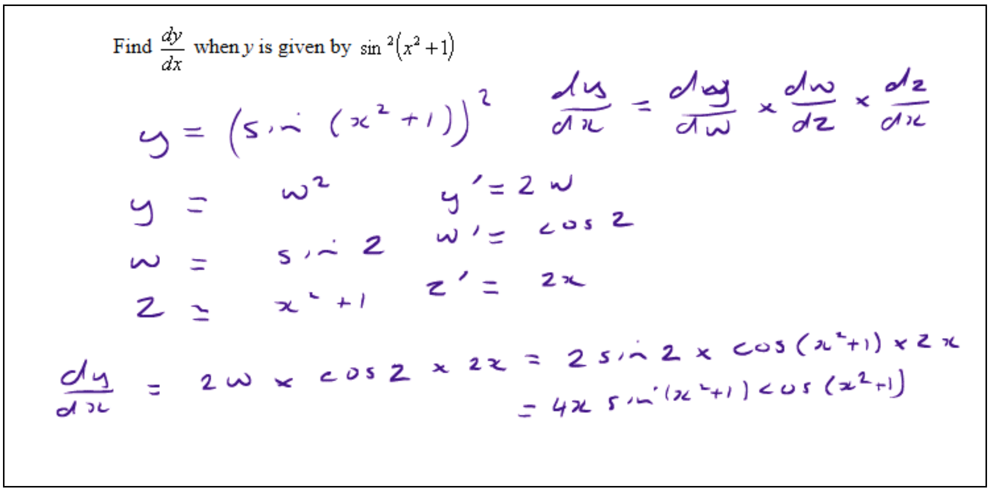


Figure 2: Step through example

5. Evaluation of the use of a Tablet PC within the Lecture Theatre Environment

The evaluation of the use of a Tablet PC within the Lecture Theatre Environment was essentially undertaken to determine whether the program's learning outcomes, media used to present learning material, teaching/learning experiences, content, resources, and assessment were appropriate and achievable. The evaluation also provided the opportunity to identify issues in using the Tablet PC that may not have been considered or not sufficiently considered.

A questionnaire was used to gain information from students about their perceptions of the Tablet PC within the Lectures. The questionnaire was given to students in the second last week of the 12 week course, Engineering Mathematics.

Engineering Mathematics is offered at Central Queensland University in second term of first year. This course introduces essential concepts of differential and integral calculus in an applied engineering context. Topics covered include sequences and series, differentiation and applications, integration and applications, introduction to differential equations. The students enrolled in Engineering Mathematics are from the Engineering, Applied Science and Education faculties at Central Queensland University. The Tablet PC was used for the delivery of all content in the lectures. Consideration is given to the skills, processes and problem solving strategies central to the effective and efficient use of the Tablet PC as a learning tool in mathematics.

A summary of the results from the questionnaires are presented in Figure 3. There were 35 responses from the 62 students in the course. The evaluation data also provided preliminary evidence that the use of the Tablet PC was an appropriate tool in assisting the learning of course content. The comments form students showed that the Tablet PC package provided a number of opportunities for students to interact with the content and to engage in active learning through a range of activities (see Figure 3). Attempts were made to explicitly link the theory, examples and activities for each topic area. These activities were sequenced in incremental stages to introduce new content and reinforce integral mathematical concepts. As indicated from the results and comments in Figure 3 the Tablet PC proved to be a successful method to provide students with a sound theoretical and practical foundation from which to develop their understandings. Using the Tablet PC in the lecture helped to guide students through the content. This was provided through scaffolding to nurture their developing skills and knowledge as students worked through the module. Using the Tablet PC in the lecture was integral in providing a variety of visual cues to guide the students through the progressive stages involved in solving questions

6. Conclusion

After using a convertible tablet PC in practice for 2 years, the authors have come to depend on it. We are able to provide clear and more helpful examples to explain course materials and have come to depend on it. Preliminary evaluation of the use of a Tablet PC has shown that it is attuned to the students and impacts upon their successful engagement, which in turn can lead to better understanding of mathematical content.

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1. Did you find the tablet PC a reasonably useful learning resource?	Yes No (Go to Question 2.) 100%...0%
If yes, please comment on the following features.	
The level of development of the theory is:	Excellent Good Not Good 37.5% 32.5% 0
There are sufficient worked examples.	Yes No 100% 0 %
The worked examples are all easy to follow.	Yes Some Too difficult 74% 26% 0%
The level of difficulty of the exercises is right.	Yes Some Too difficult 64 36% 0%
The tablet PC is generally a good aid to learning.	Yes No 100% 0%
Would you like to see Tablet PC continued to be used for next year?	Yes Undecided No 91% 9% 0%

2. Please list brief reasons for your answer.

Nice and easy to follow, being able to highlight rules is a good feature good. Allows worked examples to be done on screen. Eliminates small writing on whiteboard

You were very easy to understand and you made it very clear on what you wanted.

Clear, concise have not been required opened text book at all. Only for rules

The tablet was great. It allowed us to see exactly have to solve the problem, step by step and allowed the teacher to explain his thinking as he went along. It is always better to see it worked out in front of you than to look at the already made answer and try and decipher itVery good idea. Lecturer hand writing not too good

Good. It has good example and teaches well

It is great as it can be seen easily from all seats unlike a whiteboard. Makes it really simple to see how things are done. Keep the tablet PC. It was great

Good. It is easier to read than when written on whiteboard

It is great as it can be seen easily from all seats unlike a whiteboard. Makes it really simple to see how things are done.

it's easier to follow than a whole heap of notes with writing very easy to learn as there is a visual example in front of you

Really good. Easy to follow and easily related back to the notes very useful easy to follow much easier to follow than textbook

Great, as the example were done right in front of us and easy to follow. Very easy to follow. Steps are clear and unambiguous It is easy to learn from

It was an excellent way of teaching, step by step example worked through, more than enough examples. I'd definitely love to see it next year!!!

Maybe with more notes. Pen makes course simple and precise

Writing a bit hard to read sometimes, other than that all good

Is easy to follow when you can read what is being written.

Lecturer doing examples and working trhough whilst explaining is good. Use of lightin in room is also good.

Good to see the lecturer actually doing the examples in front of you and can highlight all the important information.

It is good

The method of "show how to do it" then "do another one" then "let everyone have a go" really works. I left every lecture understanding what was happening a how to do it

Like that in class example, extrta notes etc are stored and available later. Also it is easier to read than a whiteboard good class continuity, eg. not stopping for dead pens, or worse, continuing with a bad pen on a whiteboard Love the pen

Figure 3: Summary of questionnaires for the use of Tablet PC in Mathematics Engineering (n = 35)

Enhancing undergraduate statistical skills within non-statistical disciplines

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In teaching Statistics it is well-understood that conducting and analyzing a well-designed real experiment assists students' comprehension of the corresponding statistical concepts and techniques. When such an experiment is related to a major area of study, it also increases the students' appreciation of the relevance of statistical techniques to their core subjects. Single semester statistical courses are not able to accommodate many statistical methods, or cover all techniques students in any particular discipline may require. Ideally, every experiment conducted in a major course should be supported by some sort of statistical analysis, elementary or advanced. This requires integration of Statistics teaching into other areas of the degree program. This study demonstrates that the cross-disciplinary links between Introductory Statistics and Food Chemistry can be successfully used to enhance students' skills in designing and analysing experiments.

1. Introduction

The modern research in teaching Statistics suggests that the best way to teach students to appreciate the importance of design and analysis of experiments is to let them conduct and discuss a well-designed experiment in an Introductory Statistics course [1, 2]. To be able to effectively apply statistical techniques, students have to understand, to a certain extent, the nature of their experimental data, i.e. the physical and/or chemical processes of the experiment [3]. Similarly, in any statistically designed experiment, incorporated into a non-statistical course, students have to understand the principles of the design of the experiment and the corresponding data analysis [4]. The level of complexity of the analysis in such experiments is thus limited by the students' knowledge in both biophysical and statistical areas [1].

Although the main purpose of an Introductory Statistics course is to serve the core courses, one cannot use experiments from these courses to support teaching Statistics, because such experiments are often too advanced for the students' current level of understanding of biophysical processes. This leads to a situation when university students don't have the ability to carry out an appropriate statistical analysis of a practical experiment [5, 6]. Thus to solve this problem for any undergraduate program, one has to find a way to connect practical problems and statistical methods [7]. One way to go is to utilise simple experiments related to the major disciplines within the Introductory Statistics course and further integrate Statistics into the curriculum in such a way that it re-appears at the time the students are ready to appreciate its value in designing and analysing particular experiments in their core disciplines. This is similar to the method used in teaching Statistics to research scientists [6]. This method allows one to decide what statistical concepts and techniques should be taught at the introductory level and what may be revised and introduced later within the core courses.

One of the main problems with teaching Introductory Statistics is how to determine how much Statistics should be taught [8]. Nowadays, it seems that lecturers in Statistics tend to either select a few of the most fundamental concepts and techniques or introduce as many techniques as their colleagues from major disciplines happened to mention to be useful [9]. The first way leads to students being exposed only to a small amount of statistical methods, not seeing any relevance of Statistics to real life and thus not retaining the knowledge for any reasonable length of time [5, 10]. The second way often leads to increasing the level of statistical anxiety, already known to be high among undergraduates in Applied Science degrees [9]. It is obvious a new way of teaching Applied Statistics to modern undergraduate students not majoring in Mathematics should be developed.

The nature of Applied Statistics is to increase the human ability to understand the information about the real world, encoded in the experimental data. Therefore, the most natural way of teaching Statistics is to make its principles apparent to the students and connect its techniques to experiments. A teaching approach depends on what Applied Science program is to be addressed. The literature on teaching Statistics and Biophysical Sciences suggests that learning-from-action is one of the most suitable teaching/learning strategies for an applied discipline in a modern university [3, 11-13]. The strength of links of Applied Statistics to the major disciplines is crucial to the students' understanding and retention of statistical knowledge and methods [3, 5, 6, 14].

This paper describes a study that has been conducted over Semesters 1 and 2 2005 in the School of Land and Food Sciences at the University of Queensland, Australia. The study aimed to assess the possibility of revising and upgrading students' statistical skills outside the Introductory Statistics course. The background of this study is discussed in the Background section of the paper. The Food Science's experiments conducted in this study are described in the Experiments section. The reflection on students' results and experiences are presented in the Discussion session. Some further developments of this study are given in the Conclusions section. An example of student reports and the set up of one of the experiments are given in appendices.

2. Background

The research team consists of two biometricians and two Food Scientists. In the current curriculum of Bachelor of Applied Science (Food Science and Nutrition), SLAFS, NRAVS, UQ, Introductory Statistics is taught as a one-semester course (36 hours of lectures, 26 hours of tutorials) in the middle of the program (1st semester of the second year) and not supported by any further formal training in statistics. The current demand that graduates have to gain extensive training in a broad range of Food Science and Technology disciplines shifts the teaching focus in the School towards practical courses and industrial placements. Therefore, at the moment, it is not feasible to introduce a separate course on experimental design in the curriculum. Commonly, a core discipline would invoke some application of elementary statistical techniques at the analysis stage of an experiment but not at the design stage. Neglecting the statistical principles of experimental design in the common laboratory practicum often leads to students developing the misconception that Statistics is a computational machine rather than a powerful design tool. This problem reaches its peak in the final year of the undergraduate program, when the students have to design and implement an industrial or laboratory experiment but have very limited time to learn how to approach the design task.

An introduction to design and analysis of experiment is presented in a three-week block (6 hours) in the Research Project course for final year and honours students, almost 2 years after completion of the Introductory Statistics course. The introductory material, presented in the Research Project course, improves the students' understanding of the designs of their specific experiments. However, the students often find it difficult to fully recall the concepts they were taught in the Introductory Statistics course and thus fail to grasp the new techniques they need for their projects. The authors believe that the statistical component of the Research Project course would be more beneficial if the concepts of experimental design were regularly invoked in other courses of the program and the relevant new statistical techniques were taught in non-statistical courses.

In the current curriculum, not every practical course has a statistical component, and those that have merely require students to apply some elementary statistical techniques at the stage of the data analysis. As a result, a large amount of time is wasted by both the supervisor and biometricians in assisting the students with their experimental design and interpretation of data at the completion of the experiment. Another problem that appears to arise from lack of statistical training is that students sometimes struggle to communicate effectively with biometricians as to exactly what they wish to achieve with their experiments.

From our experience in teaching Statistics and Food Chemistry, we believe that the integration of statistics into Food Chemistry is very important as it builds on and refreshes the knowledge gained by students in the Introductory Statistics course. The ability of students to apply basic statistics to chemical data as part of the learning regime in Food Chemistry practical classes means that students gain a hands-on understanding of the importance of statistics to experimental design and analysis. In addition, the integration of statistics into the teaching of Food Chemistry at an early stage helps students understand the science behind the basic Food Chemistry concepts being studied during practical classes.

We decided to introduce a simplified Food Chemistry experiment based on the Maillard reaction to the Introductory Statistics course in Semester 1, 2005, and add a statistical report component to the Maillard reaction practical in the Food Chemistry course in Semester 2. We also decided to add a new statistical method, which the students had to learn independently, to the standard data analysis. We were interested in seeing how much of the statistical material taught in the first semester the students had retained, and how well they would be able to understand and apply the new statistical method. As this work was a pilot research in how to improve teaching statistics in the school, no formal assessment of the efficiency of the methodology was undertaken. The evaluation of the intervention was based on the staff's observations, interviews with students and a small, 5-question, questionnaire administered towards the end of the second semester.

3. Experiments

3.1. Maillard reaction: Food Chemistry experiment in Introductory Statistics (Semester 1)

3.1.1. Time constraints At the time of this experiment (Week 9 of a 13-week course), the students had been taught t-tests and simple ANOVAs and the elementary principles of experimental design (randomization, replication and control of experimental error). The time-frame of the course allowed us to allocate two two-hour tutorials and a one-hour lecture for the course laboratory project: the first tutorial was supervised laboratory work; the second was a computer class and the lecture time allowed students to present their results.

3.1.2. Setup of the experiment The setup of the experiment is discussed in detail in [12]. The design of the experiment was a one-factor completely randomised design, 5 levels, 3 replications each.

3.1.3. Data analysis The experimental data were analysed by using the one-way ANOVA technique with post-experimental multiple comparisons, Tukey's method. The analysis was conducted with Minitab 14.

3.1.4. Work mode The students worked in groups (4 ? 8 students) for two hours in a Food Chemistry laboratory under the supervision of a technician and biometrician. Following the experiment, the students performed the data analysis in Minitab 14 and, in the lecture, a fortnight after the experiment, delivered short oral group presentations. The students worked on the data analysis and presentations in the same group in which they conducted the laboratory experiment. However, the final reports (including Minitab project files) were submitted individually.

3.1.5. Learning objectives The following learning objectives were associated with the experiment:

- Enhance students' understanding of the basic principles of statistical design: replication, randomization and control of experimental error.
- Improve students' experience in handling the standard techniques of one-way ANOVA: computations and discussion of the assumptions (normality, independence and homoscedacity of the experimental error).

- Accentuate that the p-value and coefficient of determination (R2) of the experiment are statistics, i.e. random values that depend on the data observed in an experiment.

3.2. Maillard reaction: Statistical Analysis of Food Chemistry experiment (Semester 2)

3.2.1. Time constraints The experiment was conducted within a standard three-hour practical session of the Food Chemistry course. Additionally, up to two hours consultation time was allocated in the week following the practical to answer students' questions regarding their data analysis as well as to help them revise how to conduct the analysis and generate reports in Minitab. In total, it required the biometricians to allocate 15 hours to the consultations and supervision.

3.2.2. Setup of the experiment The setup of the experiment is discussed in detail in Appendix A. The design of the experiment was a one-factor completely randomised design, 6 levels, 2 replications each.

3.2.3. Data analysis The experimental data were analysed by using the one-way ANOVA technique with post-experimental multiple comparisons, Hsu's method. The analysis and the final report were generated with Minitab 14.

3.2.4. Work mode The students worked individually in small groups (3-4 students) for two hours in a food science laboratory under the supervision of the lecturer in Food Chemistry, a technician and biometrician. The students worked on the data analysis and report presentation individually.

3.2.5. Learning objectives The following learning objectives were associated with the experiment:

- Enhance students' understanding of the basic principles of statistical design: replication, randomization and control of experimental error.
- Revise students' knowledge of one-way ANOVA: computations and discussion of the diagnostics.
- Introduce the students to Hsu's test.
- Upgrade students' skills in using Minitab and presenting the results of statistical analysis.

3.2.6. Student evaluation of the intervention At the end of the second semester, students were asked to give their opinion on how useful they thought it was to have a statistical analysis as a part of the common practicum of FOOD2002; students were also asked to answer the following questions:

- How did you find the revision of experiment in Laboratory practicum 8? (Easy/Relatively easy/Hard)
- Do you think that the knowledge of statistics is important to your degree? (Very important/Useful/Not important)
- In your opinion, what courses of your program would benefit if the design and analysis of experiment was made a routine part of their laboratory practicum? (Please list the courses)
- Did you find doing statistical analysis in laboratory practicum 8 of FOOD2002 was meaningful? (Yes/No, if Yes: It helped me revise my stats/I understood better the errors in the experiment/I learned a new statistical test/ It helped me generate my laboratory report)
- Would you support the idea of introducing more statistical design and analysis into the standard laboratory practicum of FOOD2002? (Yes/No)

4. Discussion

Both experiments met their learning objectives. Experiment 1 increased students' enthusiasm about learning Statistics and improved their understanding of the concepts of randomization, replication and control of experimental error [12]. Students' feedback on the experiment was positive with the main comment being that many statistical techniques became clearer after the experiment. The final exam revealed that there still remained some difficulties in understanding multiple comparisons techniques. In the second intervention, we aimed to improve the students' understanding of multiple post-experimental comparisons. We also wanted to check how much knowledge of the ANOVA's techniques the students retained from the first semester, and how confident they would be about using ANOVA at the end of the second semester.

Having the experience of compiling a formal statistical report in the first semester obviously improved the students' skills in communication to biometricians. In the second semester it was good to see that the students easily followed our explanations about the design and analysis of the experiment. The students remembered the randomization techniques they used in the previous semester and did not object to doing a formal statistical analysis of the experiment. In the end-of-semester questionnaire, 16 out of 19 students answered that it was easy or relatively easy to revise the statistics they learned in the Introductory Statistics course.

We have noticed during the first experiment that some students who did not perform well in statistics easily manipulated the statistical concepts once they put them into a concrete setting. For example, a student who had difficulty in understanding abstract setups of the completely randomised and randomised complete block designs in lectures, commented that the groups' results could be combined to be analysed as a block design as 'each group has completed the whole experiment independently using the same glucose solutions (i.e. treatments)'. It appeared that the level of statistics anxiety decreased once statistics became relevant to some practical exercises. We wanted to see whether we would observe the same phenomenon in Semester 2.

In Semester 2, the students were not anxious about doing statistical analysis although they admitted that they were not able to immediately recall the techniques. They were confident in their ability to recall these techniques quickly once they were able to look up their lecture notes. However, they appreciated the extra help with the report preparation. The students remembered the concepts of hypothesis testing and standard error and learning Hsu's test did not cause any problems; although many students found the explanations provided in the Minitab helper insufficient and confusing.

Although we revised the practicum of FOOD2002 to introduce a proper randomization and replication procedure, we did not change the questions and objectives of this laboratory experiment. As a result, the statistical component was not appreciated by many students as being relevant (in the end-of-semester questionnaire, 11 out of 22 students commented on the lack of relevance of the statistical analysis to the practical questions). Nevertheless, 13 out of 25 students supported the idea of introducing more statistical analysis into the standard practicum on the condition that the analysis is to be made interesting and meaningful.

These experiments provided useful information that will be used to improve the efficiency of teaching Statistics in both the Introductory Statistics and Food Chemistry courses.

5. Conclusions

To enhance statistical skills of undergraduate students in an Applied Science program, the experimental practicum in a major discipline has to include a proper discussion of design of an experiment and require an appropriate statistical analysis of the experimental data whenever feasible. By including appropriate statistical designs and analyses in undergraduate experiments students are required to put into practice the concepts and techniques learnt in lectures in Statistics. This not only reinforces these concepts but demonstrates that statistics have a useful place in both experimental design and interpretation of results.

From the two experiments conducted in this study with the Food Science students this year it is clear they have gained a better understanding of Statistics. However, more importantly the students were able to more easily recall statistical concepts and effectively apply them to a practical situation. It is obvious that by expanding the above approach within the Food Science and Nutrition program undergraduate students will become more adept at applying and handling statistics in their experimental work in the future.

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Appendix A – Setup of the Laboratory Experiment in Semester 2

Appendix B – Student statistical reports of the experiments

The appendices for this paper are available at

http://www.maths.uq.edu.au/delta05/files/kravchuk.pdf

Supporting mathematics learning of engineering students: the HELM project

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HELM (Helping Engineers Learn Mathematics) was a three-year curriculum development project funded by the Higher Education Funding Council for England. The HELM project sought to address difficulties experienced by engineering students learning mathematics due to problems with working from textbooks, making useful notes from lectures, weak motivation due to lack of perceived relevance and lack of sustained application. This paper describes the HELM learning resources and assessment regime for both formative and summative assessment of engineering students learning mathematics.

1. Introduction

The HELM (Helping Engineers Learn Mathematics) project [1] has developed a strategy for addressing some of the practical difficulties currently encountered in the teaching of mathematics to UK engineering undergraduates which have been widely reported e.g. [2]. Specific problems have been the difficulty experienced by many students in working from standard textbooks, in constructing accurate, comprehensible notes from lectures, in seeing the relevance of their mathematical studies to their engineering courses, and being motivated to keep working on the development of their mathematical skills throughout the academic year rather than just 'cramming' for examinations. The HELM strategy has comprised the production and dissemination of high quality extensive teaching and learning materials supported by a comprehensive Computer Aided Assessment (CAA) testing regime. Historically the HELM project has developed its resource based on the earlier work of Loughborough's Open Learning Mathematics Project (OLMP) which developed learning materials in the period 1996-2000. The success of the OLMP encouraged staff to seek external funding to develop further this work, which resulted in the HELM project which was supported by a 250,000 FDTL grant from the Higher Education Funding Council for England for the three-year period Oct 2002-Sept 2005.

2. An overview of the HELM project

The HELM team consisted of staff at five UK universities: Loughborough (the lead institution), Hull, Manchester, Reading and Sunderland. The project's aim was to improve and greatly extend Loughborough's original OLMP materials, and thoroughly trial all resulting materials. This was achieved by

- writing additional Workbooks
- incorporating engineering examples in many Workbooks utilising the mathematics presented
- including a Workbook of engineering case studies illustrating mathematical concepts
- extending and improving the CAA question databanks
- promoting widespread trialling.

The HELM project's output consists of Workbooks, Interactive Lessons and Revision Questions segments, and a CAA Regime which is used to help 'drive' the student learning. (To view sample materials visit the HELM website at http://helm.lboro.ac.uk.) With the emphasis on flexibility, the Workbooks may be integrated into degree courses either by selecting units to complement other materials, or by creating a complete scheme of work for a semester, a year or two years. The Workbooks may be used to support lectures or for independent learning, or a mixture of the two.

Interactive Lessons and Revision Questions segments are provided for topics covered in typical UK first year undergraduate engineering mathematics courses, covered by twenty of the Workbooks.

The banks of CAA questions developed by the HELM project contain approximately 5000 questions. Original questions are held in Question Mark Perception version 3.4 format.

3. The HELM project workbooks

The main project deliverables are the Workbooks which are subdivided typically into four or five Sections, each begining with statements of prerequisites and desired learning outcomes. As far as possible, each Section is designed to be a self-contained piece of work that can be completed in a few hours. In general, a whole Workbook represents about two to three weeks' study. Each Section consists of an introduction and the presentation of mathematical concepts, simply explained, interspersed with Examples, Tasks and Exercises. The Examples are mostly purely mathematical but some are presented in an engineering context. A fully worked solution immediately follows each Example (see Figure 1).

Included in each Section are student Tasks (see Figure 2) which include space for students to write their working and answers, and, where appropriate, guide them through problems in stages (see Figure 3).

Additional Exercises are included, usually at the ends of Sections, with answers but usually without worked solutions.

HELM has the following Workbooks and Guides:

- 34 Workbooks covering undergraduate engineering mathematics.
- 12 Workbooks covering undergraduate engineering probability and statistics.
- 1 Workbook of Engineering Case Studies ranging over many engineering disciplines.
- 1 Workbook of miscellaneous topics including Dimensional Analysis and Physics Case Studies
- 1 Student's Guide providing advice, commentary, formulae and a comprehensive index.
- 1 Tutor's Guide, providing commentary on each Workbook and associated CAL and CAA resources, relating good practice derived from extensive trialling.

Workbooks vary in length from around 24 to 100 pages, with a median of around 40 pages, available in hardcopy and electronic formats. A list of all Workbooks is given in Appendix 1. Engineering and Physics Case Studies are included in two dedicated Workbooks (see Appendix 2 for an example).

4. The HELM project CAL materials

The project has 80 Interactive Lessons and Revision Questions segments written using Authorware [3]). These link to the more elementary Workbooks, typically covering Year 1 mathematics of an undergraduate engineering degree. These segments are webbased versions of parts of the Workbooks containing interactivity, audio and animations which can enhance student interest. Revision exercises with randomly generated questions are provided. These segments have been found to be especially useful for supporting students of moderate mathematical ability, for revision and for illustrating lectures.

5. The HELM project assessment regime

5.1. CAA

Assessment is normally an integral part of learning, and this was recognised by the HELM project. The HELM assessment strategy is based on using CAA to encourage formative self-assessment (which many students neglect) to verify that the appropriate skills have been learned. The project's philosophy was that assessment should be at the core of any learning and teaching strategy.

HELM provides an integrated web-delivered CAA regime for both self-testing and formal assessment. Students at Loughborough University are typically tested four or five times each semester with questions delivered over the web. Students are encouraged to engage in their own learning by allowing them unlimited practice tests before taking a one-attempt summative test worth from 5% to 10% of the module mark. Students are motivated in their studies, improving achievement and progression.

The adoption of QM Perception (QMP) [4] at Loughborough University has enabled delivery of tests to large numbers of students over the web since October 2000. Web-delivered CAA is a convenient method of delivery, and Loughborough has also developed an alternative implementation based on CDs. All of the CAA questions and associated feedback fit onto one CD which also holds a number of predetermined tests using those questions. Students provided with such a CD can do the required work and complete the tests on the CD, without an internet connection. This is very easy to implement if only formative self-testing is required; formal summative testing would be more challenging to implement.

HELM question banks are available in a QTI XML format [5] and can be used with computer based assessment systems other than QMP thus enhancing the transferability of the HELM CAA resources.

5.2. Underlying structure of HELM CAA questions

HELM CAA questions have been designed to match particular mathematical concepts in support of the topics covered by the Workbooks. The questions relevant to each mathematical (or statistical) concept have been structured into two sets, one nominally designated formative the other summative. In almost all cases, each of the two sets contains 10 questions cloned from a designated master question, thereby ensuring a comparable level of difficulty is maintained, and justifying the random selection of questions from each set for test purposes. Several mathematical concepts may be selected and appropriate questions chosen randomly and presented as a customised test.

In many cases feedback to students shows the specific worked solution or an example solution, while in simpler cases a generic solution may be presented. The importance of providing specific feedback for the benefit of the weaker learner was evident from student comments.

5.3. HELM numeric input CAA questions

The simplest response required to a particular CAA question is the input of a numerical value, which may be either a whole number or a decimal. An advantage of numeric entry is that it is simple to construct and allows for the easy generation of clones with which to populate the relevant question library and it is unlikely that learners will be able to guess the correct response. However, there are disadvantages too. It is a common occurrence for a learner to understand all of the mathematics related to a question but simply to fail to round their numeric answer correctly!

Initially marks were awarded for precise answers only, in the belief that it was important that engineering undergraduates understood the need for precision. However, feedback from students indicated that this policy was of concern to them, particularly when taking summative tests,. For example:

> "With the CAA tests, being off the answers by 0.01 can result in your answer be-

> ing incorrect, causing you to lose a lot of marks even though your method and approach are correct."

To address this problem responses are acceptable within a tolerance where rounding is required, for example, 0.01 for questions requiring 2 d.p. accuracy. Learners are alerted to this in the feedback in casses where they appear to have rounded incorrectly.

5.4. HELM multi-input CAA questions

Single numeric input for CAA questions has limitations with certain mathematical concepts. For example: in the case of complex numbers, it is desirable to check both real and imaginary parts of the answer so marks can be given according to the accuracy of each component; and, where there are two complex numbers involved, such as asking for the roots of a quadratic equation, it may be useful to mark individually the real and imaginary parts of each complex number and that means allowing for entry of four separate numbers as the answer. Figure 4 shows an example of a HELM multi-input question.

Multi-input type questions can also be used in place of existing MCQs to eliminate the associated guesswork element. For example, if a particular cubic interpolating polynomial is the expected answer, instead of presenting (say) five possible cubics using an MCQ approach, four numeric inputs can be requested representing the coefficients of the terms of the cubic (with appropriate rounding tolearance).

With multi-input questions, the feedback given to the student can be designed to indicate which components of the answers were correct (and marks can be allocated accordingly). However, setting conditions for computerised marking becomes difficult compared to single input numeric entry type. Unless the input areas are specifically labelled, students might input the answers in the boxes in any order, and automated marking conditions have to be set keeping this in mind.

5.5. HELM multi-stage CAA questions

A disadvantage inherent in single stage questions (where the final answer only is expected) is that a wrong response does not give credit for any correct work that might have been done by learners prior to submission of their answer. In some questions the loss of all credit seems very unfair, and was an issue commented upon by students:

> "The computer tests should be developed so that marks for workings can be given. Currently it is very easy to obtain low marks, despite having a good grasp of the subject."

In an attempt to address this situation some questions were written in a Multi-stage format, whereby partial credit is given for a correct response at each of several stages within a question. The learner, having perhaps submitted an incorrect answer at an intermediate stage, is subsequently presented with sufficient information at the commencement of the next stage to allow him or her to continue, thus giving the opportunity to gain partial credit within a complex question.

5.6. Assessment

In a typical HELM-based TLA regime, students are given a Workbook for a mathematics topic for two weeks' work. followed by a summative CAA test in the third week. A practice version of the summative test is available on the web for the week before the summative test which students may access at any time within this period anywhere there is an internet connection. It can be practised as often as the student desires. Many students work in small groups sorting out difficulties with the practice test and some seek help from support staff. This is a valuable learning mechanism and the evidence is that students betetr engage with the learning process at some level throughout the semester. The summative test is then available for two days, which students may access at anytime within this period and from anywhere. However, they are only allowed to take this test once.

The practice and summative tests have identical form, questions being selected randomly from previously created question banks (one practice, one summative) covering aspects of the topic just covered in lectures. Analysis of student logs has shown intensive activity during this practice test period. Feedback from students has demonstrated how much this aspect of the assessment is appreciated.

Evaluation exercises have shown that this testing regime is generally popular with both staff and students. Students particularly like the flexibility this method of assessment offers.

A legitimate concern is formal tests being unsupervised. There are benefits to the students to allow this but feedback indicates that at least some supervised tests would be preferable. This has resource implications for the academic or institution.

6. Trialling and evaluation

Over sixty individual academics from higher and further education institutions contacted the project with a view to trialling learning and assessment resources. Some wished to use the resources as a replacement for their current approach to teaching mathematics to engineers whilst others wished to have access to an additional resource to which they can direct students having difficulties coping with their mathematics.

The evaluation of the learning resources involved a number of activities. The first stage of the process was to assess the accuracy of the material in the resources and the appropriateness of the level to the audience being addressed, which was achieved through the use of 'critical readers', drawn from members of the consortium and triallists. This feedback was collated and often substantial changes made by the HELM team.

All triallists were requested to complete an evaluation questionnaire at the end of each semester or teaching period, in order to establish who was actively using the materials, which resources were being used, and to obtain comments about the Workbooks and students' reactions to them. The major advantage of the Workbooks was seen as the delivery of a clear and complete set of notes to students, which allows students to focus on understanding during lectures rather than furiously scribbling notes. The major advantage of the CAA, where used, was seen as the imposition on students of a structured learning regime with regular assessment and feedback.

All triallists were regularly contacted by email and telephone and a number were visited by a member of the HELM team to discover how the resources were being used and to gather detailed student feedback. Where possible, focus groups of approximately 4 or 5 students were held. Of two such focus groups conducted in a institution, one group was not required to have studied mathematics past GCSE level (age 16) to qualify for entry to their course whereas the other group was required to have achieved at least a grade C at A level (age 18). It was interesting to note that the groups' responses were very similar. Both groups praised the standard of the Workbooks, with a few suggestions made to improve their layout and increase the number of worked examples and exercises. Both groups found the CAA regime helped to focus their attention on the material being covered and thought that this approach should be extended to engineering modules. Some of the students even proposed that the testing should be even more frequent as they admitted that they did not open their Workbooks until a day or two before the tests, thus confirming the cynical academic's view of student motivation!

7. Summary and conclusions

HELM has provided a flexible learning resource: workbooks available in web-based or paper-based forms; web-based CAL segments; an assessment regime which enables regular testing to drive the student's learning. The large CAA question bank, which can be used for formative and summative assessments, is available either web-delivered or on CD.

The large number of participating UK triallists (33 further and higher education institutions), and enquiries from abroad, bear testimony to the usefulness of the HELM materials. Most triallists have indicated their intention to continue to use HELM materials, citing improved motivation and in some cases, improved scores in written examinations. The grant of a further 25,000 from HEFCE to aid transferability during 2005-6 means that the future looks bright for the uptake of HELM resources.

Acknowledgments

The Higher Education Funding Council for England (HEFCE) for support through the Fund for the Development of Teaching and Learning phase 4 (FDTL4).

References

[1] HELM Project: http://helm.lboro.ac.uk/

[2] "Making Mathematics Count", Report of Professor Adrian Smith's Inquiry into Post-14 Mathematics Education, February 2004, DfES publications: http://www.dfes.gov.uk/index.shtml

[3] http://www.macromedia.com/sofware/authorware/

[4] http://www.qmark.com/

[5] http://www.imsglobal.org/question/index.cfm

Engineering Example 1

Water wheel efficiency

A water wheel is constructed with symmetrical curved vanes of angle of curvature θ. Assuming that friction can be taken as negligible, the efficiency, η, i.e. the ratio of output power to input power, is calculated as

$$\eta = \frac{2(V - v)(1 + \cos\theta)v}{V^2}$$

where V is the velocity of the jet of water as it strikes the vane, v is the velocity of the vane in the direction of the jet and θ is constant. Find the ratio, v/V, which gives maximum efficiency and find the maximum efficiency.

Solution

Mathematical statement of the problem

We need to express the efficiency in terms of a single variable so that we can find the maximum value.

$$\text{Efficiency} = \frac{2(V - v)(1 + \cos\theta)v}{V^2} = 2\left(1 - \frac{v}{V}\right)\frac{v}{V}(1 + \cos\theta).$$

Let $\eta =$ Efficiency and $x = \frac{v}{V}$ then $\eta = 2x(1 - x)(1 + \cos\theta)$.

We must find the value of x which maximises η and we must find the maximum value of η. To do this we differentiate η with respect to x and solve $\dfrac{d\eta}{dx} = 0$ in order to find the stationary points.

Mathematical analysis

Now $\eta = 2x(1 - x)(1 + \cos\theta) = (2x - 2x^2)(1 + \cos\theta)$

So $\dfrac{d\eta}{dx} = (2 - 4x)(1 + \cos\theta)$

Now $\dfrac{d\eta}{dx} = 0 \Rightarrow 2 - 4x = 0 \Rightarrow x = \frac{1}{2}$ and the value of η when $x = \frac{1}{2}$ is

$$\eta = 2\left(\frac{1}{2}\right)\left(1 - \frac{1}{2}\right)(1 + \cos\theta) = \frac{1}{2}(1 + \cos\theta).$$

This is clearly a maximum not a minimum, but to check we calculate $\dfrac{\mathrm{d}^2\eta}{\mathrm{d}x^2} = -4(1 + \cos\theta)$ which is negative which provides confirmation.

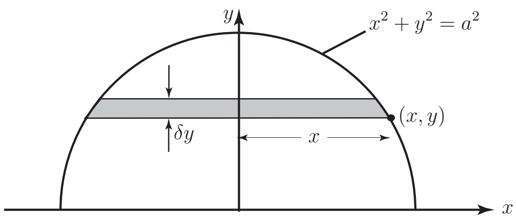
Interpretation

Maximum efficiency occurs when $\dfrac{v}{V} = \frac{1}{2}$ and the maximum efficiency is given by

$$\eta = \frac{1}{2}(1 + \cos\theta).$$

Figure 1: A typical engineering example with worked solution

Find the position of the centre of mass of a uniform semi-circular lamina of radius a, shown below.



A typical horizontal strip is shaded.

The equation of a circle centre the origin, and of radius a is $x^2 + y^2 = a^2$.

By symmetry $\bar{x} = 0$. However it is necessary to calculate $\bar{y}$.

The lamina is divided into a number of horizontal strips and a typical strip is shown. Assume that each strip is rectangular. Writing the mass per unit area as ρ, state the area and the mass of the strip:

Figure 2: A typical task

Your solution

Answer

$2x\delta y, \qquad 2x\rho\delta y$

Write down the moment of the mass about the x-axis:

Your solution

Answer

$2x\rho y\delta y$

Write down the expression representing the sum of the moments of all strips and the corresponding integral obtained as $\delta y \to 0$:

Your solution

Answer

$\sum_{y=0}^{y=a} 2x\rho y\delta y, \qquad \int_0^a 2x\rho y \mathrm{d}y$

Now since $x^2 + y^2 = a^2$ we have $x = \sqrt{a^2 - y^2}$ and the integral becomes:

$$\int_0^a 2\rho y\sqrt{a^2 - y^2}\mathrm{d}y$$

Evaluate this integral by making the substitution $u^2 = a^2 - y^2$ to obtain the total moment.

Your solution

Answer

$\frac{2\rho a^3}{3}$

The total area is half that of a circle of radius a, that is $\frac{1}{2}\pi a^2$. The total mass is $\frac{1}{2}\pi a^2\rho$ and its moment is $\frac{1}{2}\pi a^2\rho\bar{y}$.

Hence

$$\frac{1}{2}\pi a^2\rho\bar{y} = \frac{2\rho a^3}{3} \qquad \text{from which} \quad \bar{y} = \frac{4a}{3\pi}$$

Figure 3: Layout for the student's step by step solution of the task

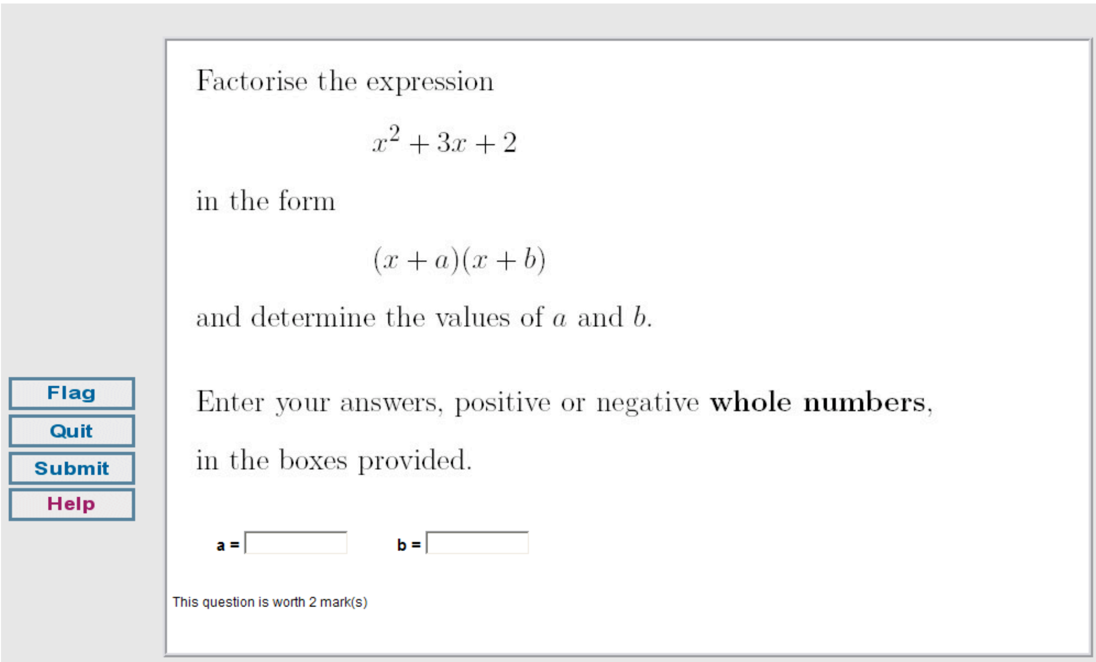


Figure 4: Helm CAA question requiring two numeric inputs (in either order)

Appendix 1 - The HELM Workbooks

No.	Title	No.	Title
1	Basic Algebra	26	Functions of a Complex Variable
2	Basic Functions	27	Multiple Integration
3	Equations, Inequalities and Partial Fractions	28	Differential Vector Calculus
4	Trigonometry	29	Integral Vector Calculus
5	Functions and Modelling	30	Introduction to Numerical Methods
6	Exponential and Logarithmic Functions	31	Numerical Methods of Approximation
7	Matrices	32	Numerical Initial Value Problems
8	Matrix Solution of Equations	33	Numerical Boundary Value Problems
9	Vectors	34	Modelling Motion
10	Complex Numbers	35	Sets and Probability
11	Differentiation	36	Descriptive Statistics
12	Applications of Differentiation	37	Discrete Probability Distributions
13	Integration	38	Continuous Probability Distributions
14	Applications of Integration 1	39	The Normal Distribution
15	Applications of Integration 2	40	Sampling Distributions and Estimation
16	Sequences and Series	41	Hypothesis Testing
17	Conics and Polar Coordinates	42	Goodness of Fit and Contingency Tables
18	Functions of Several Variables	43	Regression and Correlation
19	Differential Equations	44	Analysis of Variance
20	Laplace Transforms	45	Non-parametric Statistics
21	Z Transforms	46	Reliability and Quality Control
22	Eigenvalues and Eigenvectors	47	Mathematics and Physics Miscellany
23	Fourier Series	48	Engineering Case Studies
24	Fourier Transforms	49	Student's Guide
25	Partial Differential Equations	50	Tutor's Guide

Appendix 2 - Example of an Engineering Case Study

Engineering Case Study 2

COMPLEX REPRESENTATION OF SOUND WAVES AND SOUND REFLECTION

Mathematical Skills

Trigonometry	[4]
Complex Numbers	[10]

Introduction

Complex exponential expressions turn out often to be more convenient to handle than the trigonometric functions (sine and cosine) to represent sound waves and their interaction with surfaces. The time dependence in a single frequency sound wave may be written

$$\mathrm{e}^{-\mathrm{i}\omega t} = \cos(\omega t) - \mathrm{i}\sin(\omega t)$$

So $\mathrm{Re}[\mathrm{e}^{-\mathrm{i}\omega t}] = \cos\omega t$

If A is real then, $\mathrm{Re}[A\mathrm{e}^{-\mathrm{i}\omega t}] = A\cos\omega t$.

Similarly, using $\mathrm{e}^{-\mathrm{i}(\omega t+\phi)} = \cos(\omega t + \phi) - \mathrm{i}\sin(\omega t + \phi)$ then

$$\mathrm{Re}[A\mathrm{e}^{-\mathrm{i}(\omega t+\phi)}] = A\cos(\omega t + \phi)$$

The angle ϕ is called the **phase** of the sound wave and determines the amplitude when $t = 0$, i.e. $A\cos\phi$. A **plane sound wave** is a wave in which the wave fronts (contours of equal phase) are plane and parallel. If a plane sound wave is reflected at the boundary between two media, then, in general, the reflected wave has different amplitude and phase from the incident wave. At the boundary, the incident wave with amplitude A_i may be represented by the complex exponential expression

$$p_i = A_i\mathrm{e}^{-\mathrm{i}\omega t}$$

The reflected wave may be represented by the complex exponential expression

$$p_r = A_r\mathrm{e}^{\mathrm{i}(\omega t+\phi)}.$$

The reflected wave has the same angular frequency ω, amplitude A_r and differs in phase from the incident wave by the phase angle ϕ.
The ratio of the reflected wave to the incident wave (p_i/p_r) is called the **reflection coefficient** (R) of the boundary and is a *complex number*. It depends on the physical properties of the two media either side of the boundary.
The complex plane wave reflection coefficient is given by

$$R = \frac{p_r}{p_i} = \frac{A_r\mathrm{e}^{-\mathrm{i}(\omega t+\phi)}}{A_i\mathrm{e}^{-\mathrm{i}\omega t}} = \frac{A_r}{A_i}\mathrm{e}^{-\mathrm{i}\phi}$$

If the reflection coefficient of a boundary is written as $a + \mathrm{i}b$, i.e. a is the real part and b is the imaginary part, then

$$a + \mathrm{i}b = \frac{A_r}{A_i}\mathrm{e}^{-\mathrm{i}\phi} = \frac{A_r}{A_i}(\cos\phi - \mathrm{i}\sin\phi)$$

Hence $\quad a = \frac{A_r}{A_i}\cos\phi$ and $b = -\frac{A_r}{A_i}\sin\phi$

Dividing the second of these expressions by the first, gives

$$\tan\phi = -\frac{b}{a} \quad \text{or} \quad \arctan(-\frac{b}{a}) = \phi \tag{1}$$

Also

$$|R| = \sqrt{a^2+b^2} = \sqrt{\frac{A_r^2}{A_i^2}\cos^2\phi + \frac{A_r^2}{A_i^2}\sin^2\phi} = \frac{A_r}{A_i} \tag{2}$$

Equations (1) and (2) relate the complex reflection coefficient phase and amplitude to its real and imaginary parts.

Problem in words

For a particular boundary, the reflection coefficient is $0.8 - 0.4\text{i}$:

(a) Find the ratio of the reflected amplitude to the incident amplitude;

(b) Find the phase difference between the reflected wave and the incident wave

(c) What is implied about the phase shift at a surface if the reflection coefficient is $+1$ or -1?

Mathematical statement of problem

(a) Use equation (1) to determine the reflection coefficient amplitude;

(b) Use equation (2) to determine the phase;

(c) Use equations (1) and (2) to determine corresponding phase changes.

Mathematical analysis

(a) $|R| = |0.8 - 0.4\text{i}| = \sqrt{(0.8)^2 + (0.4)^2} = \sqrt{0.8} \approx 0.8944$

(b) $\phi = \arctan(-\frac{b}{a}) = \arctan(\frac{0.4}{0.8}) = \arctan(0.5)$

Hence $\phi \approx 0.4636$ radians $\approx 26.57°$

(c) If $|R| = +1$ then $b = 0$. So $\phi = \arctan(-\frac{b}{a}) = \arctan(0) = 0$

If $|R| = -1 = \text{e}^{\text{i}\pi}$ then $\phi = \pi$

Interpretation

(a) The magnitude of the reflection coefficient is 0.89. This means that the amplitude of the reflected wave is 0.89 of the incident wave.

(b) There is a phase change of $26.57°$ at the boundary

(c) $\phi = 0$ means zero phase change at the boundary. A boundary at which there is no change in amplitude or phase on reflection is called **acoustically hard**.

$\phi = \pi$ means that there is $180°$ phase change at the boundary. A boundary at which the reflected wave is $180°$ out of phase with the incident wave but there is no change in amplitude is called a **pressure release boundary**.

Reforming business mathematics: an attitude adjustment

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We report on a project intended to reform the mathematics courses taught by our mathematics department for students obtaining baccalaureate degrees from the College of Business Administration. This project requires no significant course, class, or content restructuring. Nor does it require additional resources: physical, technical, or personnel. All that is required is a change of focus away from the traditional emphasis on mathematical formalism and towards an emphasis on applications in the context of the business world. We discuss the impetus for our reform project and briefly describe its implementation, providing examples of the pre and post reform expected outcomes in the form of sample final exam questions. We then attempt to assess the impact of our project on student success rates by comparing grade data for the three years prior to its implementation to that for the three years after the implementation of the project

As is typical at most large, state-supported, multipurpose US universities, all students pursuing baccalaureate degrees from the College of Business Administration (COBA) at our university are required to successfully complete (with a grade of A, B, or C) a two course sequence of mathematics courses taught by the Department of Mathematics and Statistics, which is in the College of Arts and Sciences. The first course typically includes assorted topics under the general heading of 'Finite Mathematics,' while the second course includes topics from differential and integral calculus. Typically the same text book is used for both courses and contains the phrase 'for business, economics, life, and social sciences' in its title. Our reform of the first course, designated as MATH 1330 at our university, is the topic of this report.

1. Impetus for reform

In the summer of 2001 our university sent a team consisting of representatives from the Department of Mathematics and Statistics, the Office of the Provost, and COBA to participate in the American Association of Higher Education (AAHE) Sixth Annual Summer Academy [1] held at Breckenridge, Colorado. In keeping with the general goal of the AAHE Summer Academy program [2],

> ... provide an environment rich in ideas, examples, and conversation within which motivated academic leaders can create a new vision of learning for themselves, their colleagues, and their students, as well as develop strategies for action upon return to campus.

The team adopted as its project the development of 'a replicable model for redesigning high enrollment undergraduate classes.' The team chose MATH 1330 as the particular focus for the project.

The model developed by the team would consist of large-section lectures conducted by experienced faculty, and supported by a myriad of custom on-line resources and computer-laboratory learning centers. The curriculum would focus on 'a broad spectrum of modern business mathematics topics,' which would 'enhance significantly the relevance of the course content' and increase the 'level of student interest, involvement, and interaction.' There would be less dependence on the 'traditional lecture format' and more on student reading, exploration, and problem solving. Armed with this exciting model, the team returned from the mountain top to our flatland reality: 1611 students enrolled in MATH 1330 for the fall semester 2001; classrooms with maximum seating capacity of 40; 80 mathematics graduate teaching assistants (GTAs), most with little or no college level teaching experience, but all having to be supported by the department; two mathematics computer labs with 30 workstations each; and a full time faculty of 48 (plus 10 adjuncts) stretched thin trying to cover all the undergraduate (including MATH 1330) and graduate courses in which over 7000 students were enrolled. The team's response was predictable; they asked the Director of Undergraduate Programs (DUP) for the Department of Mathematics and Statistics (this author) to initiate a project aimed at reforming MATH 1330.

After considering the proposed model in the context of our reality, we decided to focus our reform efforts on just the issue of course relevance, leaving all other issues to be addressed by some later DUP. We now turn to the project itself.

2. The reform project

We decided to reform MATH 1330 in the fall of 2001 as the course proceeded. To this end the DUP was assigned as the fall 2001 course coordinator for MATH 1330, teaching one section and coordinating the nineteen GTAs and three adjuncts who were assigned to teach the other 46 sections. Using a fulltime instructor to teach one section and coordinate the GTAs and adjuncts who teach all the other sections is the model for all freshman-level mathematics courses taught in out department. Course coordinators meet with their assigned instructors once prior to the start of the term and then regularly (usually for one to two hours every two weeks) throughout the term (about 15 weeks). For MATH 1330 we met weekly in sessions that lasted from two to three hours each. During these sessions we would discuss, often intensely, exactly what material to cover during the next week and how it should be covered.

All discussions were guided by the single project objective: *change of focus away from the traditional emphasis on mathematical formalism and towards an emphasis on applications in the context of the business world*. There were two basic ground rules.

1. Assume the students have the prerequisite knowledge and skills coming into MATH 1330. For example, they are familiar with the Cartesian coordinate system and know how to derive the equation of a line given two points. They know what the slope of a line is, how to graph lines, and how to find the intersection point of two lines. They know how to graph quadratic polynomials and find the roots of them. And they have been introduced to logs and exponentials and the basic properties thereof.

2. When beginning a new section go immediately to the 'Applications' section at the end of the exercise set and determine the mathematical skills and concepts needed for the applications. Then cover that mathematics using the applications as motivation, and assign all the application exercises to be done for the next class.

These two tenets were the heart and soul of our reform project. Their use is best demonstrated via a specific example. The text we were using was *College Mathematics for Business, Economics, Life Sciences, and Social Sciences* by Barnett and Ziegler [3]. Section 3 of chapter 2 is called 'Logarithmic Functions' and covers pages 107-119. The applications section of the associated exercises starts with exercise 97 on page 118. A typical such application problem is the following:

> Investment. A newly married couple wishes to have \$20000 in 8 years for the down payment on a house. At what rate of interest compounded continuously (to three decimal places) must \$10000 be invested now to accomplish this goal?

The only concept from section 3 of chapter 2 needed to solve this problem is that of the natural log, $\ln(x)$, and the only property needed is $\ln(x^p) = p\ln(x)$. Yet the text devotes seven pages to what we call *traditional mathematical formalism*, the discussion of logs to different bases and properties thereof (to which the first 96 problems of the exercise set are devoted). The GTAs and adjuncts were instructed to simply remind their students of the definition of $\ln(x)$; namely,

$y = \ln(x)$ means $e^y = x$ (the number e to several decimal places arose naturally in the context of compound interest calculations in the previous section). Then remind them of the above exponential property and proceed to show them how to solve some of the application problems and assign the remainder. The whole section should take at most 20 minutes of class time.

The remainder of the term proceeded in this way with a section by section discussion of each or the first six chapters of the text. The only exceptions to this routine occurred at the three times during the semester when the instructors were expected to give midterm exams. Each instructor was (and still is) responsible for writing and administering midterm exams to her, or his, classes. For the purpose of our project, the GTAs and adjuncts were required to present their proposed midterm exams to the group for discussion and the coordinator for approval. The topic for discussion and criterion for approvability was always the extent to which the questions on the exam were consistent with the project's guiding objective. We should mention that the final exams in all our freshman level courses are uniform exams written by the respective course coordinator and administered to all sections of the respective course. The DUP had clearly stated at the beginning of this project that the final exam for MATH 1330 would be 100% application problems. (Examples are provided below.)

When the discussion turned to chapters 4 and 5 (Systems of Linear Equations; Matrices, and Linear Inequalities and Linear Programming) it became clear to the DUP and other instructors that the material simply degenerated into algorithms and formalism with little likelihood of having much meaning to the students of MATH 1330. The decision was made to simply omit chapters 4 and 5 from the course material and replace them with Chapter 8: Markov Chains, with the understanding that the students could use their calculators to compute the necessary matrix multiplications.

In the next section we discuss the changes that occurred in MATH 1330 as a consequence of this project. We present comparative pre and post project examples to demonstrate changes in instructional focus, as well as expected student outcomes as exhibited by the uniform final exam questions.

3. Pre and post project examples

At the beginning of each term all instructors in our department are presented with a handbook containing, among other information, a brief description of the content that should be covered in each lower division, multi-section course. The content for MATH 1330 according to the 2001, and earlier, handbook was the following :

Chapter 1 (Elementary functions) .... 5 periods
Chapter 2 (Additional elementary functions) .... 4 periods
Chapter 3 (Mathematics of Finance) .... 4 periods
Chapter 4 (Systems of Linear Equations: Matrices) omit section 7 .... 7 periods
Chapter 5 (Linear Programming) omit sections 36 .... 2 periods
Chapter 6 (Probability) .... 6 periods

As the project progressed, the DUP revised the handbook content description and had a completely new handbook description ready for the spring 2002 instructors of MATH 1330. The revised description is too long to be included in total in this report. Recall that chapters 4 and 5 were replaced with chapter 8 on Markov chains. Here are the revised content descriptions from chapters 1, and 3.

Chapter 1.
- Piecewise linear functions
- Cost function= fixed cost + variable cost
- Linear functions representing price-supply and price-demand
  - Equilibrium
- Quadratic functions representing revenue and profit
- Breakeven analysis

Chapter 3. (all required formulae provided)
- Simple and compound interest
  - Future value
  - Present value
  - Doubling time
- Future value of an annuity ("savings plans")
  - Periodic investments (Payments)
- Present value of an annuity
  - Periodic payments (amortization)
  - Interest calculations

Notice the emphasis on the use of terminology from Economics and Math of Finance and lack of emphasis on time periods.

Assuming the uniform final exam should indicate what the students are expected to bring out of the class, we now provide some representative examples from the final exam for fall 2000 and fall 2001 (the exam from the project). The following problems are taken directly from the fall 2000 MATH 1330 final exam and are representative of 58% of the problems on the final:

- Sketch a graph of $2x - 3y = 15$. What are the intercepts and slope of the line?
- Find the coordinates of the vertex and the maximum or minimum values of the function given by $F(x) = 2x^2 - 8x + 5$.
- Solve for x: a) $9^{2x-3} = 3^{x+1}$ b) $\log 3x^2 = 2 + \log 9x$

Only two of the fourteen problems on the exam were application problems related to finance or economics.

The final exam for MATH 1330 for the fall 2001 consisted of 10 applications problems, each with format very similar to the following two:

> **Supply and Demand. (Equilibrium point)** At a price of \$2.50 per bushel, the annual U.S. supply and demand for corn are 8.5 and 9.8 million bushels, respectively. When the price rises to \$3.30, the supply increases to 10.5 million bushels while the demand decreases to 7.8 million bushels. Let x represent the price per bushel and assume the price-supply and the price-demand are linear functions of x.
>
> A. Find the equation for supply as a function of x.
> B. Find the equation for demand as a function of x.
> C. What is the price per bushel that causes the supply to equal the demand?
> D. How many bushels are sold at the price per bushel found in part C?
>
> **Markov Process.** The 1990 census reported that 58% of the households in Alaska were homeowners and the remainder were renters. During the next decade 2% of the homeowners became renters and the rest continued to be homeowners. Similarly 23% of the renters became homeowners and the rest continued to rent.
>
> A. Draw a transition diagram.
> B. Write the transition matrix.
> C. What percentage of the households were homeowners in 2000?
> D. If this trend remains valid over the long term what percentage of the households should we expect to be homeowners in the long run?

E. Explain how you arrived at your conclusion in part D.

4. Sustaining the reform

The DUP had no interest in becoming the perpetual course coordinator for MATH 1330. In order to sustain the reform movement, the DUP carefully recruited the course coordinators for MATH 1330 for all subsequent spring and fall terms. Each coordinator must understand and embrace the guiding objective, and be willing to sell it to the other instructors. The DUP monitored the final exams for each subsequent term and had follow-up conversations with each subsequent course coordinator. While the exams vary somewhat from term to term, the DUP calculates that each consisted of 75% to 100% application problems similar to the two given above. He believes the reform is still alive and is confident that it is gradually becoming part of the culture of MATH 1330.

All faculty involved with MATH 1330 believe it to be a much richer course now than it was prior to the fall of 2001. The primary concern of the DUP going into this project was that the students might not be mathematically strong enough to handle such a course. The prevailing wisdom among the faculty was that these students struggle with such application problems. The obvious question is 'What has been the effect of this project on the student success rates in MATH 1330?'

5. Effect on student success rates

To attempt to determine the effect of our project on student success rates we looked at the grade distributions for MATH 1330 for all instructors, for each term from the fall of 1998 through the fall of 2004. It is generally accepted that the fall term is the 'on' term for MATH 1330 and the spring term is the 'off' term. A significant percentage of those taking it in the spring are people who were not successful in MATH 1330 the preceding fall or people who had to take developmental math to prepare for it. Thus we chose to compare fall terms to fall terms and spring terms to spring terms. The fall and spring grade distributions for MATH 1330 are presented in Tables 1 and 2 respectively. 'Passing rate' is defined as the ratio of the number of students receiving a final grade of A, B, or C divided by the total initial enrollment. The grade point average (gpa) is based on the average using a 4 point scale (4 for A, 3 for B, etc) with those students who withdrew or received no grade being omitted from the count.

Table 1: Fall grade distributions for MATH 1330 (all sections)

Fall Term	#As	#Bs	#C's	#Ds	#Fs	#Ws	Initial #	Pass rate	gpa
2004	430	345	236	91	70	118	1317	0.788	2.812
2003	419	381	262	90	124	154	1430	0.743	2.690
2002	577	346	233	71	128	160	1515	0.763	2.866
2001	416	421	345	105	150	174	1611	0.734	2.590
2000	366	391	304	174	154	189	1578	0.672	2.461
1999	340	273	203	79	91	136	1122	0.727	2.702
1989	380	344	260	100	87	155	1326	0.742	2.709

Table 2: Spring grade distributions for MATH 1330 (all sections)

Spring Term	#As	#Bs	#C's	#Ds	#Fs	#Ws	Initial #	Pass rate	gpa
2004	146	157	110	50	67	130	660	0.626	2.500
2003	137	144	138	73	82	136	710	0.590	2.315
2002	166	187	132	55	56	157	753	0.644	2.591
2001	293	224	163	59	67	87	893	0.761	2.766
2000	161	148	146	57	82	168	762	0.597	2.419
1999	226	202	142	58	58	93	779	0.732	2.700

The aggregate totals for the pre and post project fall and spring comparisons are provided in Tables 3 and 4 respectively. (This does not include the project term, fall 2001)

Table 3: Pre and post project fall comparison

fall	#As	#Bs	#C's	#Ds	#Fs	#Ws	Initial #	Pass rate	gpa
post	1426	1072	758	252	322	432	4262	0.764	2.791
pre	1086	1008	767	353	332	480	4026	0.711	2.610

Table 4: Pre and post project spring comparison

spring	#As	#Bs	#C's	#Ds	#Fs	#Ws	Initial #	Pass rate	gpa
post	449	488	380	178	205	423	2123	0.620	2.469
pre	680	574	451	174	207	348	2434	0.700	2.645

It would be tempting to simply compare the average gpa pre-reform to the average gpa post-reform in Tables 3 and 4 using the well-known two-sample t-test. Unfortunately, the two-sample t-test assumes that each and every observation in both samples is independent of each and every other observation, and that is certainly not the case here. Although it would be safe to assume that the grade observed for a student under one instructor is independent of the grade for a student under a different instructor, the same cannot be said for students who have had the same instructor. To account for this lack of independence we used a more sophisticated model known as the *two-stage nested design*. Essentially, it is an analysis of variance (ANOVA) model with two factors: reform period (pre- and post-), and instructor. In the data, there are 121 unique instructors represented. Of these, 67 taught only in the fall semesters, 35 taught only in the spring semesters, and 19 taught in both fall and spring semesters. So the fall data included 86 instructors and the spring data included 54. (The transition period, fall of 2001, was excluded from the analysis)

The reason that the design is known as a *nested* design is because of the instructor factor. The vast majority of instructors in both semesters taught only during the pre-reform period or the post-reform period, but not both. (There are 82 such instructors that taught in only one period or the other for the fall and 50 such instructors for the spring.) From a strict standpoint, the data is only "partially-nested" since there are four instructors from each semester that taught during both periods. Since this number is so small relative to the total number of instructors we are going to treat the analysis as though it is fully-nested. (This is done for simplicity's sake. The conclusions differ very little for any of the other even more sophisticated designs that treat it as "partially-nested".) For a more thorough description of these types of models, we refer the reader to the text by Montgomery [4] (Chapter 13, in particular), one of the standard texts on the subject.

To get right to the point, the increase observed in the grade point average from the pre-reform to post-reform periods for the fall semester is statistically significant (p-value = 0.0178), but we observe no significant difference for the spring semester (p-value = 0.3734). As would be expected, there are significant instructor-to-instructor differences (p-value < 0.0001 for both spring and fall). Note that the SAS software was used to perform the analysis.

6. Project assessment

The goal of our reform project was to change the focus of MATH 1330 away from the traditional emphasis on mathematical formalism and towards an emphasis on applications in the context of the business world, in effect produce a major attitude adjustment on the part of instructors and students. Logistically the project was straight forward within our existing instructional structure, and the change in emphasis was relatively easy.

Our initial concerns about the possible adverse effect on students' success rates appear to have been unfounded. In fact the pass rate improved and there appears to have been a significant positive effect on the gpa for those students who initially took the course in the fall term.

The pass rate did go down and it is possible there was a negative effect on the gpa of the typically weaker students who take the course in the spring; however, such a conclusion is not statistically supportable.

In retrospect, our experience with improved success rates is consistent with generally accepted education theory. See Davis' remarks on motivating students [5]. Relevance of the material and classroom incentives satisfying their own motives for enrolling in a course are effective internal motivators that promote students' learning. Moreover, according to Davis, 'a teacher's expectations (high but realistic) have a powerful effect on a student's performance.'

A similar project aimed at reforming the calculus course required of COBA students is proving to be somewhat more problematic, and is ongoing. The authors hope to be able to report on the second project within the next year or so. The authors also hope to be able to collect and analyze qualitative data to assess the extend to which improved success rates in MATH 1330 translate into COBA faculty perceptions of the mathematical readiness of COBA students to take upper level business classes.

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The evolution of the Delta community

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Now that we are embarking on the fifth Delta Symposium, it is appropriate to look back at our achievements in order to plan for the future. Trends in the topics discussed are identified and significant contributions are highlighted, particularly as they relate to the themes of the symposia. The influence of technology on mathematics teaching and learning is shown to be a very important issue. In addition, there is a trend away from describing individual courses to more theoretical, broader perspectives on mathematics education.

1. Introduction

Delta was born in 1997 when a small group of university mathematics educators from several Queensland institutions met at QUT to discuss the possibility of holding a conference, (later called a symposium) at which they and their colleagues could discuss the teaching and learning of mathematics at university level. The meeting was organized by Patricia Cretchley, recently arrived from South Africa, who had a dream to bring together her colleagues from her old and new countries. Delta has achieved many things in the last eight years but there is no doubt that the dream has come true.

The participants in the first symposium in 1997 came mainly from Australia, South Africa, New Zealand and the United States. This led us to start thinking about ourselves as a southern hemisphere group of educators and the words "southern hemisphere" appeared in the titles of the 2001 and 2003 symposia.

Delta is not an organization, so there are no formal aims or rules. It is, as the title of this paper suggests, a community of academics committed to improving mathematics teaching and learning at university. The community has evolved and grown in response to the changing needs of the participants. The concept of change in university mathematics inspired the name, Delta and it has proved a wise choice as one of the things this paper will demonstrate is that some of our interests are indeed changing.

2. Themes

Each conference has had a different location and a different theme and both these factors have influenced the issues that were presented and discussed. The theme for the first Delta, "What can we do to improve learning", embodied our basic objective but did not specify any particular direction. Unsure of what kinds of issues would emerge as important for our colleagues, we aimed to encourage as many different points of view as possible. The 1999 theme was, "The challenge of diversity", and this inspired many papers to consider the diverse backgrounds of our students such as [1] in which Swedosh looks at the problem of students' misconceptions. Students' diverse thinking and learning styles and the kinds of courses appropriate for them were described by Steyn, de Boer and Jakobus but unfortunately did not appear in the proceedings. There are also papers on courses that offer students diverse modes of study such as [2] in which Bulmer provides flexibility in a large statistics class and [3] in which Coutis, Farrell and Pettet take a similar approach in a large engineering mathematics class. Finally in [4] Pierce notes the diverse responses of students in an introductory course using Derive, where she notes that, “This learning style benefited some students but certainly not all”.

Both the first symposia were held in Queensland but the third which was held in South Africa, attracted participants from many African countries for whom recent history magnified the problems of teaching and learning, often in a second language as in [5], where Bohlmann is attempting to find out if improving English reading skills, will improve mathematics learning. For the third Delta in 2001 the theme was, “Gearing for flexibility” and as with the previous theme of diversity, many papers addressed flexibility which was frequently made possible by the use of technology. In [6] Farnsworth describes a web based course designed for flexible use and in [7] Felix describes a flexible course using graphing calculators in which students showed very little flexibility in using the technology. The most recent Delta in 2003, the largest Delta to date, had the theme, “From all angles” and this appeared to encourage a very wide range of topics treated in a variety of ways, from the detailed statistical analysis presented by Freislich and Bowen-James [8] to the in-depth interviews in the work of Bartholemew and Rodd [9].

3. Major areas of interest

At the outset it must be made clear that the classification in Table 1 is a personal one and as has been described in [10], we all see the world around us differently and attach importance to different things. These data are an underestimate as not all abstracts/papers were represented in the Proceedings. Panel discussions and workshops have not been included.

Table 1: Frequently occurring topics

	97	99	01	03
Total number of papers	62	50	70	85
Technology	23	19	32	25
Bridging/Foundation/Support	4	2	9	5
Calculus reform	10	3	-	2
Courses for teachers	1	1	6	11
Statistics	2	4	5	9
Assessment	8	4	4	5
Modeling	4	3	2	4

We can see from Table 1 that interest in some topics such as technology, bridging, assessment and modeling has been consistent, while other topics show a trend courses for teachers and statistics upwards and reformed calculus downwards. This may not mean that calculus reform is no longer of interest but rather that some of its principles have been incorporated into courses and are now taken for granted. Many of the courses for teachers in the last two Deltas involved familiarizing pre or in service teachers with the use of technology in teaching mathematics. Papers [11] and [12] both involved dynamic geometry software while [13] and [14] involved computer algebra systems. The theme running through these courses for teachers and others that did not involve technology, such as Barnes [15] was that teacher training was often inconsistent and therefore did not necessarily lead to teachers adopting new approaches to their teaching.

Delta’03 was the first symposium to include the word “statistics’ in its title and had an invited speaker, Chris Wild, who is a statistician. This had the effect of increasing the number of statistics papers, a trend that will probably continue. The large number of Bridging/Foundation/Support programs discussed at Delta’01 was largely the result of the location in Africa where universities were faced with large numbers of potential entrants who had not had the opportunity of quality secondary education. This situation has led to a great deal of debate about what mathematical skills are really necessary and how best these can be provided to large numbers of students. Unfortunately several of these excellent papers did not appear in the proceedings.

4. Technology in more detail

The various ways in which technology can be used in mathematics education is a subject that has received considerable attention for many years now, for example Keitel and Ruthven [16]. However, the classification in Table 2 is mainly about which technologies presenters discussed. A number of papers involved more than one type of technology and some abstracts, (with no accompanying paper) did not specify the technology used so the numbers do not "add up" when compared to the data in Table 1 The word "packages" is used very broadly and includes software, CD ROMs and even videos.

Again there are some trends evident. Clearly, existing packages are here to stay and people who have the resources will continue to produce packages specific to their needs. As the internet becomes a way of life, its use for class administration and course delivery is increasing. Also increasing, are papers which compare different packages and also investigate how and if students actually use technology when left to their own devices. The distinction between calculators and computer algebra systems is disappearing.

Table 2: Types of technology use

	97	99	01	03
Using existing packages	15	9	18	14
Producing own packages	6	4	5	4
Using the Web	1	3	8	7
Calculators	3	3	3	1
Comparing packages or comparing use with non-use	1	1	4	6

Two interesting examples of the last category in the table are Kawski [17] which compares certain educational benefits of Matlab with those of Maple and Mathematica and Cretchley [18] which investigates factors that predispose students to adopting technology.

5. Approach of papers

The vast majority of papers at all the Deltas described individual courses, materials prepared for them and the responses of students to them. This is not surprising as most participants work at the coalface in university education, teaching or facilitating courses, trying to improve their work and to help others do likewise. Interestingly, papers from U.K. participants tended to focus on cross-university studies and course materials, while in Australia and South Africa, most of the courses described were specific to one institution. However there has been an increasing trend towards papers taking a broader, more theoretical perspective towards teaching, learning, course design and assessment and this has not been the prerogative of the invited speakers. This trend is shown in Table 3.

Table 3: Papers taking a broad or theoretical view

	97	99	01	03
Broad, theoretical aspects	11	14	20	29

6. Conclusion

There are many other ways in which our short history could have been classified, for example, according to level - foundation, first year, later years, or according to the type of innovation that was described, for example, cooperative learning, flexible assessment or web-based instruction. Since our history is still quite short, it did not seem to be appropriate at this time to attempt such detailed classifications or cross-classifications. The above broad classifications were chosen in the hope that they would be helpful to participants involved in the current

Delta and more particularly to those planning Delta symposia in the future.

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Functions: Looking ahead beyond calculus

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The concept of a *function* is a unifying foundation of modern mathematics. Increasingly, the function concept also guides middle school algebra curricula, and ubiquitous research studies analyze its learning. Such recent changes have helped bridging the traditional gap between high-school algebra and college-level calculus. But teaching and learning functions in a narrow context opens and widens the rift between calculus and both post-calculus courses and real-world applications. This note provides several examples to illustrate that some of these efforts miss what mathematicians consider key characteristics of functions. It is based on the observation that these school curricular changes and the majority of research studies focus almost exclusively on (real valued) functions of a (scalar) continuous variable. We focus on composition as the distinguishing operation on functions, issues relating to domain and codomain, injectivity and surjectivity, and the interplay with alternations of universal and existential quantifiers. This note is to stimulate discussion about how much teachers of algebra and curriculum designers at all levels need to understand about functions themselves, and about how deeply should students at which level understand functions.

1. Introduction

This note is motivated by personal experiences when teaching a wide variety of post-calculus courses at a large public North American research university, by discussions with middle and high school mathematics teachers, and impressions at numerous colloquia and conferences on research in mathematics education. In a nutshell, the author welcomes the increased attention given in early grades to the fundamental and unifying concept of a *function*, partially implementing recommendations made in the *Principles and Standards of School Mathematics* [8]. It is a great improvement if middle school algebra teachers do not just teach algebra for the sake of teaching algebra, but instead if (s)he looks far ahead and understands the importance and meaning of the subject for all students, whether they study university level science or not. But the author is worried that the reformed curricula do not adequately develop some the most important characteristics of functions. This becomes particularly clear in discussions with teachers, curriculum designers and in numerous colloquia and conference talks where the focus is solely on (piecewise smooth, real-valued) functions of a (scalar) continuous variable. Worse yet, at numerous talks, the speakers, when questioned, did not even seem to understand why mathematicians would not be perfectly happy with what is taught under the name function at the earlier levels. As an instructor of courses named vector calculus (VC), differential equations (DE), linear algebra (LA), introduction to writing proofs, advanced calculus / introduction to analysis, partial differential equations, abstract algebra, and differential geometry, this author, semester after semester, encounters large groups of students who demonstrate almost fatal misunderstandings of what characterizes a function. All too often, this author wonders whether it might not have been better if these students' teachers had not used the word function at all, not caused misconceptions that are extremely hard to undo. These experiences are reminiscent of the foundational work [6, 7] that identifies the need to first wipe the slate clean by explicitly addressing common misconceptions in mechanics, before even starting with new material.

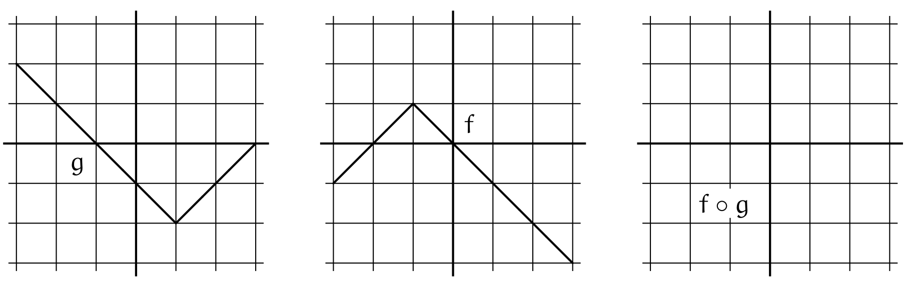
Paralleling the historical migration of other mathematics subjects from being university courses to becoming schools courses, calculus is becoming a high school course: In 2004 in the United States, 175,000 and 50,000 students took the advanced placement (AP) exams in calculus at the AB and BC levels, respectively [3]. Much larger numbers of students enroll in high school advanced placement courses without taking the exams. The success of the AP program is being analyzed in many studies, e.g. [10,11]. This migration facilitates continuity between middle school algebra and calculus. But we fear that it may also be a cause for a widening chasm between calculus and follow-on subjects. This is exacerbated by calculus even at the universities now often being taught exclusively by full-time instructors, where in the past research faculty always had an eye on how to keep calculus connected with the next levels.

There are so many studies and publications on the function concept that it is impossible to list even a representative fraction in this place. We limit ourselves to a few remarks about closely related literature. Since the well-known article [18] many studies have analyzed various aspects of the function concept, applicable to the entire range from the elementary levels to beyond *introduction to analysis* (advanced calculus), see e.g. [1, 16, 17] and the references therein for particularly deep analysis. A large number of studies by others focus on real-valued functions of a single continuous variable with primary focus on the notion of covariation, see e.g. [2, 12]. While these are close to the immediate needs of teachers, other mathematicians are concerned that the resulting policy recommendations miss important features.

The subsequent sections provide examples and test cases from different contexts that illustrate shortcomings of students' understandings in selected features of functions: composition as the characteristic operation on functions, domain and codomain, preservation of structure (e.g. monotonicity and linearity), injectivity and surjectivity, invertibility, and alternating quantifiers. For lack of space we do not address the important interplay of function as taught in most mathematics courses with algorithms and functions found under various names such as *subroutine, procedure, method* in FORTRAN, ISETL, C, MATLAB, MAPLE, JAVA. As argued in other places, e.g. [4, 9, 17] the use of computer languages can be a powerful tool to construct stronger conceptual images of the function concept.

2. Composition

The set of functions Y^X from a set X to a set Y inherits many algebraic structures from the codomain Y. Even at advanced levels such as abstract algebra, many students have never consciously reflected on this basic observation. If the codomain Y is a ring or a vector space then the set of functions inherits this structure under pointwise operations. The true meanings of pointwise definitions of operations on functions such as $(f+g)(x) = f(x) + g(x)$ are often poorly understood. What distinguishes functions from the algebraic objects that students have encountered previously is that functions may be composed. Some aspects of compositions are readily taught, such as evaluating compositions (of formulas) at numbers, and basic symbolic manipulations. Students also quickly learn how to use the chain-rule for finding derivatives of expressions such as $\sin(x^2)$. But these are very limited levels of understanding of compositions. Some of the author's favorite questions on his *first-day-of-class diagnostic tests* are given below. The first question illustrates that even evaluation of compositions of functions is rarely mastered when combined with the need for even elementary graphical reasoning.



Question 2.1. Given the graphs of two functions f and g above sketch the graph of their composition $f \circ g$. Use a minimal number of function evaluations.

Generally poorly understood are more general properties of compositions of functions, as illustrated typical answers to:

Question 2.2. Suppose that f, g are monotonically decreasing functions. Decide whether the inverse f^{-1} and the composition $f \circ g$ are decreasing, increasing, or whether both are possible.

This question is applicable in the broader context of order preserving/reversing functions between partially ordered sets as commonly studied in post-calculus courses. The next task illustrates that chain (and product-)rule are often only mastered for specific explicit formulas, not for functions, hinting at a superficial understanding of compositions.

Question 2.3 Suppose that $f, g\colon \mathbf{R} \mapsto \mathbf{R}$ are differentiable. Use rules of differentiation to simplify $(f \circ g)''$.

Let us change gears, and consider the following three tables. The first represents the percentage of the recommended daily allowance (RDA) of vitamins A and C and iron for one serving each of various fruit. The second table lists the contents of two different kinds of dessert baskets offered at a cafeteria, the third lists the number of baskets that three students consumed today for breakfast, lunch and dinner.

	Orange	Apple	Kiwi	Banana
A	0	2	2	0
C	130	6	240	15
Iron	2	2	4	2

	Basket 1	Basket 2
Orange	1	1
Apple	1	0
Kiwi	0	1
Banana	0	1

	Ann	Bob	Jill
Basket 1	1	0	2
Basket 2	2	3	0

Where are the functions? What class are we in? There are many places where we consider tables and matrices as functions of two variables: The notation makes all the difference, we write $f(x, y)$ and a_{ij} but should be able to understand just as well $a(i, j)$ and f_{xy}. Alternatively, each row and each column in each table represents a function, as does each table in its entirety – think about the practical interpretation, the domain and the codomain!

A typical use of these tables in the first course of linear algebra is to answer: Why do we multiply matrices the way we multiply them? Most students at this level have learnt *how*. But who understands *why*? Given these tables *with* their explicit labels, it is an elementary school task to figure out what to do with them, how to meaningfully combine them into new tables that answer the obvious questions: How many percent of her RDA of vitamin A did Ann get from the fruit she ate today?

We have three tables – and hence we may multiply $(A \cdot B) \cdot C$ or $A \cdot (B \cdot C)$. Do we get the same result? Why? It was clear to my children that the resulting numbers must be the same, simply because each number has meaning. Yet from a linear algebra perspective, we want a more formal argument why matrix multiplication is associative. The mathematician thinks: multiplication by a matrix is a function, multiplication of matrices corresponds to function composition and function composition is associative – the most fundamental property of function composition. Are our students prepared to understand these functions? When their teachers teach functions, do they understand the fundamental importance of associativity of function composition? These tables also nicely illustrate the importance of domain and codomain – here we need matching dimensions (sizes), but even then some products are utterly meaningless (though algebraically legal). As an immediate corollary we get that matrix multiplication is not commutative.

It is challenging to teach students, who have never really thought of multiplication by a number as a function, that *multiplication by a matrix* is a function. All too often the notation $y = 3x$ gets in the way – where we really need to think *"times* 3". Too many students think it cannot be a function if there is no x. Indeed, all too often the use of the letter x is an obstacle. Why not say that the derivative of sin is cos (MAPLE uses this language), or that the derivative of *"squaring"* is *"multiplication by 2"*? In the same vein, when in multi-variable calculus we create a table of values for the function $f(x, y) = xy$ it seems that our students never thought of multiplication (of two numbers) as a function – whose inputs are pairs of numbers. This is one of the first functions ever encountered. Do the teachers understand how to sow the seeds for a comprehensive understanding of the concept of functions?

3. Domain and properties of the image

Many students confuse a formula with a letter x with the concept of a function.

Question 3.1. Does $y = \begin{cases} 1-x & \text{if } x < 0 \\ x^2 & \text{if } x \geqslant 0 \end{cases}$ define a (*one*) function?

The majority of second year college students votes that this is two functions, not one. This becomes very painful in DE courses where natural forcing functions typically are piecewise smooth.

Question 3.2. Find the derivative of $y = \log(\log(\sin x)))$ with respect to x and overlay the graphs of y and y'.

Every single time that I asked my post-calculus students, the large majority of them correctly (?) applied derivation rules. Yet they did not even see the problem: What does it mean to differentiate a function with an empty domain? Even more curious, how can its derivative have a nonempty domain? But then the antiderivatives of the derivative also must have a nonempty domain – but how do these relate to the original? It is clear that these students confuse the (algebraic) derivate of a formula (symbolic algebraic expression) with the derivative of a function. Most introductory textbooks begin with well-intentioned pictorial and tabular examples of functions with discrete domain (and codomains) – yet what is learnt in these lessons is quickly superseded by an confusing functions with symbolic formulas. See [1] for a related discussion.

Students venturing beyond calculus have trouble recognizing functions when they appear as vector fields in vector calculus, and as differential operators in the first DE course. In the first case the problem is that not only is the input not a number, but a point in the plane, in 3-space, on a curve or a surface, but also its output is not a number, but a vector. In the second case, both the input and the output of the function (operator) are functions themselves.

In its most simple example, the step from the DE $y'' + 4y' + 3y = g(t)$ to the differential operator $L\colon y \mapsto y'' + 4y' + 3y$ is as important and powerful as the step from the quadratic equation $x^2 + 4x + 3 = 0$ to the quadratic function $f\colon x \mapsto x^2 + 4x + 3$. The graph of the first consists of the two points -1 and -3 on the real number line, whereas the graph of the latter is a parabola. An easy way to recognize the power of this step is to consider the quadratic equation $x^2+4x+2.9 = 0$. The first view offers no hint. From the second picture it is immediate that the solutions are approximately -3.05 and -0.95. (The x^2 term has unit coefficient, and hence the slope one unit to either side of the vertex is $\pm 2\ldots$) Precisely this kind of thinking is needed in the context of DEs. It is a huge step from the algebraic equation to the function. Do the teachers understand how important it is, and that it is repeated later, again and again?

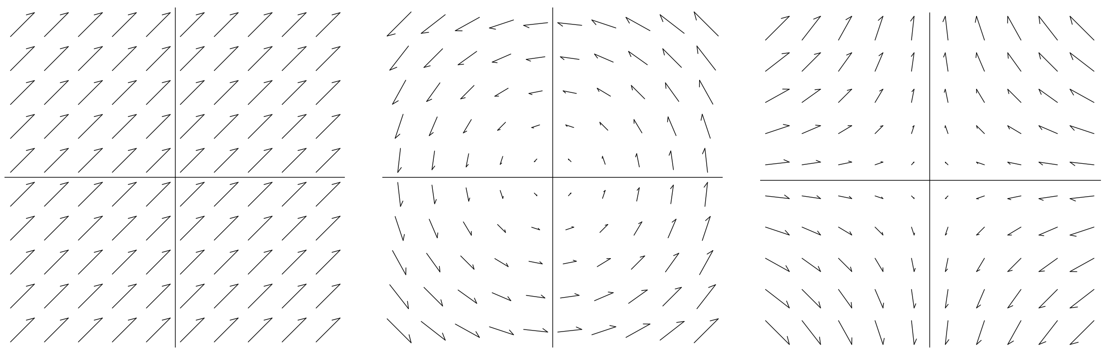
Many mathematicians prefer to only work with the vector field defined by a DE since it is a function (not an equation) and hence has much more desirable properties. At its simplest, there is a precise notion of equality: two functions f and g are equal if they have the same domains, same codomains, and if for every input their outputs agree. On the other hand, are the equations $y = x$, $x = y$, and $y - x = 0$ the *"same"*? On the algebra and calculus level teachers and students manage to muddle their way through: the right answer is the one in the back of the book. This misses a golden opportunity to *"sell"* functions (for their precision) to students clinging to equations, and to explicitly address the common quandary of *same function, different formula*.
For now the first DE course stays an *equations* course, with the function notions (painfully) employed only when absolutely needed as in the form of differential operators. Our students

simply arrive with such a warped sense of function at this course, that it would take too long to wipe clean the slate before being able to start. The connection with vector fields – which our students do not recognize as functions – remains on the fringes, with consequent inability to connect DEs to curl and divergence.

At all levels we would like our students to have a sense for the depth of the question which structures are preserved by which functions. Focusing too much on only (real valued) functions on the real line muddies the waters in a different way, as the real numbers simply have too much structure: Arithmetic properties, order properties, and topological properties (closeness). Often it is easier to study properties in more abstract settings as this allows one to focus on the essentials, what makes things work. This is beautifully accomplished in the new *Introduction to Analysis* textbook *Closer and Closer* [13].

Arguably the most important algebraic notions are linearity and multiplicativity. Of course, these may also be mixed, as, e.g., the exponential function maps the additive group $(\mathbf{R}, +)$ to the multiplicative group $(\mathbf{R}^+, \ \cdot \ )$, and hence its inverse allows one to use a *slide-rule* to perform multiplication by simple adding. For an in-depth study of teaching multiplicative properties at the school level see [15]. At the precalculus level students study linear functions and their graphs ad infinitum, and they also learn rules how not to simplify e.g. $\frac{77+6x}{7+3x^2}$, $\sqrt{x^2+9}$, $\sin(\alpha+\beta)$. But do they and their teachers understand the fundamental importance of *linearity*, understand the importance of differentiation and integrations being linear?

Question 3.3. Which of the vector fields depicted below is linear?



Even among university faculty the answers are all too often false (correct: the first is not linear, the other two are linear). The universal definition of linearity is: A function $f\colon X \mapsto Y$ is linear if for all scalars c and all $p, q \in X$, $f(cp+q) = cf(p)+f(q)$. (For this to make sense, one must be able to add and take scalar multiples in both domain and codomain, i.e. both need to be vector spaces, or modules.) In their first year after calculus, students encounter this in linear algebra, DEs, mechanics (superposition principle), linear electric circuits, ... Do the teachers of linear functions understand that $y = mx + b$ is a lot more than just one step before studying more complicated formulas? A recent parent-teacher meeting with my daughter's algebra teacher again indicates otherwise. Although he proudly told her that in 7th grade she would *"learn about functions, a concept that will get her prepared for calculus"*, he had no understanding of the role of functions outside calculus.

Next are *increasing*, *decreasing*, *convex* and *concave* functions – well understood to be a core calculus topics. All too often students *define* a function to be increasing (or convex) if its (second) derivative is positive. This is the first hint that increasing (convexity) have not been taught properly. Another hunch is the earlier question 2.2 whether inverses and compositions of decreasing functions are decreasing. Monotonicity comes in many different forms, at many levels – e.g. the modern mathematical models of many biological systems are rooted in the theory of *monotone systems of DEs* [14]. Likely the best preparation for later uses is a strong verbal understanding of *increasing* and *increasing at a decreasing rate*. We would like to see more familiarity of *order preserving/reversing* functions on partially ordered sets. Well-tested activities start with directed graphs that represent tournament results / rankings.

Another key property is continuity, called *stability* in the applied sciences, loosely defined as: small changes in the input yield small changes in the output.

Question 3.4. Using this figurative notion, decide whether $f\colon \mathbf{R}\setminus\{0\} \mapsto \mathbf{R}$, $f(x) = \frac{1}{x}$ and $g\colon \cup_{n=-\infty}^{\infty}(2n, 2n+1) \mapsto \mathbf{R}$, $g(x) = \log(\sin x)$ are continuous.

Every mathematician will agree that both functions are continuous – the figurative criterion matches the precise topological / analytical definitions. Yet, the commonly used criterion of being able to draw the graph of the function without lifting the pencil from the paper fails – due to lack of *connectedness of the domain*. This is a delicate topic. What should the secondary school teacher know, and what should (s)he tell her/his students?

4. One-to-one, quantifiers and related topics

The question whether one can go back, *undo* a function, leads one to analyze when functions are one-to-one (injective) and onto (surjective). Most textbooks include a few pictures of functions between discrete sets to illustrate these concepts – but due to lack of follow-up and making repeated connections to other settings, these lessons seem to have little lasting impact.

It has been sown that it is possible to successfully teach secondary school students functions and injectivity in the precise language of alternating universal and existential quantifiers (and negations of quantified statements). But in most school classes one finds years of very casual treatment of the precise definitions of a function and of injectivity. We advocate that all secondary school mathematics teachers have a profound understanding of functions and of injectivity so that they can conduct regular activities that promote thinking along the lines of modern mathematics, thereby opening the pipeline to more students, and cushioning the shock when students step outside calculus.

In repeated courses of teaching *Introduction to Mathematical Reasoning*, the author has found it beneficial to sidestep real-valued functions of a real variable for this purpose, as the majority of students just carries too much baggage of misconceptions and misunderstandings about functions. At some time these will have to all be addressed, and painstakingly cleared up – this is not unlike the situation in university physics described in [6, 7]. Together with colleagues who taught the same material at the school level to both teachers and students, we found counting functions to be a particularly amenable area for developing notions of function, one-to-oneness and bijection. These are bijective functions whose domain is either the set of positive integers or an interval $\mathbf{Z} \cap [1, n]$. We explicitly count permutations, combinations, pairs of numbers, rationals, and arrangements of playing cards (both sets and sequences) – which nicely show the mathematics behind stunning *card tricks*. Algebraic formulas are obtainable – but often these are unattractive. The focus is on the structural properties that the constructed objects are indeed (counting) functions (one-to-one and onto). Our experiences indicate that many of these activities are just as well suited for younger students at the schools level – and they would allow students at an early age to develop profound and broad understanding of the central concepts that characterize functions, as opposed to the currently prevalent tendencies that solely emphasize covariation of scalar continuous variable with a narrow focus on conceptual understanding of single-variable calculus.

From a more advanced perspective, functions are routinely employed, especially in analysis, to simplify statements that otherwise are hard to read due to large numbers of alternations of existential and universal quantifiers. A typical definition from calculus reads: *"A function $f\colon \mathbf{R} \mapsto \mathbf{R}$ has a limit at $a \in \mathbf{R}$ if there exists a number $L \in \mathbf{R}$ such that for every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every $x \in \mathbf{R}$, if the distance from a to x is less than δ then the distance of $f(x)$ from L is less than ε"*. Using functions this can, for general metric spaces X and Y, be rewritten as: *"A function $f\colon X \mapsto Y$ has the limit $L \in Y$ at $a \in X$ if there exists a continuous, strictly increasing function $\varepsilon\colon \mathbf{R}_0^+ \mapsto \mathbf{R}_0^+$, with $\varepsilon(0) = 0$ such that for all $x \neq a$,*

$d(f(x), L) < \varepsilon(d(x, a))$."

A particularly successful notion is that of a $\mathcal{KL}$-function which has a similarly simple definition and which allows one to avoid complicated alternations of quantifiers in the characterization of *asymptotic stability* of a dynamical system. For lack of space we cannot elaborate the details here, but encourage the reader to contact the author who in turn plans to give a detailed description of the use of functions to encapsulate alternating quantifiers, thereby making otherwise too complicated statements understandable.

5. Summary and conclusion

In many places efforts are made to teach and develop the concept of function at earlier levels, while research studies analyze the learning of this concept. This note gives examples that suggest that some of these efforts focus on a very narrow slice of this concept, leading to far from desirable levels of understanding of functions at various levels.

Sometimes it makes sense to start with one special case, and gradually generalize a concept. In other cases harm is done by too much focussing first on a special case – the exclusive study of linear DEs being a case in point where students confuse what remains true and which methods don't apply to nonlinear problems. We leave it for discussion in which of these two ways to introduce functions, and which aspects of functions should be studied at which level. A follow-up task is to agree on model activities that develop currently neglected critical aspects of functions at various levels. This requires a close cooperation between curriculum developers at the schools level, education researchers, and faculty with broad teaching experiences at postcalculus levels.

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A School and University partnership to support elementary teachers' professional growth

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We describe an ongoing and growing professional development project for upper elementary/middle schoolteachers (primarily grades 3-6) at the University of Minnesota. This project aims to deepen teacher mathematics knowledge of central mathematical concepts, increase understanding of the mathematical connections to middle and high school curricula, and explore how literacy best practices can enhance student performance in mathematics. The central activity is a carefully designed summer mathematics course, "Algebraic Processes and their Connections to Geometric Structures", primarily for current and emerging teacher leaders in a selected group of partner districts. Opportunities are provided for curricula and instructional leadership, using teacher-generated enrichment modules based on its deep mathematical content focus. Classroom visits, module dissemination of web-based versions, and final workshops round out the basic program. This paper will provide more details about these completed activities, and a description of new follow-up courses and academic year professional development opportunities to begin in Summer 2005.

1. Introduction

In this paper, we describe an ongoing and growing professional development project for upper elementary/middle school teachers (primarily grades 3-6) in the Institute of Technology Center for Educational Programs (ITCEP), University of Minnesota. This project aims to deepen teacher mathematics knowledge of the underlying central mathematical concepts in this curriculum, increase understanding of the mathematical connections to middle and high school curricula, and explore how some best practices in literacy can enhance student performance in mathematics. The central activity during the first three years has been a carefully designed, intensive two-week summer mathematics course, "Algebraic Processes and their Connections to Geometric Structures." This course is intended for current and future teacher leaders in a selected group of partner districts, and provides opportunities for curricula and instructional leadership using teacher-generated enrichment modules within its deep mathematical content focus. Academic year classroom visits, web-based availability and dissemination of refined versions of the enrichment modules, and closing workshops round out the basic program. During the initial three-year period (2002-05), 67 teachers representing a rich cross-section of Minnesota's elementary teacher population have completed this program, and many continue this profession growth via peer mentoring and other professional opportunities available through other ITCEP instructional programs. This paper will provide more details about these completed activities, as well as a description of new follow-up courses and academic year professional development opportunities to begin in Summer 2005.

2. The Professional Development Model

As previously noted, the project is aimed at existing and emerging teacher leaders who wish to strengthen their mathematical teaching capacity by acquiring a deeper understanding of the underlying K-8 mathematics concepts and content, and a better understanding of connections among elementary school topics and middle school topics. The typical teacher participant is knowledgeable about many mathematics topics at a reasonable level, very successful in teaching mathematics when their understanding was solid, and interested in becoming a more powerful learner of mathematics in order to improve their instructional capabilities. An extensive survey of teachers in the Twin Cities indicated a modest total number (< 3) of mathematics courses completed and frequently taken more than five years ago. This profile was valid even among the teacher leaders identified as possible participants. When knowledge of core algebraic and geometric concepts in the elementary curriculum was assessed, significant numbers of project participants in pre-course surveys over all three years showed gaps in their knowledge.

Table 3: Survey responses –200 teachers from five districts, and *Mathematics Within* teacher participants pre-course responses: algebra and geometry knowledge

Survey item	Fall 2002 200 district-teachers "Do not know"	2002-03 23 teacher participants "Do Not Know"	2003-04 24 teacher participants "Do Not Know"	2004-05 20 teacher respondents "Do Not Know"
Knowledge of algebraic items concerning lines	37%	52%	46%	35%
Knowledge of algebraic items: linear relationships	54%	57%	62%	45%
Knowledge of algebraic items: non-linear relationships	68%	70%	21%	50%
Knowledge of geometric reasoning and visualization in mathematics	11%	13%	17%	15%
Knowledge and use of geometric properties of 3-D figures in science	12%	13%	24%	25%

Thus, our projected model was central to the professional development needs of even this selected group of teachers.

An important secondary issue is the current political and economic climate in Minnesota specifically, and the United States in general. Several years of major state budget deficits and significant cuts in K-12 school funding has removed virtually all state/district professional development opportunities, except for several associated with the increasing emphasis on state standards, high stakes testing, Assessments of Academic Year Progress (AYP) and other components of the federal No Child Left Behind (NCLB) movement. Thus, many teachers question the value of professional development and growth when their classroom and student performance is increasingly judged within this narrow framework.

Especially in this overall K-12 academic climate, an important problem/barrier was the challenge of building teacher buy-in to the idea of improving their mathematics content knowledge as an effective method of professional development. This approach is distinctively different from the more typical recent methods of providing presentations on pre-packaged math materials to plug into lesson plans, in lieu of actual development. Improving student achievement in math within the complex, standards-based curriculum takes time and demands a high level of content knowledge from the teachers. This challenge is compounded by the diversity of the students they teach, and specifically by the range of student literacy levels. We quickly learned that teacher commitment to a professional development program depended upon obtaining teacher buy-in as early as possible. We accomplished it by including educators and especially teachers at every stage of project planning. Once teachers realized that they would (a) have opportunities to identify and pursue their own needs, (b) contribute to the design of the coursework, (c) participate at various levels of the project, (d) do hands-on development of enrichment materials for their classrooms and (e) join with a network of like-minded others, they became excited about the course and project. Over time, many teachers became enthusiastic network participants as they experienced gains in mathematics knowledge, developed new connections with peers and applied this knowledge to develop their own enriched materials for children.

3. The Course

The two-week summer course, *Algebraic Processes and their Connections to Geometric Structures*, has evolved slightly in details over the three years of the project, but has kept the same overall focus. The overall goal is to provide teachers with an in-depth understanding of the number operations and geometric structures that were core to *any* solid elementary mathematics curriculum, and to study these topics using geometric models and representations whenever possible. As an example, the geometric area model for representing the distributive property of multiplication over addition was explored in several ways to show its mathematical utility in middle school and high school algebraic reassuring. Using a deductive model to show how, for example, 3^2 is obtained from 2^2 by decomposing a 3×3 square, the teacher participants learned and seemed to understand the formula $(n+1)^2 = n^2 + 2n + 1$ for integers, and in fact for any real variable n. This newly developed sense of understanding the connections between teachable number patterns and more advanced mathematics, appeared to be a major component in improving their mathematical self-esteem and confidence. Moreover, the same "algebraic" area model was subsequently used to provide a solid algebraic proof of the Pythagorean Theorem using a square of side length $a + b$, as well as deriving and using the difference-of-squares formula $a^2 - b^2 = (a - b)(a + b)$ to discover Pythagorean triples. Other themes, such as fractions and numerical operations, were developed in a similar fashion.

Some unusual consequences developed from these approaches. The course stressed the role of the participants as powerful learners, and the issues of what constitutes good "habits of the mind" for learning mathematics became important. Models for how to investigate a mathematical idea were discussed, with the major theme being how a mathematician explores an idea or investigates a problem. In teaching new ideas, we tried as instructors to reveal our mental picture and reasoning approaches in examining the concepts in the courses, and the reasons why we were excited about some approaches and connections. This type of mathematical revelation and bonding led to a genuine atmosphere of trust and comfort among the teacher participants and the instructors, resulting in to a wonderful classroom atmosphere for studying and learning mathematics. As instructors, we were sometimes in awe of the wonderful pedagogical ideas and insightful approaches used by some of these teacher-leaders as they prepared lessons and showed their ideas on teaching these modules to the entire class.

From a formal viewpoint, we tried to make this class very attractive. Teachers obtained two graduate credits in mathematics for completing the summer course, presenting a model lesson in the academic year classroom visit, and attending a final half-day workshop. In addition to the clearly-defined mathematical goals, we aligned the course with both the Minnesota State Standards and with the NTCM Standards. The textbook used in the course appeared on the approval lists of both sets of standards. Teacher participants provided additional linkages through their discussion about the specific needs and curricular requirements of their schools and districts. We referenced Minnesota Comprehensive Assessment (MCA) mathematics test items for grade 3 and grade 5 as background material to inform the course design process. Finally, we made sure that the high quality of the food selections for snacks and other aspects made the teachers feel a bit special!

4. Assessment

Assessing the impact of the project in a short time period is clearly difficult and subject to interpretation. Nevertheless, we gathered some preliminary data and tried to assess two aspects: teacher beliefs/knowledge (i.e. professional development and growth) and student achievement. For the professional development aspect, pre-course, post-course and final surveys were used. The post-course survey for the Summer 2004 course is shown in Table 1.

Table 1: Post-Course Survey, Summer 2004

ITQ Summer Course, *Algebraic Processes and Its Connections to Geometry* Evaluations Summary

A. The participants (N = 27) rated the following qualitative statements on a scale of 1= not valuable to 5=very valuable:

I. From this section, the highest rated categories were instructor-led presentations-(26 participants rated 4-5), enrichment connections to your curriculum (25 participants rated 4-5) , enrichment lesson development, lesson presentations to your team, and final lesson presentations.

	Rated 4 or 5	% Rated 4 or 5	Overall Mean
1. **Instructor-led mathematics presentations**	26	96%	4.8
2. Practice problems/homework	21	78%	4.2
3. Integrated connections to student literacy	23	85%	4.3
4. Mathematics enrichment sessions	23	85%	4.5
5. Presentation on the new math standards	16	59%	3.8
6. Enrichment connections to your curriculum	25	93%	4.5
7. Enrichment lesson development	24	89%	4.3
8. Lesson presentations to your team	24	89%	4.5
9. Final lesson presentations	24	89%	4.6
10. Critiquing of lesson presentations	21	78%	4.2

II. The highest rated categories in this section were statements #1 and #2. Twenty-four participants rated "deepened my mathematics knowledge" as 5, and twenty participants rated "engaged me in new ways to *think* mathematically" as 5.

	Rated 4 or 5	Overall Mean	Rated 5	% Rated 5
1. The course content **deepened** my mathematics knowledge.	27	4.8	24	89%
2. The course instructors engaged me in new ways to ***think*** mathematically.	27	4.9	20	74%
3. The course methods **increased** my awareness of how literacy influences the learning of mathematics.	25	4.5	15	56%
4. The Mathematics Enrichment content was **very interesting** and **motivational.**	23	4.3	16	59%
5. The Mathematics Enrichment content was **very challenging**.	24	4.5	11	41%

In the survey narrative responses about whether the course contributed to their understanding of mathematics, most respondents reported gaining a deeper understanding of mathematics. The following quotations typify elective responses describing how they made new connections, gained insights or saw new ways to present ideas:

- "I started making connections with fractions, division, and algebra that I never made before."
- "The Area Model is fabulous. It is possible to use this with students, not just the teacher."
- "I know why 'negative means negative' is "true" now and hopefully will be able to transfer this knowledge to others."
- "When we applied the Pythagorean Theorem in class, it was a pure wonder. It is another way of making connections in math topics and ideas."
- "Multiplication and addition as operations on sets was something I had not thought very much about before. I was not one who generalized easily in mathematics, but this exploration forced me to generalize. It was uncomfortable but I learned from doing it."
- "I like the proofs because it gave me better understanding of all the concepts."
- "[All the methods/approaches] opened my eyes to the necessity of exploring ideas in many different ways."
- "It was very exciting to spend ten days discussing math and being exposed to so many connections."

Preliminary analysis also indicates that the summer course, although focused primarily on deepening understanding of mathematics content, has also assisted teachers in bringing their new math knowledge into their own classrooms. Based on participants opportunities to develop enrichment lessons that utilize their new math knowledge, they have now started to use these materials in their own class. About half of the respondents indicated that their increased understanding helped them make new connections, gain new insights, or see new ways to present ideas applicable to their elementary curriculum:

- "The course increased my confidence as a math teacher. I feel more comfortable answering my students' questions and letting them explore. It has really changed the way I approach math instruction. I would recommend it to any teacher."
- "[The] Area Model is a model that helps to increase understanding in both algebra and geometry and will be very useful to me in teaching multiple models that increase student understanding."
- "Negative numbers [contributed to my understanding of mathematics] because I needed a better understanding of how to work with them so I can better explain them to my students. I also know what is going on in the students' brains when they have that 'I don't understand' look on their faces because I had that look many times!"

Several other project outcomes also stand out. The evaluations revealed a renewed sense of professional enthusiasm for some participants. This enthusiasm, according to district partners, also influenced teachers' peers, many of whom are skeptical of the benefits of discipline-specific coursework. One immediate impact is an emerging cohort of mathematically-trained teacher leaders who have and can serve as peer mentors in district projects. Finally, many participants also reported feeling more capable in successfully applying new mathematical approaches to classroom practice, such as the enrichment lessons they developed or other curricular ideas. There teachers believe that this new sense of confidence after participating in the *Mathematics Within* project will positively impact student learning in their classrooms.

5. Future Direction for the Program

The success of the Algebraic Patterns course and the subsequent networking/professional development opportunities during the first three years has led the project to plan and to programmatically develop the next steps. One aspect is additional content coursework with two options, based on the needs and interests of the 67 alums of the first course. One option is a course parallel to the Algebraic Patterns, but with a reverse structure: a primary emphasis on geometry and measurement and their related algebraic processes. This course, "A Deeper Study of Algebraic Processes and Their Connection to Geometric Structures," is targeted to those alums who want to emphasize deepening and broadening their knowledge of the geometry side of the grades 3-6 mathematics curriculum in the same way and at the same level as the Algebraic Patterns courses, and will be the natural second course in the sequence for most teachers. The proposed content topics will be chosen from the following list:

Content Topics

A. Geometry and Measurement

- Characteristics and classification of two and three-dimensional shapes.
- Use of descriptive and representational systems for shape, position, and movement: Coordinate systems and two-dimensional representations of three-dimensional objects.
- Investigation of mathematical situations through pattern, symmetry and transformations, using symmetric constructions to inter-relate types of triangles and quadrilaterals
- Units of measurement: length, area, volume, weight and angle and their conversions. Changes in measurements as shape or size is varied.
- Geometric vocabulary and definitions. Problem solving, reasoning and proof, using symmetry, construction angle facts, inducting and deduction. Communication, connections and representation of mathematical concepts.

B. Topics from Algebraic Processes

- Geometry of lines and connections to division and equivalent fractions. Algebraic representations of lines, their properties and geometric relationships.
- Diameter versus circumference of a circle (proportional), and diameter versus area (non-linear). Approximating circles by triangles and rectangles. Indirect measurement.
- Pythagorean Triples. Definitions. Geometric experimentation and use of number patters in finding Pythagorean Triples. Other applications of the Pythagorean Theorem: Real world uses and mathematical formulae for distances with coordinates.
- Application of the area model for multiplication to algebraic formulae and patterns. Multiple representations (functional, geometric and symbolic) for determining algebraic facts and formulas.

The second option is to pursue more deeply the connection aspects in the middle school (grades 5-8) mathematics curriculum and provide a more foundational and somewhat more mathematically-precise analysis of key elementary and middle school mathematics topics. There would be greater emphasis on deeper problem solving skills and more appreciation of the role of proofs. The teacher audience would likely come from the strongest and most mathematically mature of the first-year alums – a group of about 27 participants. This course, "A Capstone View of Algebraic and Geometric Concepts in Middle School Mathematics," or some variant would likely become the third course in the sequence for many of the teachers. The proposed content topics for will come from the following:

Content Topics

- **Order of Operations:** Explore why we need an order of operation, the rules and ambiguities of the current order, the possibilities of other orders, such as reverse polish notation.
- **Division by 0:** Explore the properties and representations of division of real numbers to ultimately answer the question why division by zero is undefined.
- **Divisibility Tests:** Investigate why certain simple tests on whole numbers are necessary and sufficient to determine divisibility by another number. Participants will be able to create their own divisibility tests afterwards.
- **Negative Times Negative:** Investigate the mathematical reasons that the product of negative numbers is positive. We use this topic to investigate why patterns and models are not sufficient as the exclusive mathematical arguments.
- **Inclusive Definitions of Quadrilaterals:** This unit answers why we have the definition that a square is a rectangle but a a rectangle is not necessarily a square. We will explore how the inclusive definitions we use in geometry are useful to understanding the qualities of polygons.

- **Exponents:** Explore the properties of exponential functions, such as why multiplication of common-base exponentials translates to addition of exponents. We will use this to answer why any number to the zero power is one.
- **Repeating Decimals:** Investigate why the decimal point nine repeating is equal to one. We will use this opportunity to understand some of the fundamental properties of real numbers and the implications to decimal representations.
- **Infinity:** Introduce some definitions of infinity and the implications to elementary school and middle school curriculum.

A second and extremely important aspect is the establishment of a new extensive academic year professional development and networking component. While a fundamental priority of the project is to improve the math achievement of students, it is equally critical for teachers to have continuing opportunities to deepen their mathematical content knowledge and to clarify the K-12 MN Academic Standards mathematic expectations for grades 3-6. There is general agreement among participants, schools and districts, and instructional staff that teachers need professional mentoring and peer support as they use newly acquired knowledge and improved professional development experiences to 1) lead peer professional development workshops, 2) assist in refining and developing specific approaches to reach all students, and 3) creating, refining and disseminating classroom enrichment lessons. To support this growth, the Academic Year Professional Development Network (AYPDN) will include a variety of content, mentoring, and leadership activities during the 2005-06 academic year. One key component will be two days of professional development programs for course participants at a school site or at the University of Minnesota campus. Activities will include: a) opportunities to co-instruct in campus- or school-site teacher in-service workshops, in which project activities will be disseminated, b) presentations by mathematics faculty on additional mathematics topics, and c) assessment of the impact on student achievement.

The AYPDN will also provide additional learning, teaching and networking opportunities for current and past course participants as well as other teachers. The exact mix will depend on network participants' interest and available University opportunities. The listing below is the proposed core of anticipated activities:

i. Teacher observation in a classroom with immediate feedback to help teachers refine, critique, and disseminate lessons and materials currently or previously developed in the summer courses.
ii. Mentoring by University faculty and staff to help teachers present and disseminate their own materials as well as other participants' materials in their own school and district. The AYPDN will maintain a website linked to the existing ITCEP website, which contains all materials and lessons to facilitate this process.
iii. Two half-day participant "fairs," where AYPDN members can present new and refined materials and activities, informally network with other members and university instructional staff, and discuss plans/directions for continuing AYPDN networking activities. Discussions on state standards and state tests will also take place. These fairs will also help to develop organizational and leadership capabilities for some of the AYPDN participants, who in turn will assume leadership roles in developing future programs.
iv. Four mathematical seminars, uniquely designed for teachers, will be offered during the academic year. Some will be on new mathematical concepts, and other seminars will be linked to curriculum developed for Saturday Morning enrichment workshops offered to students in grades 3-7. Teachers will receive the curriculum before the seminar, do an in-depth study of the math concepts during the seminar, and have the opportunity to teach the curriculum in a future enrichment program workshop. Together with other AYPDN teachers, they will discuss and critique the curriculum and how they taught it immediately following the enrichment presentation.
v. Opportunities for AYPDN participants to formulate and design additional workshop materials for the ITCEP academic year and summer Enrichment Programs.
vi. AYPDN workshops to validate the funding of the evaluation and to ensure statewide dissemination. The teacher participants will 1) complete summative evaluations, 2) review

student data, and discuss ways to increase AYPDN impact on student achievement and AYP rates, 3) discuss the impact on student achievement findings and refine enrichment lessons for subsequent teacher use, and 4) post relevant findings on ITCEP's website.

Support for these activities would initially come from grant funding. As of April 1, 2005, the project has obtained funding for pilot versions of the two new courses and seed funding for the AYPDN. Other grant prospects seem promising, and it appears likely that the AYPDN will be fully funded to commence in the 2005-06 academic year.

Tablet Technology in First Year Calculus and Linear Algebra Teaching

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Although tablet technology has been around for more than a decade, it seems to be used mainly in niche markets such as arts and design, and appears to have bypassed most of academia. The purpose of this paper is to show the usefulness and effectiveness of tablet technology in undergraduate mathematics teaching by sharing experiences made using a graphics tablet for lecturing a large first year class. It is meant to encourage discussion of current teaching tools, for instance for lecture delivery, preparation of on-line material and student consultation at a distance. This technology allows writing by hand on the computer as well as keeping a record of what has been written, which makes publication of this material on the course website very simple.

1. Introduction

Mathematics lectures have traditionally been presented on blackboards. As Townsley [9] points out, *"One reason we all use blackboards to write down mathematics is the symbols with which mathematics is communicated. Writing the symbols down gives the student a chance to read what has been spoken and thus access the content via several senses."* Apart from mathematical symbols, students also need to be shown step by step how to work out a problem, and how to write down a solution in a clear and precise, mathematically correct way. Students need to learn mathematical explanation [4].

Nowadays, the blackboard is often replaced by overhead projectors (OHPs). The advantage of this is that the lecturer can refer back to what was written earlier, and may be able to keep a record. It is possible to cover more material faster, as lecture notes can be prepared in advance, printed or written onto slides and presented more quickly in the lecture.

In the last decade, computers have found their way into the classroom, as presentation software has become more user friendly and easier to use. However, programs like PowerPoint used on a standard computer are non-interactive tools [10], where the lecturer shows one slide after the other, maybe revealing one line at a time. In an interactive learning atmosphere, students see how the lecturer develops a solution from scratch and they can contribute to a particular path. This is lost when everything is typed before the lecture begins. One popular way to overcome this is the combination of computer-based notes and OHPs or blackboards. The lecturer displays the theory on the computer, and evolves proofs or worked examples separately by writing on the other medium. While this method enables interactivity and allows covering more material, no electronic copy is kept of what was written during the lecture.

There is affordable hardware and software technology available now that allows direct handwriting on the computer, and saves a copy for later reference or for posting on a website. This technology is the graphics tablet, or, more expensively (and sophisticated), the Tablet PC, in combination with software that allows annotating of text or simply drawing. To date, few Tablet PCs have been sold in academia [10], and even fewer universities seem to have introduced tablets into mathematics teaching. A notable exception is the University of Colorado, where for a range of courses both voice and image are recorded and presented to external students via MathOnline [1]. Another promising tool, used to tape lectures before publication on the web to external students is Audiograph [4]. This multimodal software [7] records voice as well as pen movement and is used with an emphasis on teaching mathematical explanation [4]. Both of the above use speech and pen recording for teaching external (distance) students.

In this paper, I will describe my experiences using a graphics tablet for lecturing a large first year mathematics class of internal students, by writing in the electronic lecture workbook during class. Advantages and disadvantages will be mentioned, student perception will be described and an outlook will be given into the potential of such technology for teaching both internal and external students.

I would like to emphasize that this paper presents a survey of current methodology and a report of the initial exploration of the usefulness and advantage of tablet technology in course delivery. It is meant to encourage others to consider this technology for their teaching setup and make them aware of its potential.

2. Brief History of Graphics Tablet and Tablet PC

Graphics tablet technology has been around for a number of years, with the first tablet input device introduced by Wacom [11] in 1988. These days tablets are available in various sizes, where bigger is not necessarily better, as many people prefer a smaller tablet to minimize arm motion [2]. Tablets usually connect to the USB port and are powered by the computer.

Tablet PC's were first offered in the early 1990s. In 2002, Microsoft introduced a specialized Tablet PC operating system, Windows XP Tablet Edition, which is a full version of Windows XP Professional with tablet specific additions such as handwriting recognition. Both have been used mainly in niche markets such as manufacturing, graphics arts and design. They have not yet found widespread acceptance for university teaching. A very informative online overview of tablet technology can be found in [3]. It concentrates on Tablet PCs and their use in education, and gives reviews and results of trials. Detailed information about graphics tablets is given in [2].

3. Tablet technology in teaching first year mathematics

3.1. Course specifications

Calculus and Linear Algebra I (MATH1051) is the first mathematics course taught at university level, offered to internal students in two semesters each year at the University of Queensland. Students are enrolled in programs such as Engineering and Sciences, and this course is usually compulsory for their studies. About 500 students enrol in semester 1 (taught in two streams), and 300 in semester 2. Every student attends three hours of lectures a week, one tutorial hour and one practical class (Matlab based).

In semester 1, 2004, the lecture notes were available for the first time electronically in the form of a workbook (PDF format, also for sale in print at the bookshop), which contains the relevant material and includes a large number of blank boxes (see Appendix B for an example page). Students fill in these boxes during lectures. Material to be filled in can be a theorem, a proof, part of a proof, a definition, a graph, or an example. The boxes are distributed with the intention that a lot of material can be covered quickly, but at the same time allowing enough time to be spent solving interesting examples in detail by writing out the steps, or defining and explaining new and important concepts.

In addition to the student version of the workbook, there is also an instructor version, where the blank boxes are filled with the relevant material in type. The introduction of the workbook has made it easier for lecturers new to this course to see what is covered, and to prepare for lectures. It has lead to a standardized way of presenting the material. Since the introduction of the workbook, MATH1051 has been taught by presenting the student version of the workbook on the computer projector and writing solutions by hand on blank transparencies, or by displaying workbook and solutions on the computer, where typed solutions were revealed one line at a time.

In recent years students have begun to ask for complete lecture material to be posted on the web following the lecture, to allow them to compare notes or to catch up on missed lectures. Either the completely typed solutions or scans of transparencies were published on the course website.

3.2. Why use graphics tablet technology?

When I took over lecturing of MATH1051 in semester 2, 2004, I decided to fill in the blank boxes of the workbook by writing in the lecture. I believe that it is important not only for students to see how the solution of a mathematical problem evolves by writing out each step by hand, but that it also increases my own motivation and engagement for the course. Students are required to set-out their solution by hand in assignments and exams, and will benefit from seeing the lecturer "work out a problem". It can be tiring to sit passively in a lecture, and I admit that I tend to go faster than intended if there is nothing to write. Hand-writing is more dynamic and thus gives the flexibility to draw additional graphs, explain concepts further, show alternative methods of solving a problem or pose additional problems, for example when answering student questions.

In the first few lectures of the semester, I attempted to project the workbook onto the screen and wrote the contents of the boxes on transparencies. I encountered unexpected difficulties, as recent refurbishment left the lecture theatre without sufficient wall space to use computer and overhead projector simultaneously. As soon as the computer projector was turned on, a large screen was lowered from the ceiling, covering any other available wall space. The computer projection was conveniently centred on the screen, leaving about half the necessary space on either side for an overhead projection. My decision to change the method of lecture delivery was also driven by the brightness of the OHP. I found it impossible to keep eye contact with the students because my eyes did not adjust quickly enough and I simply could not see them in the relative darkness of the lecture theatre.

3.3. Experiences made using a graphics tablet

I decided to try a very small graphics tablet (about $100) for writing directly on the computer, in combination with Adobe Acrobat Standard. Since the workbook is available in pdf format (converted from LaTeX), this seemed to be the obvious choice. Acrobat Standard is an elegant solution as it incorporates a commenting function that allows one to hand-write on an existing pdf file, as well as to include images such as Matlab graphs. Comments are objects that can be added on top of the underlying document without changing it, and they can be edited until the document is printed to a file. This file is then readable by Acrobat Reader, a program available free of charge for all operating systems. Comments may be written in various colours, for example to point out changes or additions made after the lecture.

Although writing on a tablet requires a certain level of hand-eye coordination, as the user writes blindly on the tablet while looking up at the screen, I found it did not take long to get used to writing with the pen, both standing in front of the class and sitting at my desk. I was in fact so impressed by the ease of use of both the pen and the software, that I began exploring further use such as keeping a record of the material to be posted on the website, where students were able to view an exact copy of what I had presented. Preparing material for the website took no more than five minutes, and this includes improving my handwriting in places where I thought it was difficult to decipher because I had written too fast. I quickly found other areas where pen-writing proved to be helpful, such as posting additional handwritten notes on the website to further explain topics students found difficult to understand. I found it easy to reply to student questions with an image or pdf file attachment containing the solution in hand-writing. Writing on the tablet saved a lot of time, as I no longer needed to type notes in LaTeX.

One may argue that posting lecture notes on the website will decrease student attendance. I had the impression that students regarded the web material as a very helpful addition to, rather than a replacement of lectures. During my experiences with MATH1051, I claim that posting lectures on the web had no effect on attendance, which is an observation that was shared in [5] and [1].

After getting used to the tablet I was able to borrow a tablet PC for a few lectures. Tablet PCs are quite expensive compared to graphics tablets that connect to existing computers. However, they offer the advantage that the user can see pen movement while writing, as writing takes place on the screen. Overall, the tablet PC seemed to be even easier to use than a tablet, and it requires less space since the writing surface is integrated.

3.4. Student Perception

Graphics tablets had never before been used to lecture in mathematics at the university. Students were asked for feedback on the workbook, writing on the computer and the course website where lecture material was posted after the lecture. Out of the 65 participating students,

- 89% agreed that "writing during lectures helps my understanding"
- 80% said they prefer if the lecturer writes on the computer (12% didn't care)
- 92% agreed that "it's great to have computer-generated lecture notes on the website"

Students were also asked to write comments about the course set-up. Some of these comments can be found in Appendix A. Overall, students seemed to like the graphics tablet, and it was much preferred to the OHP. Students noted that the computer projection was easier to see than the overhead projection. They found the website very helpful, as they could download lecture notes whenever they liked.

This student survey was meant to give immediate feedback on a new teaching tool half-way through the semester. It does not give an accurate view of how students judge effectiveness of the use of this technology. I agree with Lowerison [5] that it is possible that the technology used may be perceived as a means to deliver the material and not as a means to promote learning.

4. Conclusions and potential of this technology

The graphics tablet is an affordable and efficient teaching tool that is very easy to use. It was preferred to the OHP by most students. I have got used to it in such a way that I would not want to have to teach this course without. I believe it replaces the need for blackboards and OHPs completely, while still allowing hand-writing of mathematics. Although writing takes place on the stationary tablet, the lecturer does not need to remain in one spot for the whole time. I found it natural to walk between the computer and the screen to further explain concepts, which kept the lecture dynamic.

New technology is often used in the same way old technology was used, and not to its full potential, because of lack of knowledge and comfort of familiarity on the user's part. Tablet technology should not simply replace the need for an OHP. Its advantages need to be identified and appropriately used, for example the advantage that notes written on the computer can be saved and published on a course website, without the need to use a scanner. With Adobe Acrobat Standard, notes can be saved directly in Acrobat Reader readable pdf format. Tablet technology can also be useful when answering student questions by email or in discussion groups, as an image can be attached that contains the worked out solution step by step. Additional information, complementing the lectures, can easily be posted for download on the website. The amount of time saved by not typing in LaTeX is enormous.

There is huge potential for tablet technology in distance education. McCloskey wrote that tablet PCs *"should be a boon for distance educators who will be able to mark up electronically submitted papers with Ink and e-mail the hand-notated assignments back to students"* [6].

Some discussion group software allows the use of whiteboard space. An instructor's writing could be read in real time by students who are signed-on. However, the full potential of this application would only be used if both sides, instructor and students, were using graphics tablets. Graphics tablet are now available for less than AUD 100, and *"it may become appropriate to require students at a distance to purchase such devices in order to better interact with the instructor"* [6].

Literature on the effectiveness of tablet technology for mathematics teaching appears to be scarce. A comparison of the various software applications suitable for handling hand-written mathematics needs to be performed. Only Adobe Acrobat Standard was explored in this paper, as the course material had been prepared in LaTeX and conversion from LaTeX to pdf is natural. It presented a fast and reliable way of writing in lectures which had to be found because of shortcomings in lecture theatre equipment. Better and cheaper software may exist, and may be in use elsewhere.

Acknowledgments

I would like to thank Michael Bulmer for purchasing the graphics tablet from his teaching funds and for generously lending it to me for the whole of the semester. I would also like to thank Phil Pollett, for allowing me to try his Tablet PC for a few lectures.

The explorations of tablet technology took place while I was appointed as a temporary lecturer at the University of Queensland.

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Appendix A - Student Comments

The following is a selection of comments made by students on the survey form.

4.1. Writing in lectures

- Perfect lecture set up. Perfect course for that matter, I've really enjoyed this subject, each maths subject should have this setup

- Keeps me awake in lectures
- Writing assists with comprehension
- We work through problems together
- It is an incentive to come to lectures, you can learn more by writing it down
- It makes the lecture more active, rather than passive listening
- The lecturer's writing is easy to understand
- It's easy to see how the concepts are applied. Shows how to set out problems/working
- Allows you to concentrate a bit more on what is being said instead of copying notes down all lecture
- The graphics pad is easier to see than the OHP
- Good to see use of technology in a lecture theatre!
- The computer is heaps better than overheads
- Love the graphics pad! Very useful and modern, 1000x better than OHT
- I can't follow along when writing. I have to go back to read them later to understand
- Though the lectures are better on the computer they would be easier to understand if they were typed up
- Computer-written notes are sufficiently easy to read, but not best for proofs

4.2. Availability of notes on the course website

- I know exactly what I miss in lectures (if I miss them)
- The writing on graphics tablet is fine, especially as can get it of net easily
- It's good to make sure what you wrote down is correct on the website. I doubt that people who don't come to the lectures can understand from just the material on the web
- Very useful if you miss a lecture
- It's really helpful
- Website is essential
- Able to catch up on notes

Appendix B - Extract from Lecture Material

This is an extract from the lecture notes published on the course website. Sarrus' Rule was not covered in the lecture material and was briefly introduced for this example as an alternative method.

11.13.4 Example

Calculate the determinant $\begin{vmatrix} 2 & 5 & 6 & 1 \\ 1 & 2 & 3 & -2 \\ 3 & 2 & 4 & 4 \\ 0 & 2 & 4 & 6 \end{vmatrix}$.

Workbook Solutions 11.30:

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ \vdots \end{matrix}$$

$$\begin{vmatrix} 2 & 5 & 6 & 1 \\ 1 & 2 & 3 & -2 \\ 3 & 2 & 4 & 4 \\ 0 & 2 & 4 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 5 \\ 1 & 2 & 3 & -2 \\ 0 & -4 & -5 & 10 \\ 0 & 2 & 4 & 6 \end{vmatrix} \quad \begin{matrix} (r_1 \to r_1 - 2r_2) \\ (r_3 \to r_3 - 3r_2) \end{matrix}$$

$$= (-1) \cdot \begin{vmatrix} 1 & 0 & 5 \\ -4 & -5 & 10 \\ 2 & 4 & 6 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 0 \\ -4 & -5 & 30 \\ 2 & 4 & -4 \end{vmatrix} \quad (c_3 \to c_3 - 5c_1)$$

$$= (-1) \cdot (+1) \cdot \begin{vmatrix} -5 & 30 \\ 4 & -4 \end{vmatrix} = -(20 - 120) = +100$$

$$\begin{vmatrix} 1 & 0 & 5 & 1 & 0 \\ -4 & -5 & 10 & -4 & -5 \\ 2 & 4 & 6 & 2 & 4 \end{vmatrix} = 1 \cdot (-5) \cdot 6 + 0 \cdot 10 \cdot 2 + 5 \cdot (-4) \cdot 4 - [2 \cdot (-5) 5 + 4 \cdot 10 \cdot 1 + 6 \cdot (-4) \cdot 0]$$

$$= -30 - 80 - (-50 + 40)$$

$$= -110 - (-10) = -100$$

Sarrus' rule

P.D.E. real-life problems: visualization and intuitive approach

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Differential Equations is a very interesting and motivating topic in Mathematical courses, but analytical solutions can not always be interpreted intuitively by the students. In fact, when the solution has an integral representation, it is not easy to check and/or visualize it. In this paper, an experience of introducing P.D.E. non-trivial, real-life problems to last year secondary school and first year university students, at an introductory level, is commented. Problems and solutions are briefly described here, but the focus is put in the visualization process and the students' reactions. The main outcomes are commented and taking into account this results, conclusions and recommendations are suggested.

1. Introduction

Differential Equations courses usually consist in lectures corresponding with the theory and tutorials where routine exercises are solved by students with help of teachers. In this traditional way of teaching subjects like O.D.E. and P.D.E., there is no opportunity for students to get in touch with real-life problems. This was exactly the situation in the Chemistry Faculty (Universidad de la Republica, Montevideo, Uruguay) until 1996, when things began to change.

In 1996, the Differential Equations course was complemented with "application-problems", that is, a small list of non-routine problems strongly connected with other disciplines like Physics, Chemistry, etc. Several proposed problems, were obtained from textbooks of Mathematics [1], or directly form texts of other subjects (for example [2] and [3], among others).

In 1997 and 1998, there were new changes in the Differential Equations course and gradually it became based on problem-solving corresponding with real-life situations [4].

Finally, from 1999 to 2001, a new change took place when technology was used in the visualization of the solutions of several O.D.E. and P.D.E. problems. Bi-dimensional and tri-dimensional graphics were obtained using Excel and MathCAD [5]. Most of the mentioned problems required the use of non-elementary mathematical tools such as Laplace and Fourier Transformations, Distributions such as Dirac's Delta, Bessel Functions, etc. [6], so they were restricted to advanced students and it was impossible at that time, to present them to a wider auditory.

Since 2002 there were some changes in the teacher's team and a new challenge was proposed, consisting of presenting these problems to students of different educative levels and different areas in an intuitive approach. For this purpose, it was necessary to choose carefully the problems to consider and also it was important to predict how these problems can be visuallized by students of this new wider group.

In this paper, these experiences [7] will be commented and the focus will be placed in the role of visualization to solve successfully this interesting challenge.

2. The real-life problems

As it was mentioned before, several problems obtained form textbooks, both of Mathematics and other subjects, showed strong connections with other disciplines, but they were not exactly "real-life problems". So it was necessary to consider a new source of mathematical problems where industrial procedures were modelled using O.D.E., P.D.E., Laplace and Fourier Transform, etc. In these cases the solution (and the problem itself) recquired a didactic transposition in order to present it in the classroom [8].

Most of these problems were obtained from a P.D.E. post-graduate course, where the students (teachers and researchers of the Chemical Engineering Institute) were encouraged to apply the mathematical tools taught in the course, to real-life problems of their own interest. Their original problems were simplified for undergraduate courses, but as it were mentioned before; even these simplified versions needed non-elementary mathematical tools [6].

In order to present them to a wider auditory, new process of didactic transposition [8] was required. These new versions considered only the "core" of the problem, i.e., a brief description of the main ideas of the model and an intuitive and visualizable solution.

So, the principal purpose was to introduce these problems to different students, from different areas, at different levels, trying to develope their intuitive thinking and showing how mathematical tools can be used to obtain in a rigourous way the formal solution.

Finally, this solution can be contrasted with the intuitive graphical one.

3. An example: Drying vegetables form their faces.

Suppose for example that carrots, apples, etc., are cut in slices, which will be dried using air with a low humidity rate from both faces (in fact, this is the problem when most of the spices like Laurel or Wild Marjoram are dried).

This situation can be represented as shown in Figure 1:

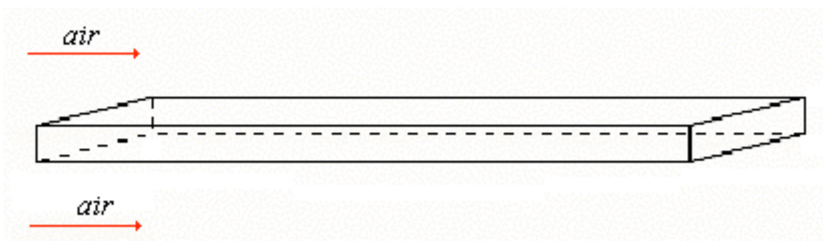

Figure 1: Drying vegetables cut in slices

When students are asked about how humidity (C) varies with time (t), all of them say that humidity decreases as $t \to \infty$, but they do not know which the correct function is, corresponding to this variation. At this moment, the teacher usually asks them how much it diminishes. The answer in most cases is that it vanishes, but there is always one of the students that realises that humidity in the slice cannot diminish under the value of the air humidity.

Following this first idea, it is possible to identify in the graphic of concentration versus time, two points: one of them, corresponding with the initial condition and the other, corresponding to the "minimum" humidity in the slice (in fact, it is an asymptotic value, not a minimum). These two points can be marked by the students as it is shown in Figure 2.

If nobody realises that humidity cannot be zero, students will mark the minimum humidity point over the t axis. This is not an important mistake, because at this stage the focus is put in the variation itself, not in the values.

In their first approach, the students tend to draw a straight line joining both points in the graphic, so it is necessary for the teacher to ask more questions in order to help them. For

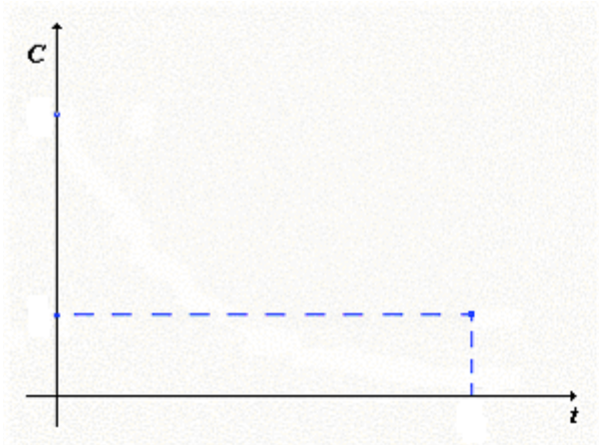

Figure 2: Points marked by students

this purpose, it is recommendable to ask the students if the humidity leaving the slice at the beginning is the same as the one leaving the slice at the end of this process. Their typical answer is that at the end, there is not so much water, so, less humidity leaves the slice when the experiment is finishing. This fact represents a contradiction with the linear behaviour predicted at the beginning, and as a consequence, they realise that it must be another function, not the linear one.

At this moment, if the students are encouraged to "draw a curve of humidity against time" (note that the variable, z, position, is not considered in their intuitive solution), most of them postulate a decreasing curve. Moreover, they tend to draw an exponential curve, as in other problems studied in secondary school (e.g. radioactive decomposition or a capacitor discharge), so the most common graphic is like the one shown in Figure 3.

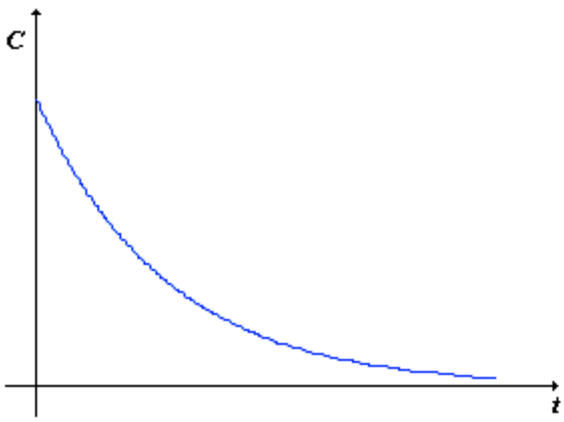

Figure 3: Exponential behaviour proposed by students

Now, it is probably the best moment to mention that there is one more variable: the position "z". Then, the graphical representation of the slice can be modified, as in Figure 4.

It is important to observe that if the students are asked about where to locate the "zero" of this "new variable", most of them will suggest to put it at the bottom of the slice. Once more, the teacher should guide them with relevant questions, in order to obtain an optimal location for $z = 0$. For this purpose, it is important to note that the air is drying both sides of the slice at same time, in a symmetrical situation. Then, an interesting question will be: Where does the humidity reach the maximum value? Even better, a more complete question can be asked to the students, about the internal variation of the humidity in the slice. Intuitively, they can

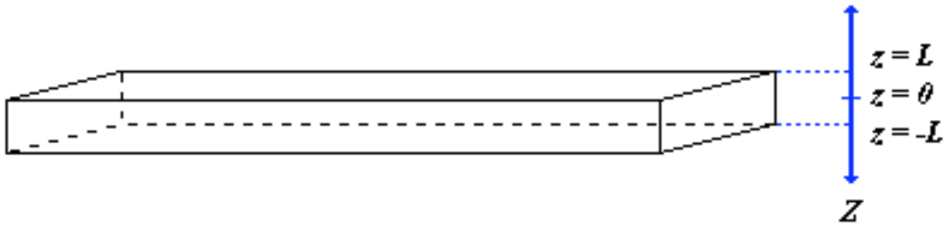

Figure 4: The slice and the "new" variable z

observe that humidity reaches its maximum value at the middle of the slice and it decreases as $z \to L$ or as $z \to -L$. At this moment a general agreement about the zero of the z axis can be obtained.

Probably, the students had never seen before these kind of functions (i.e., functions with two real variables t and z). As a consequence of this fact, it is better to focus in the dependence of C versus z, considering t as a constant. If they are asked to draw this function the most common drawing is something like that shown in Figure 5.

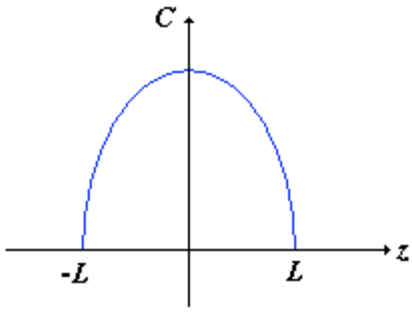

Figure 5: Parabolic behaviour for concentration versus position

In other words, students postulate a parabolic behaviour, regardless of the t value considered. At this point it is convenient to guide their reasoning towards the initial situation. Concretely, if $t = 0$, then C_0 is the humidity in the slice and C_e is the air humidity, so if the internal humidity is considered the same in all the points of the slice, and the external humidity is lower than the internal one, then the resulting graphic is not a parabolic one. Moreover, it is a discontinuous function, like the one shown in Figure 6.

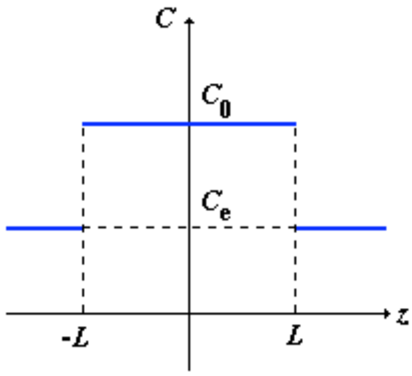

Figure 6: Discontinuous function corresponding with initial conditions

Once more, a "contradiction" (i.e., the unexpected non-parabolic function) strikes the intuitive thinking of several students. In fact, the "first" graphic corresponds to a discontinuous function but "the others" seem to be parabolic curves. Not all of the students get confused, it is interesting to observe that always in a group there exist a few who suggest something like a "deformation" of the discontinuous function into a parabolic one, i.e., the gradual transformation of graphic 4 into graphic 3. On the other hand, as it was mentioned before, there is always someone that notes that humidity cannot vanish, because there exists a minimum value corresponding with air humidity.

At this moment, not earlier, a graphical representation as in Figure 7 can be presented to the students:

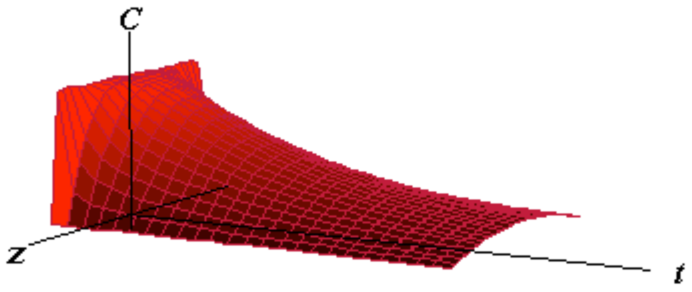

Figure 7: Three-dimensional graphic: concentration as a function of time and position

It is important to note that all the sequential approximations agree with this last figure. In fact, if this graphic is analysed focusing on plane C t, an exponential decrease can be observed and also, the asymptotic value for concentration C , when $t \to \infty$ will be C_e, i.e., the air humidity. Finally, if planes parallel to C z are considered, the parabolic behaviour predicted seems to be reasonable, obtaining maximum values decreasing with time (if $t > 0$) and an expected discontinuous function when $t = 0$.

This graphic was obtained using WinPlot, but other software (for example MATLAB or MathCAD), can be used. In fact, Figure 7 shows a graphical representation of the trucated Fourier Series corresponding with the solution. This series is

$$C(z,t) = C_e + \frac{4(C_0 - C_e)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \cos\left(\frac{2n-1}{L}\frac{\pi}{2}z\right) \exp\left(-\frac{(2n-1)^2}{b+1}\frac{\pi^2}{4}\left[\left(1 + D_0\frac{t}{L^2}\right)^{b+1} - 1\right]\right)$$

where D_0 is the equivalent diffusion coefficient and b is an adimensional exponent [6].

If the problem is introduced to last year secondary school students or first year university ones, it is better to consider a simplified version with $b = 0$, so the series become:

$$C(z,t) = C_e + \frac{4(C_0 - C_e)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \cos\left(\frac{2n-1}{L}\frac{\pi}{2}z\right) \exp\left(-\frac{(2n-1)^2\pi^2}{4}\frac{D_0 t}{L^2}\right)$$

The graphic of the truncated series (for example putting $\sum_{n=1}^{500}$ instead of $\sum_{n=1}^{\infty}$) is like the one of Figure 7, if the parameter values are accurately chosen.

This selection of numerical values and the corresponding graphics can be done successfully by students of different educative levels, after a short introductory explanation about how to use the software properly. In fact, this can be done with a graphic calculator such as a TI/92, among others.

This is an interesting excercise where variation of functions with different parameters can be visualized using technology in an interactive and motivating way of learning. As a final remark, the P.D.E. problem can be presented to the students. In this case, the problem is [6].

$$\left\{\begin{array}{ll} \frac{\partial C}{\partial t} = \frac{\partial}{\partial z}\left(D_0\left(1+\frac{D_0 t}{L^2}\right)^b \frac{\partial C}{\partial z}\right) & \text{PDE (diffusion in a plane geometry)} \\ C(L,t) = C_e \quad \text{instantaneous equilibrium reached at } z = L & \left.\right\} \text{Boundary} \\ \frac{\partial C}{\partial z}(0,t) = 0 \quad \text{Humidity flux equal zero in the middle of the slice} & \left.\right\} \text{conditions} \\ C(z,0) = C_0 \quad \text{vegetable initial humidity } \} \text{ Initial condition} & \end{array}\right.$$

It is not necessary to explain which the sense of each equation is, because the main objective is that they can realise how derivatives can be used for modelling real-life problems. The same situation takes place when the analytical solution (i.e., the Fourier Series) is presented in the class. The unique purpose is to show how series, trigonometrical functions or exponential functions are useful to solve industrial engineering problems.

Students react positively to this kind of applied approach of Mathematics, where different concepts as functions, derivatives, series, etc., are widely used in a more motivating context.

4. An example solved by university students

The Uruguay River is a very big river (several kilometres width in its last part), and is very important from economical and environmental points of view [9].

The mathematical model for contaminant dispersion in the Uruguay River can be represented by the following equation (a non-homogeneous parabolic P.D.E. with variable coefficients which involves the unknown function S and its two first derivatives):

$$\frac{\partial S}{\partial t} = E\frac{\partial^2 S}{\partial x^2} + \left(\frac{E}{A}\frac{\partial A}{\partial x} - \frac{Q}{A}\right)\frac{\partial S}{\partial x} + \left(\frac{1}{A}\frac{\partial Q}{\partial x} - k\right)S + \sum W$$

Next, if only a part of the Uruguay River is considered (between the cities of Paysandu and Fray Bentos), then, the last equation becomes simpler, resulting in a P.D.E. problem like this:

$$\begin{aligned} \frac{\partial S}{\partial t} &= E\frac{\partial^2 S}{\partial x^2} - U\frac{\partial S}{\partial x} - kS \\ S(x,0) &= 0 \\ S(0,t) &= c \end{aligned}$$

Using Laplace Transform with variable t (time), combined with translation and convolution theorems, it is possible to obtain an analytical solution:

$$S(x,t) = c\exp\left(\frac{Ux}{2E}\right)\int_0^t \frac{x}{\sqrt{4\pi E z^3}}\exp\left(-\frac{x^2}{4Ez} - \frac{U^2 z}{4E} - kz\right)dz$$

This analytical solution is valid at least for several parts of the river [6], where the linear velocity of water (U) and the area of the transversal section (A) remain constant and no contaminants are added to the course of water (that is $\sum W = 0$).

When this problem is introduced to first year university students, they are not able to solve it, but they can understand the problem itself. Moreover, they expect a decrease in concentration when $x \to +\infty$ and $t \to +\infty$.

An experience with first year engineering students was carried out at the end of the second semester. Concretely, the Simpson's rule was taught to them at an introductory level (just the formula and the main ideas) and they were encouraged to apply this formula to compute numerically different values of $S(x,t)$ and then, try to plot them in a 3-D graphic, using Excel.

Figure 8 shows a graphic [10] obtained by Guillermo (a first year student). In this graphic, the tendencial behaviour [11] of the solution can be observed.

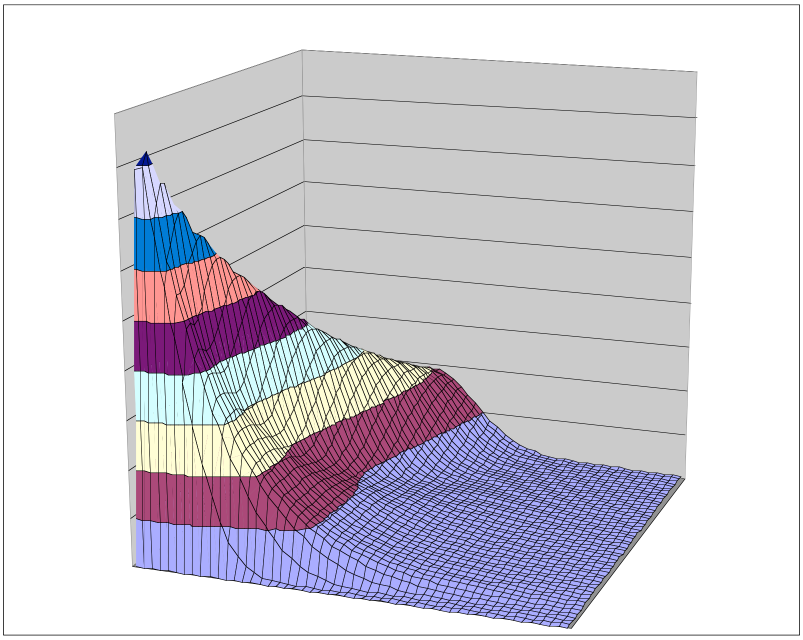

Figure 8: Graphic obtained by a first year university student

As it was mentioned before, this problem has a higher degree of difficulty, so, probably it cannot be “solved” intuitively by first year students, but it can be explained to them helping the collective reasoning with opportune questions. Moreover, the same task done by Guillermo (i.e., compute different values and plot them in a 3-D graphic) can be proposed as a project-work for individual students or even a group of them.

5. Results of several experiences

One of these problems (the first one) was presented to students of Liceo N 1 of Melo (a secondary school institution of a small city in Uruguay) in an intuitive approach, as mentioned before. This activity was part of a panel presentation with local secondary teachers and the auditory was constituted by students of the last pre-university year.

Students feel motivated by the real problems and participate actively in the “intuitive resolution” and discussion. They asked questions, proposed possible “solutions” and make all kind of interesting comments.

Their teachers became interested in how different concepts were used (derivatives, series, exponentials, etc.) and several times they participated in the “collective reasoning” of the group joining the students in the task.

In another case, in Tacuarembó (a small city in the north of Uruguay), these activities were complemented with the use of computers and graphic calculators and the participants were teachers and advanced students of teachers training courses. Once again, the group feel motivated and participate enthusiastically in all the activities proposed.

The second problem (the Uruguay River water pollution) was presented to first year engineering students and they were able to obtain graphics of the solution, getting a deep understanding of the tendencial behaviour of solutions.

6. Conclusions

Several problems (such as those presented here) have the interesting property of being intuitive enough, that the solution can be estimated even by people with no deeper mathematical studies. So, in these cases, it is possible to begin near the end of the process that is, plotting the solution. This previous visualization allows the accuracy of the formal solutions (if they can be obtained) to be checked and helps to develop the intuitive and graphical thinking. These facts have several advantages:

- it allows the student to estimate results.
- it is possible to predict behaviours of the solutions, not only with O.D.E. (as was studied by Cordero [11]), but also with P.D.E. parabolic problems.
- these problems provide interesting activities related to analytical study and graphical representation of functions of several variables.
- this approach can be used to introduce the student to the use of several packages (Excel, MATLAB, Winplot, etc.), electronic calculators, etc.

Finally, it can be noted that this way of thinking, that is, first estimate and plot and then try to solve formally the problem, is more realistic. Moreover, in a textbook with mathematical models for Chemical Engineers [12] can be read the following sentence: "We survey the plot, then draw the model".

It is interesting to note that the author of that sentence is not a mathematician nor a Chemical Engineering... In fact, the author was William Shakespeare.

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The interplay between the visual and algebraic abilities of students: Learning of solids of revolutions through visualisation using Mathematica

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The purpose of this study is to investigate the interplay between the visual and algebraic abilities of N6 (National Technical Certificate level 6) FET (Further Education and Training) college engineering students after learning of solids of revolutions using Mathematica. Mathematica was used to illustrate (visually) how a given area may be rotated to formulate a solid of revolution (through graphics and animations), with an attempt to make the concept concrete for the students. Students were given a pre-test after verbal instruction and then given a post-test after visual instruction. The written responses were marked and analysed qualitatively. The results of this study reveal that a significant number of students were able to identify the proper method used to compute the required volume (by rotating the area bounded by the given graphs) but tend to abandon the drawn graph when they had to calculate algebraically. Students in this case were able to recognise what was expected but failed to realise or make proper connections between the visual and the algebraic manipulations, hence knowledge transfer was inadequate.

1. Introduction And Background Of The Study

This study was done in an attempt to address the problems I encountered as a lecturer of and external marker for engineering students doing Mathematics N6 (slightly below the level of a first year university course in calculus) and as an external marker for Mathematics N6 for several years. I realised that students continuously encountered problems in dealing with volumes generated when the area bounded by graphs were rotated about the applicable axis. Because students had to imagine the rotations with different types of graphs, I decided to search for a program that could help students visualise the concept. Previously, I introduced solids of revolutions by demonstrating how a circular coin could form a sphere by spinning it. I also encouraged students to think about somersaulting (for the cylindrical disc method) and to think about the CD (music) they play at home for (the washer) as well as using an onion (layers) and a toilet paper in a toilet with one hand unrolling (rotating) while the other hand remains the axis of rotation (for the cylindrical shell). Mathematica software [6] was found to be a better model for such demonstrations. In their study on teaching and learning models for mathematics using Mathematica, Kim and Ryan [7] found Mathematica being a useful visual tool in teaching and learning 'volume of surfaces of revolution' and other concepts.

1.1. Computers in use

Presently computer software in teaching of mathematics are mainly used for "executing algorithms; illustrating multivariable, multidimensional relationships, storing and retrieving data and functions; communicating in a language of mathematics" [2]. These methods involve feeding the computer with data (by the students) so that it does all the calculations since the formulae are readily programmed. After the illustrations through Mathematica, students in my study were expected to draw the graph (if the graph is not drawn), rotate the selected strip and to use the correct formula to do their own calculations in writing, not using the computer to do it for them.

In his study, (using Mathematica) Murphy [9] revealed that in multivariable calculus, students' understanding of the 3D (3 dimensional) objects requires that they visualise.

1.2. The aim of the research

This study aims to investigate the interplay between the visual and algebraic abilities of students after learning of solids of revolutions using Mathematica. Students' written interpretations, as they translated the visual to the algebraic (from the diagram to the correct equation for computing volume) after the correct rotation of the selected strip, were explored. There was less focus on the calculations.

2. Literature Review and the Theoretical Background

2.1. What is visualisation?

Different researchers have different views on what visualisation is. Among others, McLoughlin and Krakowiski [8] associate visualisation with "symbols, pictures, graphics, simulations and animations", which is what this study focuses on.

In their study McLoughlin and Krakowiski [8] point out that the educational system tends to emphasise verbal, symbolic and numerical modes of learning, even though recently focus on 'visual literacy' is also gaining momentum. They believe that learning visually enable one to interpret experience and build understanding and that students should be encouraged to use multiple modes of learning when learning with technology. In that regard, students are able to integrate personal experience and imagination with social experience, technology and aesthetics.

In his paper titled 'A picture is worth a thousand words', Thornton [11] argues that visual thinking should be an integral part of students' mathematical experiences. He highlights that it plays a crucial part in developing algebraic understanding and in valuing a variety of learning styles and it provides a powerful problem-solving tool. When students learn through visualisation, "powerful algebraic thinking arises when students attach meaning to variables and visualise the relationship in a number of different ways" [11]. The importance of visualisation in this regard will be made explicit in this study. The development of algebraic thinking will be explored as students translate the visual (rotation of graphs) to the algebraic (equations).

2.2. Visualisation and verbalisation

In the light of the previous discussion, one can argue that verbalisation provides a link between visual and algebraic understanding [8]. That is, one should also focus on making proper links between visual representation and verbal representation. In her study involving two different classes, Woolner [13] compared the classes in terms of the visual-spatial approach and the verbal approach to the teaching of mathematics. She found that the verbally taught class scored significantly higher than the one that was taught visually. However, she emphasised that the contrasting results could be as a result of a "mismatch between their preferred learning style and the predominance of verbal teaching and assessment". This mismatch was evident in the results of a survey of the thinking styles of students aged 10 to 14 when they were solving problems. The survey revealed that verbal-logical thinkers attempted to solve all problems algebraically, without using the diagrams even if the visual representation was available, whereas the visual-pictorial thinkers attempted to solve all problems using pictures even if it was not necessary (Krutetskii in [11]).

2.3. Visualisation and cognition

Steyn and Maree [10] discuss the thinking preferences of engineering and science students, in terms of the 'four quadrants whole brain approach', as measured by the Herrmann Brain Dominance Instrument (an assessment tool), involving the functioning of the brain. This analysis was based on students learning by using computer graphing technology to visualise and explore two-dimensional graphs. Their study revealed that the majority of engineering students have thinking style preferences associated with the A quadrant (involving critical, logical, analytical and fact-based information) while the majority of the science students have thinking style preferences associated with the B quadrant (involving organised, planned and detailed information), despite the fact that the technology used required visualisation associated with quadrant D. Very few students displayed preference in quadrant D. The above mismatch may be explained using theories of transfer.

2.4. Theories of transfer

The mismatch mentioned above may be as a result of students failing to recognise and realise. Cooper and Dunne [4] define recognition rules, as a means by which individuals are able to recognise the speciality of the context they are in, that is, being able to recognise the kind of mathematics items they are dealing with. They then define realisation rules, as a means by which individuals are able to realise the required outcome, that is, producing the intended response. The students are able to recognise and realise if they are able to transfer what they learned from one level to another, better understood by using Wenger's [12] Social theory.

2.5. Wenger's Social Theory

The students in this study were participating in a community of practice, what Wenger calls a theory of learning and becoming, where learning involves both practice and identity [12]. Brown and Rodd [1] believes that as individuals engage in a world in a particular fashion, they participate in a practice and they become part of the community of practice 'reifying' the tokens of that group. In that way they begin to belong to and hence identify with that community. Issues that affect the recognition and the realisation rules and knowledge transfer will be discussed in detail in the interpretation and the analysis of data.

3. Research Design and Methodology

3.1. The research sample

The sample for this study was drawn from 15 Mathematics N6 students. Out of the 15 students, only seven wrote the pre-test as well as the post-test and were taught using Mathematica, hence the analysis of the results will mainly be focused on their written interpretations.

The sampling procedures were 'purposeful and convenient' [3], since I used the college where I was once a staff member. Permission was granted from the participants, who participated voluntarily [5] without being forced. The students were also ensured that the results will be treated with confidentiality and that their names would not be used [3].

3.2. The research instrument

Students were taught for four periods of 80 minutes each. In the first two periods their lecturer taught them in a traditional verbal way, before the pre-test was administered. In the last two periods, the students were taught by the researcher with the aid of a computer software program (Mathematica), through visualisation and verbalisation, where visualisation was the main method, with verbalisation being used for clarification of ideas. Thereafter a post-test was administered. The students responded in writing in both tests.

During the second two periods, the animations and the graphics where displayed via a projector. The intention was that after the lesson, the students would be in a position to draw the graph if it is not drawn, select the correct strip and illustrate the correct method for rotation (cylindrical disc/shell/washer).

The students were given four questions in a pre-test and six questions in a post-test, resulting in 28 responses in the pre-test (7x4) and 42 responses in the post-test (7x6), since seven students participated. Graphs were not drawn in two questions in neither the pre-test nor the post-test, but in the rest of the questions graphs were drawn. The written responses were marked, analysed and summarised. Students were also asked to give written comments about how Mathematica impacted on their visual thinking and whether they benefited from the program.

Questions in the pre-test and the post- test were discussed with their lecturer to ensure compliance with the required level. For the analysis of data, students' written responses were marked and discussed with their lecturer before the scripts were given back to the students.

4. Description of the Findings

4.1. Introduction

In this section, the students' written responses are presented. They are firstly presented in tabular form (refer to Table 1). Secondly four of responses are selected, two from the pre-test and two from the post-test and discussed individually. The students' responses will be described, interpreted and related in terms of Cooper and Dunne's recognition and realisation rules [4].

4.2. Overall responses

The students' responses are classified in the following categories:

- Rotating the strip correctly and using the correct formula (able and able).
- Rotating the strip correctly but using the incorrect formula (able but unable).
- Rotating the strip incorrectly but using the correct formula (unable but able).
- Rotating the strip incorrectly and using the incorrect formula (unable and unable).

Since this research focuses on the interplay between the visual and algebraic, the emphasis was on the correct rotation of the graph and the correct formula. The focus was on the first steps of the calculation only. So even if a student made a mistake, the student will be regarded as *able and able* as long as he/she managed to rotate the strip correctly as well as using the correct formula. In Table 1, *able and able* will be denoted by *aa* with the total of *aa* per question vertically down denoted by *aat*. The same apply to the other three categories, *au*, *ua* and *uu*. S# denotes a particular student's response, while NG denotes questions where the graphs were not drawn for the students.

Table 1: Classification of students' written responses

	Pre-Test Results					Post-test Results						
	Q1a	Q1b	Q2a	Q2b	Sum	Q1a	Q1b	Q1c	Q2a	Q2b	Q2c	Sum
		NG		NG				NG			NG	
S1	aa	aa	aa	au		aa	aa	au	au	au	au	
S2	ua	ua	ua	uu		uu	ua	uu	uu	uu	uu	
S3	aa	ua	uu	au		uu	ua	uu	ua	uu	uu	
S4	ua	uu	au	uu		au	au	uu	au	uu	au	
S5	aa	uu	au	uu		au	aa	au	aa	au	au	
S6	aa	aa	ua	aa		au	aa	uu	au	au	uu	
S9	aa	ua	uu	uu		uu	uu	uu	au	au	uu	
aat	5	2	1	1	9	1	3	0	1	0	0	5
aut	0	0	2	2	4	3	1	2	4	4	3	17
uat	2	3	2	0	7	0	2	0	1	0	0	3
uut	0	2	2	4	8	3	1	5	1	3	4	17

In overall, the pre-test results indicate that most students performed best in question 1(a), where 5 students were *able and able*. The students' responses for question 2(a) were interesting, with equal number of students falling in the aat, aut and the uut category. Question 1(b) and question 2(b) required that the students start with the graph. However, it is evident that most of the students were either unable to draw the graph or to rotate the strip correctly. The aat of 9 out of 28 responses and the uat total of 7 out of 28 responses under column 6 were found to be interesting. With this highest aat and uat, one wanted to analyse these categories further. The responses for students S1 and S6 were then analysed further.

The post-test results indicate that the highest number of responses was in questions 2 (a) and 2 (b), with 4 students in each case being able but unable. The results indicate that the students were able to rotate the selected strip correctly, but could not select the correct formula to use. The students in this case were unable to translate the visual to the algebraic. In both questions 1(c) and 2(c) as it was the case in the pre-test, it was required that the students should draw the graphs first, hence the poor performance, with most of the students being unable and unable. The post-test results indicate that most of the students performed much lower than in the pre-test, with the highest sum total of 17 out of 42 under column 13 falling in the aut category. For that reason this category was further analysed using S1 and S9 to find out why so many students were able to rotate correctly but failed to compute algebraically.

The uut for both the tests were not considered for further analysis of the responses since there were no interesting interpretations or no interpretations at all in relation to translating the visual to the algebraic.

In the following section, the actual written responses presented are for the categories aa and ua in the pre-test for question 2(a) with students S1 and S6 as well as the category au in the post test using students S1 and S9 as highlighted in table 1 above.

4.3. Pre-test results for S1 and S6

Question 2 (a)

In this question, Figure 1 shows the response for S1 where everything was correct (able and able), and the response for S6 where the strip was correct, the rotation was incorrect but the calculation was correct (unable but able).

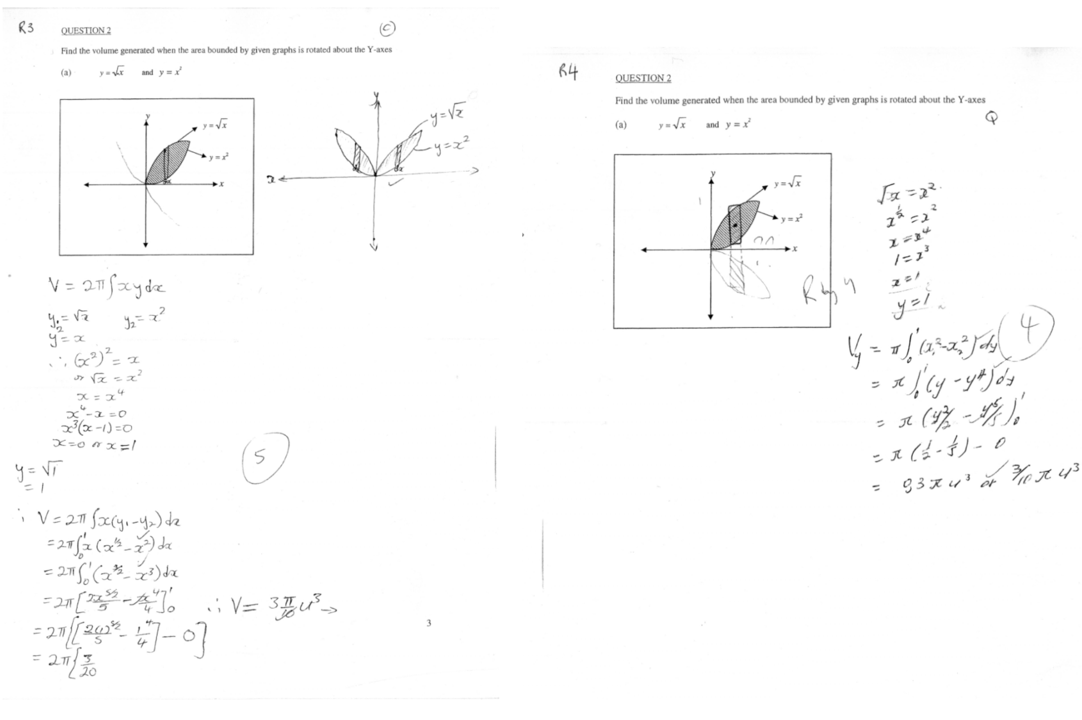

Figure 1: S1 written response (left) and S6 written response (right)

4.4. Post-test results for S1 and S9

Question 1 (c)

This question was very difficult for the majority of the students. They could not draw the correct graph, nor draw the strip. Only one student (S1) out of the 7 managed to draw the correct graph, rotate correctly but used an incorrect formula (able but unable). Refer to Figure 2.

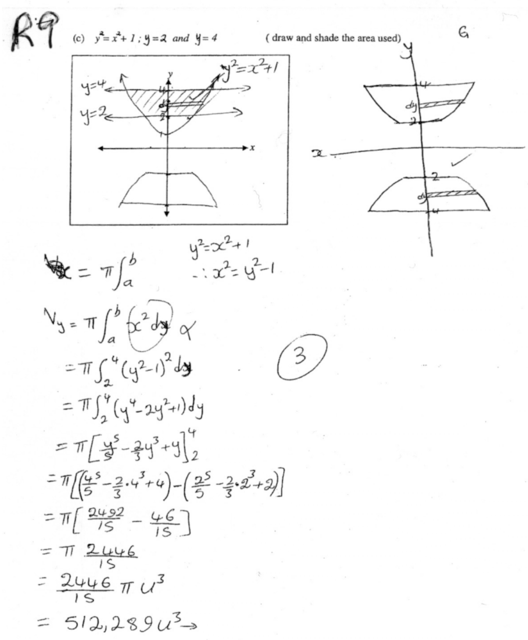

Figure 2: S1 written response

Question 2 (b)

Figure 3 shows how a student (S9) showed two different strips but still worked with one area in computing the volume. This student did not relate the diagram to what was asked (able but unable).

5. Discussion of the Results

The above results reveal some interesting aspects. The students demonstrated different ways of dealing with the given questions. There were instances where there were students who were unable to even draw the proper graph. If the graph was drawn, some students would still fail to draw the proper strip. In some cases, where the strip was correct, the rotation would be incorrect. The students would rotate it incorrectly and therefore select the incorrect formula. Even if the students in this study believed that the model that I used was beneficial to them, some of them were unable to work algebraically even after rotating the strip correctly. The students in this case were unable to translate the visual to the algebraic, hence unable to operate in different levels of representations [8]. The results of my study concur with the one done by Steyn et al [10] about the four quadrants of the brain that revealed that their students were not good visualisers.

The different levels of representation discussed can also be related to the recognition and the realisation rules and the notion of transfer. Failure to translate across is as a result of failing to recognise and realise as discussed by [4]. Students who were (able and able) were able to recognise the speciality of the context they were in and to realise the required outcome. That is producing the intended response. They recognised and realised. Students who were (able but unable) were able to recognise the speciality of the context they were in, but failed to realise the required outcome. They recognised but not realised. Students who were (unable but able) were unable to recognise the speciality of the context they were in, but realised the required outcome. They did not recognise but realised. Students who were (unable and unable) were able to recognise the speciality of the context they were in but did not realise the required outcome. They failed to both recognise and realise.

The above results reveal in some cases that the students failed to transfer. They did not fully

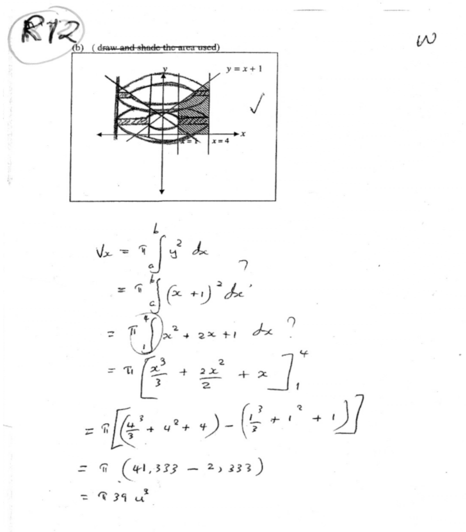

Figure 3: S9 written response

identify with the community of practice [12]. The students in this study were participating in a community of practice, what Wenger calls a theory of learning and becoming, where learning involves both practice and identity [12]. In that way they begin to belong hence identify with that community if they fully understand the rules governing that community.

6. Conclusions

6.1. Reflections on the study

The focus of this research was to investigate the interplay between the visual and algebraic abilities of students after learning of solids of revolutions through visualisation using Mathematica. The results of the study reveal that after visual illustrations, a significant number of students were able to identify the proper method that would be used to compute the required volume, by rotating the area bounded by the given graphs but tend to abandon the drawn graphs when they had to calculate algebraically. As a result, they failed to transfer. This type of results can be attributed to the fact that visualisation on its own is non significant or not useful, rather the focus is to ensure that all different modes of representations are equally important. However, introduction of visualisation will be significant as it was ignored in the past. That was evident from the way in which the students commented about the beauty of mathematics while learning through visualisation as compared to the chalk and talk method.

6.2. Recommendations for strengthening the study

One would then focus on dealing with the 'mismatch' that affect the way in which students are expected to learn and to make visualisation work. Emphasis should then be on balancing all the methods so as to ensure that the students benefit. It is crucial that visualisation be emphasised through verbalisation so as to be successful algebraically and numerically. A computer program that allows students to be interactive, using the graphs they may want to rotate and explore, may as well benefit them.

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On-Line Quizzes in Introductory University Statistics Courses

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Many students arrive at university with negative attitudes to statistics. The challenge has been to engage these students and show that an elementary knowledge of statistics can help them in their science studies. In this paper we report on the development of online quizzes which attempt to engage the students in their learning, the students evaluation of the quizzes, and what the quizzes told us about the problems students have in statistics.

1. Introduction

Students doing a basic first level statistical methods for science unit at Monash University come from a variety of backgrounds. There are those who have done no mathematics beyond year 10, and there are those who have already completed a major study in mathematics at tertiary level. There are some students, typically from certain countries overseas, who claim to have never seen any basic probability or elementary statistics before. The wide variety of backgrounds presents a challenge to statistics educators delivering this unit.

Students also arrive with a variety of calculators, from basic scientific models to graphing calculators and calculators containing computer algebra systems. Because of equity concerns, we have to focus our calculations at a basic level, rather than use the power of the more sophisticated graphing calculators.

In the past students have turned up poorly prepared for tutorial classes in statistics. Part of the problem is due to a negative attitude to statistics developed at the school level. The first few weeks of classes review and extend material already covered by the majority of students, and many students tend to disengage from the unit by not even reviewing materials covered in classes. Unfortunately there is a technical language associated with statistics [1], and students who fail to keep up suddenly find themselves lost in a strange language with $\hat{p}$, p, $\bar{x}$, μ, σ , s, confidence intervals, H_0, H_a and P-values, to mention a few. The challenge has been to engage the students in their learning. In the past we have asked students to attempt a brief one page list of questions before being allowed into the tutorial. Although this worked to some extent, the tutor did not have time to peruse the responses and give general feedback and specific feedback during the tutorial class, and most students never followed up once their sheet of answers were returned a week later.

There is growing pressure for on-line learning resources to be made available to students, to enable more flexible student-centred learning. There are now on-line textbooks for statistics, such as *Hyperstat* [2] and *Surfstat* [3] , and there have been reports comparing traditional and hybrid internet-based instruction, such as [4], and reports on teaching statistics online [5]. In the past lecture notes and other handouts have been available for students to download from the web. Monash University has decided to adopt the WebCT Vista program for delivery of materials to students, and it was decided to try to implement these preliminary questions in online quiz form where possible, and encourage students to complete them by awarding a small percentage of the total marks for them. The quizzes were to be assessed by the computer and feedback at least in terms of correct and incorrect responses given to the student.

2. Method

The existing preliminary tutorial questions were considered for implementation in suitable form for on-line assessment. The available question formats were calculated, combination, fill in the blank, jumbled sentence, matching, multiple choice, paragraph, short answer and true/false. An internal equation editor came with the package. The system was due to go live on 1 Dec 2004, but unfortunately was struggling to go live just before the start of semester 1 in 2005.

It was not an easy task to convert simple questions requiring a written answer to online questions which can easily be assessed by a machine, especially where mathematics is concerned. Indeed only eight quizzes were initially constructed. The aim of the quizzes was not purely assessment, but mainly to ensure that students understand the basics, and are given feedback on weaknesses. As a result, students were given three tries at each quiz. The first try was to have been attempted before the tutorial, and they could have up to two attempts after this to improve their mark if they wanted. They were encouraged to consult with their tutor on any questions they could not understand. The quizzes were worth 5% of the total mark for the unit.

Students were surveyed in their tutorial class in the second last week of semester, having to complete a sheet of questions. The questions asked about where the quizzes were done, how many they did and how useful they were to the students in terms of overall learning and in identifying areas of weakness. (An online survey in WebCT Vista was considered, but it was earlier found to be difficult to get the raw data from such surveys.)

3. Results

The quizzes were generally well accepted by the students. Table 1 gives the total number of students enrolled in the unit and the total number who attempted each quiz. The first two quizzes were fairly simple, and indeed there were comments to this effect in the online discussions. It may be that some students thought these first two quizzes were a waste of time. The mean number attempting Quizzes 3 to 7 was 192, representing 82% of the cohort. The numbers completing quiz 8 were down slightly, but this one was not available until near the end of semester and a few weeks after quiz 7, and students are often focused on other things at this time of semester. There was a correlation of 0.61 between the total quiz score and the final exam score (with 214 students sitting the final exam).

Table 1: Numbers of students attempting each quiz

Enrolled	Quiz 1	Quiz 2	Quiz 3	Quiz 4	Quiz 5	Quiz 6	Quiz 7	Quiz 8
235	221	207	191	197	189	192	191	164

One hundred and twenty-eight surveys were returned. Table 2 gives the numbers of quizzes completed by the respondents. Of those who responded to the question, about 75% had completed all eight quizzes. So even though they were only worth 5%, it would appear that they were of some value to the students.

Table 2: Numbers and percentages of students completing quizzes

No of quizzes completed	None	2	3	4	5	6	7	8
No of students completing the number of quizzes	10	1	3	1	2	7	16	88

Of some interest is where the students did the quizzes. Table 3 summarises the responses here. Generally, they were done on a computer at home, with 87% listing this as a response. 12% of students did them solely at university. For the "other" category, one did all the quizzes at a friend's house, and the other used home computer, university computer and "Dad's laptop while in Sydney". Clearly most students are comfortable using a home computer to do the quizzes.

Table 3: Where students did the quizzes and the numbers using each place

Where were the quizzes done?	No of students
Home	111
University	42
Other	2
Both home and uni	27
No response	1

The students were then asked to rate the usefulness of the quizzes to their overall learning in the unit, to identifying areas of weakness in their knowledge, and also asked how well they thought they assessed their statistical knowledge. The results are given in tables 4, 5 and 6 respectively below. In summary, 84% rated them as very or somewhat useful to their overall learning, 90% rated them as very or somewhat useful in identifying areas of weakness and 75% rated them moderate or very well in assessing their statistical knowledge.

Table 4: Rating the usefulness of the quizzes to overall learning

Category	No of students
Very Useful	38
Somewhat useful	68
Not very useful	17
A waste of time	4

Table 5: Rating the usefulness of the quizzes to identifying areas of weakness

Category	No of students
Very Useful	48
Somewhat useful	67
Not very useful	12
A waste of time	1

The students were asked where they sought help for answering questions they had initially got wrong. The main responses were from the text [6] (indicated by 62% of the respondents), from their tutor (indicated by 33% of respondents) and from other students (indicated by 36% of respondents). The students were also asked how much they thought these quizzes should contribute to the assessment, and responses ranged from 0 to 30%, with a median of about 10%. Some of their other units have similar online assessment components worth 10 – 20%, and they appear to be quite happy with this.

4. Conclusions

Overall the quizzes worked quite well at getting the students to do some work in advance of tutorials, gave them some basic feedback and reduced the marking load for the tutors.

There were problems with implementation of the quizzes. In giving feedback to students, multiple choice questions also gave the correct answer, so students just selected the correct response on the second try sometimes without thinking about it. The use of such questions was limited where possible in later quizzes. In the first quiz students were given the symbol (constructed using the built-in equation editor) and asked to identify its meaning; at least half the students reported that no expression was to be seen, so the equation editor, although limited compared to the Microsoft Word Equation Editor Version 3, had to be abandoned from then on, which was a great restriction. Although a ? was available as a symbol, ? was not. This was quite a problem for a statistics unit. However it is believed that even with these harsh restrictions, suitable questions could be devised in many cases.

Short answer questions came with the difficulty that student response had to exactly match the answer, so all possible answers, and the way that students would present them, had to

Table 6: Rating the ability of the quizzes to assess statistical knowledge

Category	No of students
Very well	20
Moderately well	76
Not very well	29
A waste of time	1

be given. A few of these were used to obtain numeric answers, but students had difficulty writing them in a form that was marked correct. Generally, it appeared that "Calculated" type questions were generally preferable, but only standard mathematical functions (with no calculus options) were available to use in the construction of such calculations.

What sort of questions caused the most problem to students? There were two questions that caused much student angst and consultation. The first was a multiple choice question about identifying independent events:

> Consider the experiment of rolling a fair dice. The possible outcomes are 1, 2, 3, 4, 5 or 6. Consider the following events: E = 2, 4, 6, O = 1, 3, 5, P = 2, 3, 5, L = 3,4. Which of the above events are independent?
>
> a. E and O   b. E and P   c. P and O   d. O and L

Although it was a multiple choice question and students were given the correct answer (d.) after attempting it, they invariably wanted to know why. This was not always the case with multiple choice questions, when students seemed happy to have the answer without fully understanding why. Subsequent discussions with students about independent events yielded examples such as tossing a coin several times and rolling a dice several times, but they were unable to use the definition in terms of conditional probabilities to explain why such events were independent.

The second problematic question was: "The proportion of heads in 500 tosses of a fair coin has a mean of (calculate answer)." Most students wanted to know why their answer of 250 was not marked as correct. It appears that the word "proportion" is not well understood by students. This word is another example of the "technical language" associated with the study of statistics. Students correctly identify percentage, so perhaps it is necessary to associate proportion and percentage on a regular basis so that students become familiar with proportion.

After initial troubles were ironed out, the types of questions that caused problems (with less than about 50% getting them correct) were:

1. Calculation of standard deviation of variables after a linear transformation or for two independent variables added or subtracted
2. Calculation of standard deviation of a discrete random variable (taking only 3 values!) with distribution specified by a table.
3. Calculation of means and standard deviations for the distribution of sample proportions
4. Calculation of means and standard deviations for the distribution of sample means
5. Calculating a z-statistic and a P-value for a one-proportion z-test
6. Calculating correct values of test statistics for a simple t-test for the mean or a chi-square statistic for a two-by-two table. (No quizzes asked them to calculate the test statistic for a two independent sample t test for means.)

It is worrying that so many students are unable to do a fairly straightforward arithmetic calculation using their calculator. Approximately 80% of students have a graphing calculator (or one including a computer algebra system), and the others a simple scientific calculator. A true statistics educator may not worry about this, as generally students will be using a statistical package that gives them all the right numbers (assuming the right procedures are called for). However it does put a large question mark over the numeracy skills of students today.

The quizzes are currently being amended for the next semester, and will be worth 10% of the total marks for the unit. More attention will be paid in lectures and tutorials to addressing the problems identified above.

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Teach fundamental abstract linear algebra starting from singular value decomposition

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Consider the teaching of fundamental abstract linear algebra from vector spaces to eigenvalues and eigenvectors. Many teachers and students find this the most difficult and irrelevant part, respectively, of fundamental mathematics courses. Here we explore using the sophisticated computational engine of singular value decomposition as the starting point in the learning of this linear algebra. The result is that students will be led from more concrete concepts towards the more abstract, rather than losing the students in the first abstract definitions. Topics important to a student's later professional practise will be taught and learnt earlier in the course and as a core part of the course.

1. Introduction

Let us explore the teaching of fundamental abstract linear algebra, that is: vector spaces and subspaces, basis, dimension, linear independence, orthogonality, linear transformations, orthogonal projections, eigenvalues and eigenvectors. Imagine the audience of students is to be a class of engineers taking fundamental mathematics courses from mathematics staff as occurs in many universities.

Why do we teach this material? Because it lays the foundation for addressing really large scale problems that occur in practice—even those infinite in size! *The key is to generalise 2D and 3D vectors concept and tools to arbitrarily high dimension.* Of utmost importance is that we provide tools to solve problems as a whole rather than as a collection of parts to be fudged together. By empowering students to manipulate the whole picture their ability to analyse is raised to a new and powerful level. With these concepts we create the foundation upon which students build in their professional courses to solve large scale problems. It is fundamental knowledge for at least engineers and scientists. This is widely recognised since Uhlig [1] estimates that from 100,000–300,000 undergraduate students "take an elementary linear algebra course in the USA each year".

What is the teaching difficulty? Inevitably the difficulties in teaching this material is personal, but Uhlig [1] reports "there is ample talk in the math ed literature of classes hitting a 'brick wall', when linear (in)dependence is studied in the middle of such a course". I believe that the bulk of students turn away from this material because they feel it has no relevance to them and their career: there is a large amount of abstract concepts to study before they reach results that they view as practical; this eliminates any spark of interest they might have started with; in any case, solving large scale systems is usually far outside their experience—students usually have not even solved a coupled *pair* of differential equations let alone hundreds of coupled equations. Consequently, we need to devote more time to setting up, solving and interpreting large scale problems.

> Is it life, I ask, is it even prudence,
> To bore thyself and bore the students?
>
> *Johann Wolfgang von Goethe*

Given the 'black box' approach to professional computation, do students even need to know any of this linear algebra? I believe the answer is yes. For example, the Fluent software package [2] is a standard for engineering consultants involved in computational fluid mechanics: its manual for use invokes some of the concepts taught in this linear algebra. We need to find a route to teach linear algebra material to fit a 21st century student aspiring to a professional scientific or engineering career.

Adapting the pioneering syllabus of Davis & Uhl [3] and Will [4], and the applications of Akritas & Malaschonok [5], here we explore a route, detailed in §3, which uses the sophisticated computational engine of singular value decomposition (SVD): students start using SVD as a 'black box' to solve professionally interesting problems, then we explain its effectiveness using geometric pictures of the related transformations, move on to orthogonality, subspaces and orthogonal projection, then diagonalising symmetric matrices, and finally general eigenvalues and eigenvectors, bases and linear independence. You might think this order completely screwy. But the most important topics and skills are actually taught first: for example, after the SVD this order of concepts occur in the Fluent manual in roughly decreasing occurrence, see §2—orthogonality is very important in Fluent, eigenvalues of symmetric matrices mediumly, and linear independence does not even rate a mention. Thus with this order of presentation of concepts the students generally see the most important ones first and have more time to assimilate the knowledge for their later career.

Furthermore, this proposal fits reasonably with Uhlig's contention that transforms are the best starting point for linear algebra [1]: we propose exploring the geometric interpretation of matrix transformations immediately after introducing the black box of the SVD.

Are there any glitches in this proposed order of learning? Surprisingly few. The significant glitch is that we need the concept of dimension very early on, whereas a sound definition cannot be made until bases are defined near the very end! We make do with a more intuitive concept of dimension in the interim.

At this stage we do not know how well the course can achieve its objectives of keeping the students interested and active in learning linear algebra. Mulcahy & Rossi [6] comment that "SVD is in fact a very natural and approachable topic". The course is currently being developed for implementation—one of the disadvantages of our large commitment to distance education is the long lead time between conception and implementation; a draft study guide is available for download [7]. After it has been trialled, we will compare student feedback and performance for this course with feedback for the current traditional course.

2. Demand linear algebra

The linear algebra we discuss does arise in a professional career. However, we need to be certain that the most relevant parts are given their due attention in a course.

As preliminary let us explore the application area of computational fluid dynamics. Both consulting and research engineers commonly use computer software as "black boxes" to solve their problems. One widespread software package is Fluent [2]. It comes with reasonably comprehensive online documentation in HTML format on a CD. Using Unix `grep␣-i` and `wc` I searched the entire directory of help files for the occurrence of key linear algebra terms. Table 1 lists the number of lines in which key terms occurred in the Fluent 6.1 help files. It is the common terms that professional engineers will meet in their career; we need to ensure students learn the corresponding concepts.

See that the most common term is projection. Generally it is not explicitly "orthogonal projection", but usually that seems to be what is meant. Usually the projection occurs in 3D physical space, not abstract high dimensional spaces. Nonetheless orthogonal projection appears important.

Next in occurrence are the basic terms "dot product", "diagonal" and "unit vector". These will likely have been taught to students before the abstract linear algebra discussed here, but it will be important to link to and involve these terms to reinforce the students earlier learning.

Table 1: The number of lines in the help files of Fluent 6.1 containing the specified term. (Note: one line of the HTML source typically corresponds to one small paragraph of typeset text, so these counts are roughly the number of paragraphs in which the terms appeared, not the total number of appearances.)

Term	Fluent
column space	0
diagonal	18
diagonaliz*	1
dot product	21
eigenvalue	14
eigenvector	0
inner product	2
least[-]square	9
linear combination	1
linear transform	0
linear[ly] [in]dependen	0
null space	0
nullity	0
orthogonal	8
orthonormal	0
projection	33
principal[-]ax	21
rank	0
row operation	0
row space	0
subspace	0
symmetric	12
unit vector	23
vector space	0
well[-]condition	3

The next most common terms are "eigenvalue", "least square", "orthogonal" and " symmetric"; eigenvectors seem to occur primarily as "principal axes". Our aim is to develop these as a core part of the course. Traditionally least squares is just a small application to one side of the main line of the course, and symmetric matrices and their eigenvalues often one of the last topics discussed. We recommend moving these concepts earlier and into the main course development.

Other terms are barely mentioned, if at all. We seek to only introduce them when absolutely necessary.

This survey of Fluent 6.1 is certainly just one restricted view of the material. More exploration could be done of other areas of engineering disciplines. However, in the interim, let us see what we can reasonably propose for restructuring this linear algebra.

3. Outline the proposed course

Now look at the outline of my proposed course in linear algebra. Much of the inspiration for this structure comes from Davis & Uhl [3] but modified for a prerequisite experience of using Gaussian elimination to solve sets of linear equations. The proposed course [7] aims to introduce all concepts and properties with examples. Our intention is to integrate computer laboratory exploration, but nowhere near as heavily as does Davis & Uhl [3]. A consequent crucial difference between the proposal here and that of Davis & Uhl [3] is that our course is not tied almost inextricably to any one computational engine (Davis & Uhl use MATHEMATICA exclusively), but can be readily adapted to any modern system.

> Suppose you want to teach the 'cat' concept to a very young child. Do you explain that a cat is a relatively small, primarily carnivorous mammal with rectile claws, a distinctive sonic output, etc.: I'll bet not. You probably show the the kid a lot of different cats saying 'kitty' each time until it gets the idea. To put it more generally, generalizations are best made by abstraction from experience. *Ralph Boas*

3.1. Recall assumed knowledge The background knowledge we assume includes: vectors, length, unit vectors, distance between vectors, dot products and their relation to angles and length, orthogonal vectors, linear combinations, row and column vectors, matrix transpose, inverse and some determinants. This assumed knowledge corresponds roughly to lesson MAT.01 of Uhl [8]. Knowledge of Gaussian elimination is not required; although as proposed for our university the accompanying knowledge of over-determined and under-determined systems is assumed and so is given less time here than Davis & Uhl [3] do in an extensive linear algebra course started from the SVD.

3.2. Three magic recipes with the SVD

We start with three procedures important in science and engineering and using the SVD. These procedures cannot be computed by hand as the SVD factorisation involves considerable calculation; I do not know of any graphics calculator that computes SVD. However, we certainly plan to set exercises calling for students to solve and interpret problems and solutions away from a computer given a particular SVD.

Firstly, use the SVD to solve consistent linear equations. The use of the SVD factorisation parallels Gaussian elimination. We introduce orthogonal matrices, rank, general solution and obtain the smallest particular solution by setting any free variables to zero.

Secondly, approximating inconsistent requirements introduces the justification for making minimal changes to observed data. Further, we explore how small divisors in the SVD solution perhaps should be treated as if they are exactly zero.

Thirdly, regularisation introduces that the matrix itself may have errors and may be justifiably approximated. Given adequate time, this would naturally flow on, later in the course, to a discussion on matrix norms and abstract vector spaces.

3.3. Fundamental transformations of the plane

This module establishes the vital geometric foundation of the linear algebra: it corresponds roughly to lessons MAT.02 to MAT.04 of Uhl [8]; and it uses extensive computer graphics in 2D. Basic linear transforms are used to build the interpretation of the SVD factorisation. In particular, we focus on transforming to and from new coordinate systems. This is natural start to different bases, but more accessible as it uses familiar orthogonal bases.

The topics are: Stretchers expand or contract; Rotate with orthogonal matrices; Rotate to new coordinates; Reflections; General orthogonal projections; Shear chaos[1]; Geometry of the 2D SVD; and finally the axioms for Linear transforms.

3.4. The geometry of SVD analysis

The next step is to generalise the visual 2D introduction of the previous module, to arbitrary dimension. This corresponds to lessons MAT.06 and MAT.08 of Uhl [8]; it also is analogous to the introduction described by Akritas & Malaschonok [5].

The key is that multiplication by orthogonal matrices is effectively a rotation: it does not change lengths nor angles. We introduce the axioms and properties of subspaces, in particular the null space, and the concept of span. Then we explore how orthogonal projection minimises distance. With these tools we then prove that the first two recipes, based upon SVD in any dimension, are indeed correct. In particular, the second recipe gives the least square solution [5].

3.5. Diagonalise symmetric matrices

In many applications, the eigenvalues of symmetric matrices occur more often than general matrices. With the SVD it is more natural to explore eigenvalues of symmetric matrices first. So we do. As a last step we establish a procedure for calculating any SVD. These topics roughly correspond to MAT.09 and MAT.10 of Uhl [8].

The topics are: orthogonal diagonalisation (via the SVD) and possible applications come first; eigenvalues and eigenvectors, definition, characteristic polynomial, homogeneous solutions give eigenvectors, reality and orthogonality; compute any SVD via symmetric matrices; the SVD exists for all matrices.

3.6. Eigenvalues and eigenvectors of general matrices

Move on to eigenvalues and eigenvectors of general matrices via some examples to show firstly that the definitions of eigenvalue and eigenvectors still apply, and secondly that the marvellous properties for symmetric matrices fail.

Triangular matrices are slotted in here as they are a source of simple examples. Instead of orthogonality, eigenvectors are linearly independent and so this property is introduced here. This leads onto how bases replace orthogonal coordinate axes, and finally achieve a sound general definition of dimension. Lastly we diagonalise general matrices with applications. This route develops these concepts based upon the more specific cases handled earlier. We could possibly discuss the polar decomposition [6].

3.7. Approximate matrices

Corresponding roughly to lesson MAT.07 by Uhl [8], we justify the third magic recipe of approximating matrices. A theme application to start is image compression [e.g. 5]. Then regularisation and condition number naturally follow. This module could go earlier, as Uhl does, but eigenvalues and eigenvectors appear more important.

3.8. Abstract vector spaces and transforms

Lastly we introduce the most abstract part of the course and its applications. Corresponding roughly to lesson MAT.11 of Uhl [8] we define abstract vector spaces, applied to function

[1] Shearing, especially as employed in Gaussian elimination, is briefly linked to an advanced course on chaos.

spaces, linear transforms, and perhaps apply the concepts to numerical differentiation.

This and the previous module could be omitted when time is short.

4. Conclusion

The ubiquity of modern computing power empowers us to reorganise important foundations of the mathematics curriculum. My view of the proposal outlined here is the following: the classic linear algebra curriculum is a beautiful step by step construction from the foundation up of high dimensional analysis, but most students fail to perceive its beauty; the course proposed here reverses much of the material to "dig" down, from the black box of the three magic recipes, deeper and deeper to the foundations until the same marvellous structure is taught. The difference is that starting from the top enables us to relate more closely to problems the students should recognise as important. Moreover, this reorganisation seems to to introduce early in the course the concepts that practicing professionals need to know most strongly.

We look forward to trialling this proposed course in the near future.

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Pointing the way to proof

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The habits of communication, the habits of thought and the point of view of pure mathematics represent a sort of 'mathematical culture'. Our students' first introduction to mathematical proof is also an initiation into this new culture. People who become immersed in an unfamiliar culture often have trouble knowing what is significant and what is not. They don't know what to pay attention to. So it is with our students. One of the challenges that we face as teachers is to help them focus their attention in productive directions so that they can prove theorems.

Undergraduate teaching is the primary mission at my institution, so I was taken aback, some years ago, to hear one of my colleagues say, 'I used to believe that the most important thing that happens at our college is teaching, but I have changed my mind. . . . ' After a pause, he let the other shoe drop. 'I have come to realize that the most important thing that happens at our College is *learning*'. This suggests a false dichotomy, of course. On the one hand, it is clear that the real issue is not what teachers do; it is what happens to their students as a result that is truly important. On the other hand, what teachers do is vitally important since if teachers were to do nothing there would be no 'resulting' effect on the students. It is the way that we focus our attention when we decide *what* to do that is at the heart of the phrase 'student centered teaching'.

I am on the faculty at a small, private undergraduate college in the USA. Admissions are fairly selective, so our students are generally well-prepared, hard-working, and bright. We try to structure our classes in such a way that our students have a great deal of responsibility for developing and discovering mathematical ideas for themselves. All of our mathematics majors take 'Foundations', in which we introduce them to mathematical language and proof. In Foundations, our students prove all of the theorems themselves and present them to each other in a seminar setting. Our upper level theoretical courses build on this experience. This paper summarizes insights that I have gained over the years into useful ways of supporting my students' theorem-proving efforts. I will concentrate mostly on the issues raised early in this process.

1. What we say/what they hear—culture shock in the classroom

When we speak to our students, we need to consider how they will interpret what we say. This may seem obvious, but traps await the unwary. We all know that mathematicians use a private language, a dialect of English (or French or Chinese, or whatever) with strict grammatical rules and a peculiar vocabulary all its own. The danger for the incautious teacher is that we co-opt perfectly common words and use them in seemingly ordinary ways, while insidiously ascribing to them a technical meaning that most ordinary people do not. Consider the word 'definition', for instance.

What is a definition? To a mathematician, it is the tool that is used to make an intuitive idea subject to rigorous analysis. To anyone else in the world, *including most of our students*, it is a phrase or sentence that is used to help understand what a word means. Given this, it is no wonder that our students are baffled when we say things like:

Definition: A function f is continuous if for all x in the domain of f and for all $\epsilon > 0$ there exists a $\delta > 0$ such that ...

Even in cases where definitions are not subtle and deep like the definition of continuity, we sometimes make weird choices. Consider the standard definition of pairwise disjoint:

A collection Ω of sets is said to be pairwise disjoint if for all $A, B \in \Omega$, either $A = B$ or $A \cap B = \emptyset$.

Why not the following?

A collection Ω of sets is said to be pairwise disjoint if for every pair A, B of distinct elements of Ω, $A \cap B = \emptyset$.

These are clearly equivalent, and the second is much better for coming to understand the meaning of the phrase 'pairwise disjoint'. So why do we use the previous definition? Because it is phrased in the most useful way, from a *mathematical* point of view. That is, from the point of view of someone who understands that the purpose of definitions is to allow one to make logical connections and prove theorems. (When we want to prove that a collection is pairwise disjoint, we assume that $A \cap B \neq \emptyset$ and then to prove that $A = B$. This process is more easily suggested by the first formulation than by the second.)

It only gets worse. In real life, one reads a definition, uses it in a sentence, and then discards it. In the case of a mathematical definition students probably won't be surprised to find that they are expected to draw a picture or look for examples, but they think we have really lost our grip when we make it clear that we expect them to memorize definitions precisely and word for word. (Only crazy middle school English teachers make anyone memorize definitions.) Our students are experiencing culture shock.

Communication in the other direction can sometimes be just as frustrating. Suppose one of our students says that something is 'obvious'. They likely mean that it is 'common knowledge' or at least something with which they are previously very familiar. For instance, if they are given the field axioms and then asked to prove that for all elements x in the field, $0 \cdot x = 0$, some students will wonder why they should have to prove this, as it is 'obvious'. Since we pure mathematicians use the word obvious as a shorthand for 'this can be easily established from previously proved facts', we may be flummoxed by their contempt for the assignment. In my experience it is helpful to stipulate two things: first, that $0 \cdot x = 0$ is, indeed, a pretty basic thing to prove but that they are just starting out and that people don't learn to prove theorems by immediately tackling the really deep ones. Second, that we are trying to help them to understand how mathematical ideas are connected. This fact is sort of a 'test' for the axioms. $0 \cdot x = 0$ is so fundamental that if the axioms did not allow us to prove it, we would have to assume it. The amazing thing is that the field axioms discuss only additive properties of 0; nevertheless, because we assume a relationship between addition and multiplication (the distributive property), this multiplicative property of 0 is obtained 'for free'. It is inherent in the way that addition and multiplication interact. It holds because nothing else is possible. If we can get our students to see that this is, perhaps, not so obvious, we have helped them take a *cultural* step toward becoming mathematicians.

2. What? ... Where?

Part of our job as teachers is to introduce our students to our 'mathematical culture'. That is, the habits of communication, the habits of thought, and the point of view of mathematics.[1] You may ask what mathematical acculturation has to do with helping our students prove theorems. I suggest that it is not just a pleasant or welcoming thing to do. It is absolutely central to their

[1] In the larger educational context, I believe that we must introduce them to the 'cultures' of both pure and applied mathematics. In their early introduction to proof, I am generally working on the pure mathematics side of mathematical culture and it is in this limited context that I am discussing the issue here.

mathematical education. Like anyone immersed in an unfamiliar culture, they will initially not know what to pay attention to.

Even students who have previously been very successful in their mathematics courses sometimes struggle when they are first introduced to mathematical proof. Why? The precise formulation of intuitive ideas is the key to mathematical discourse. But this does not come naturally to many students. In fact, things that we pure mathematicians do quite easily are often more complicated than we think. Our ability to handle mathematical formalism even at the most basic level requires that we be able to do several things. We have to be able to take an intuitive statement and write it in precise mathematical terms. Conversely, we have to be able to take a (sometimes abstruse) mathematical statement and 'reconstruct' the intuitive idea that it is trying to capture. We have to be able to take a definition and see how it applies to an example or the hypothesis of a theorem we are trying to prove. We have to be able to take an abstract definition and use it to construct concrete examples. Though closely related, these are actually *different* skills; fortunately, they can be learned. Teaching our students to do these things is, in fact, part of introducing them to the culture of pure mathematics; at the same time, it is an essential part of helping them learn to prove theorems.

Perhaps an example would be helpful. One day 'Karen' came to my office, purporting to be completely stuck on what I knew to be a one or two sentence proof requiring only a simple application of a definition. I made Karen look through the words in the theorem and asked her to look up the relevant definition and read it aloud. Then I asked her if she saw any connections to the statement of the theorem. Karen immediately saw how the theorem could be proved. One might be tempted to conclude that Karen didn't try very hard before giving up and asking for help. But I don't believe this was the case. In view of her previous experience with definitions, I believe that looking carefully at the definition simply never occurred to her!

You may wonder why I required that Karen read the definition *out loud*. This can feel uncomfortable and somewhat childish at first, but I find that it makes a difference. I think that when students are simply asked to look at their books, they will revert to skimming through the definition rather than carefully reading every detail. This hypothesis is supported by my observation that students often resist reading the definition in its entirety; they will just focus on the final upshot or what they think of as the 'main point'. It is part of our job as teachers to show our students that no part of a definition is superfluous. When I said before that 'looking carefully at the definition simply never occurred' to Karen, the emphasis was on the word 'carefully'. Students don't ignore definitions, they just don't start out knowing how to use them. It can take time to fully move from a long-held habit of using definitions as a general guide, to using them as the engine that drives the mathematical machine.

Charlie came by another day, with a problem similar to Karen's. But unlike Karen, Charlie was not able to make the necessary connections from just a careful reading of the definitions. Reading the definition did not help because he had no idea what it was saying; furthermore, he had no strategies to cope with this lack of understanding. We pure mathematicians do. We look at special cases. We construct examples. We look through every detail several times, working to understand its role in the definition. Just pointing Charlie in the direction of these strategies was not enough, however. These are skills that many students, including Charlie, do not bring to the table. Charlie needed a model for the necessary behavior. So I guided him through the building of an example.

CSS: The definition says 'let S be a set'. Alright, pick a set and write it on the board.

Chr: *Any* set?

CSS: Yes, any set.

(Charlie writes $S = \{a, b, c\}$ *on the board.)* ...

As we worked through the example, I gave broad, unsubtle hints, but Charlie made all the choices and drew the necessary conclusions. By the end, he was much more able to understand what the pieces of the definition were saying and how it all fit together. Even better, he had a concrete model for how one goes about constructing an example to help understand a definition. (Just providing an example for him would not have been nearly as useful, since it was going through the construction that would help him in the future.) After working through the example, Charlie was ready to use the definition in finding the proof.

Charlie and Karen needed guidance on where to focus their attention. Without it, they were at a loss.

Here is another example. My students had just started studying vector spaces in a linear algebra class. Linear subspaces were on the agenda for the day. They had all read the definition of linear subspace in the text before coming to class, so we discussed it briefly. In order to make the concept more tangible, I wanted my students to see how the definition works itself out in $\mathbb{R}^2$ and $\mathbb{R}^3$—a natural and standard first step. This is not a difficult exercise, so when I assigned it, I thought that it would work well as a short, in-class project. I divided the class into small groups and set them to work on describing the linear subspaces of the plane and of 3-space. The students easily saw that they should separately consider closure under scalar multiplication and closure under vector addition. But at that point many students were completely stymied; they had no idea how to proceed.

It took me a some time to realize what the difficulty was. This was the first time that the students had encountered the concept of closure under an operation. They focused on the familiar; glossing over the new idea. One can imagine that they perceived the breakdown this way:

closure under scalar multiplication and closure under vector addition

Once again the students' attention was on the wrong things.

In order to make headway on the problem, they first needed to consider what it *means* to be closed under scalar multiplication, and what it *means* to be closed under vector addition. They would have been much better off if I had realized this ahead of time and given them some exercises in which they first considered these questions.

For instance, to get them started, I could have had each student choose a specific vector $\mathbf{v}$ in $\mathbb{R}^2$. I would then specify an (unknown) subset S of the plane that contains $\mathbf{v}$ and that is also closed under scalar multiplication. Then I could have asked each student to find 3 or 4 vectors that must be in S. Very rapidly, the students would have seen that if a set is closed under scalar and contains a non-zero vector, it must contain the entire line that passes through the origin and that vector. Alternatively, I could have given them some specific subsets of the plane to work on, asking them which were closed under scalar multiplication and which were not. Similar exercises could have given them some insight into closure under vector addition. After doing these 'preliminary' exercises, the students would have been in a much better position to tackle the more substantial question: what are the linear subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$?

As 'cultural mentors' it is our job to carefully and deliberately point our students in the right direction. But this can be tricky; we are so steeped in our own habits of thought and mathematical point of view, that we sometimes find it hard to identify them as such. As guides introducing novices to our culture, we must first do the hard work of analyzing how we think about and choose to express ideas.

3. Sorting out the issues

The very first theorems that students are asked to prove, require only that they make logical, step-by-step deductions. There are few dead-ends or blind alleys. But things soon become more complicated. Students often stumble when there are too many issues to deal with at one time. They do not have our resources for breaking a problem up into smaller pieces. Moreover, their tendency to conflate various issues often obscures the intuition behind the mathematics, making them lose sight of the 'big picture'.

Consider, for instance, the relationship between equivalence relations and partitions. We know that every equivalence relation generates a partition of the underlying set and that every partition generates an equivalence relation on the underlying set—a tremendously significant fact. The ideas involved are not difficult, but there are many things going on at once; and this can lead to a good deal of confusion for our students.

Suppose we have an equivalence relation on a set A. What is involved in proving the 'big theorem'?

1. First we apply the procedure for forming the equivalence classes from the equivalence relation and so obtain the collection of sets.

2. Next, we prove that the resulting collection is pairwise disjoint.

3. Finally, we prove that the union over the collection is all of A.

Now this may not sound too bad until you realize that there are several hypotheses lying around (the relation is reflexive, symmetric, and transitive) and the proof must make use of the fact that the subsets were obtained in a particular way. The students have trouble attacking the problem. Though no individual piece of the puzzle is difficult, the students are overwhelmed by the many choices.

Suppose that instead of taking this traditional route to equivalence relation/partition duality, we start on the other end. That is, we start with a partition of the set A (motivated by the useful notion of 'dividing the elements of A into categories'). At first glance, this doesn't seem to make things much better. Not until we realize that with this approach, we can break things down into a sequence of manageable theorems that all add to the big result.

Once again, we have a procedure. This procedure allows us to define an equivalence relation on A from the given partition. But we make an observation. The procedure can be used on any collection of subsets of A to obtain a relation on A. (The relation may not be an equivalence relation if the collection of sets is not a partition, however.) Giving the students some exercises to help them become familiar with the procedure outside the context of partitions, helps them once they start proving theorems.

It is not too hard to show that given any collection Ω of subsets of A, the relation obtained from Ω is always symmetric. It is reflexive if and only if the union of the elements in Ω is all of A. And if the collection Ω is pairwise disjoint, then the resulting relation is transitive.[2] Once the students understand the procedure that is used to generate a relation from a collection of sets, students can easily prove that the relation is symmetric. As a separate problem, they can conjecture the condition needed to get a reflexive relation and prove their conjecture. Finally, they can prove that a pairwise disjoint collection yields a transitive relation. With this series of three very manageable results we have helped our students focus their attention on one piece of the problem at a time. In the end, we obtain as a corollary the fact that a partition of A always yields a reflexive, symmetric and transitive relation on A. As a side benefit, notice that this provides a nice motivation for the definition of equivalence relation!

Once these ideas have been established, most students are able to handle the converse, in which an equivalence relation can be seen to yield a partition.[3] Though the multiple hypotheses are still present, the students have gained some experience dealing with the relationship between collections of sets and relations. Moreover, the ideas have been de-mystified by the earlier work with partitions. When students understand why a theorem is interesting or useful, they almost always find it easier to prove. Another part of our support role is to make sure

[2] The converse of this last statement is not true.

[3] Once again, it is useful to note that the procedure used to produce the equivalence classes from an equivalence relation does not require that the relation be an equivalence relation. If we apply the procedure to any relation on A, we get a collection of subsets of A. (The collection will not, in general, partition A.) It is very helpful to the students to get familiar with this procedure out of the context of equivalence relations before they try to introduce the reflexive, symmetric, and transitive conditions.

our students have a sense of the 'big picture', the underlying motivation for the theorems they are proving.

4. Conclusions

As I have thought deeply over the years about what I need to do to help my students prove theorems, I have found that a big part of my job is to focus their attention on the right mathematical elements. They can do the rest. The mathematical instincts that tell us which pieces of a problem to concentrate on in order to attack it successfully come with immersion in 'mathematical culture' and are absolutely essential for learning to prove theorems. Therefore, it is in our job description as mathematics teachers to cultivate these instincts in our students by being their guides in our foreign culture.

In conclusion, I will add that seasoned mathematicians tend to be self-reliant and self-confident. If something is true, we pure mathematicians have the feeling that with sufficient effort we ought to be able to prove it. It is this attitude that allows us to continue working on hard problems, even when multiple attempts have not provided a solution. In some students this self-assurance seems to be inborn, in others it requires fostering. To develop it, students need to build the habit and expectation of success, which takes time. So the early stages of my campaign to make my students into mathematicians includes a lot of encouragement and moral support for them and for their efforts.

The Math Gateway: A New Portal to Online Materials

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The Math Gateway (mathgateway.org) is a new resource for tertiary and secondary mathematics education that can be found on the World Wide Web at mathgateway.org. It is a recently-released component of the National Sciences Digital Library (nsdl.org), a multi-faceted project supported by the (USA) National Science Foundation (nsf.gov).

The Gateway is an outgrowth of a series of informal annual meetings of mathematics web site project directors and others who called themselves the Mathematical Sciences Conference Group on Digital Educational Resources. Under the leadership of the Mathematical Association of America (in particular, Conference Group convenor Lawrence Moore and Publications Director Don Albers), the various web projects became "partners" in a successful MAA proposal to NSF to create the Gateway as a single point of entry for multiple collections of Web-based resources.

The initial Gateway partners, in alphabetical order, are

- College Board AP Central, apcentral.collegeboard.com
- Connected Curriculum Project, www.math.duke.edu/education/ccp
- Consortium for the Advancement of Undergraduate Statistics Education (CAUSE), causeweb.org
- Demos with Positive Impact, mathdemos.org
- Eduworks Reusable Learning Project, reusablelearning.org
- Eisenhower National Clearinghouse, enc.org
- Ethnomathematics Digital Library, ethnomath.org
- iLumina, ilumina-dlib.org
- Mathematical Sciences Digital Library (MathDL), mathdl.org
- Math Forum @ Drexel, mathforum.org
- MathWorld, mathworld.wolfram.com
- Multimedia Educational Resource for Learning and Online Teaching (MERLOT), merlot.org
- National Curve Bank, curvebank.calstatela.edu
- Virtual Laboratories in Probability and Statistics, www.math.uah.edu/stat/
- WebODE Project, www.webodes.org (coming soon)
- WeBWorK, webwork.math.rochester.edu

Of course, each of these resources can be visited individually, and most of them have their own search tools. The primary significance of the Gateway is that users will be able to search over all the collections at once, and items in the search results will be labeled by the source collection from which they come. In addition, the Gateway will feature a regular Math in the News column and will provide many other services to viewers. In particular, users will be able to construct and maintain their own personal collections of "favorite" resources.

The list of 16 partners is not a closed shop. Additional collections will be welcomed, and guidelines are in preparation for standards and procedures for inclusion.

I am involved in this project in two ways. Lawrence (Lang) Moore and I are the co-directors of the Connected Curriculum Project, and I am the founding Editor (soon to be Editor Emeritus) of the *Journal of Online Mathematics and its Applications*, the scholarly journal of MAA's Mathematical Sciences Digital Library (MathDL).

MathDL is itself a multi-component collection, i.e., a library, and it is still growing. Its components include

- *Journal of Online Mathematics and its Applications*, peer-reviewed journal (JOMA, 2001)
- Digital Classroom Resources, reviewed and tested materials (DCR, 2001)
- Convergence, a history of mathematics magazine (2004)
- Open Source Sharable Mathlets (OSSLETS, 2004)
- MAA Reviews (2005)
- Classroom Capsules (2005)

The two most recent components are online adaptations of features that have previously appeared only in MAA's print journals.

MAA Reviews will replace the full and brief reviews of books and other materials that have appeared in the *American Mathematical Monthly* and *Mathematics Magazine*. It will also replace MAA's Basic Library List by flagging BLL items for the search tools.

Classroom Capsules will replace and build on the 111-year-old feature that has been in the *College Mathematics Journal*. Initially, its content will include PDF files of classic contributions that have been archived in JSTOR. However, unlike JSTOR, which houses only print materials that are at least three years old, the Capsules collection will be brought up to date, and it will eventually include more interactive materials.

Of these six components, JOMA, DCR, and OSSLETS will remain free and open to everyone for the foreseeable future. *Convergence* is in transition to a subscription-only model. Reviews and Capsules will be offered as a privilege of membership in MAA and may be offered to the world at large for a modest fee.

The MathDL components are just a few of the sources of quality materials that will be quickly searchable and organizable into personal "libraries" through the Math Gateway. We are on the verge of taming the vast store of information on the Web and making it truly usable for mathematics education.

Teaching Elementary Inductive Inequality Proofs: A Strategy for Success

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Mathematical Induction is an important topic in a modern computer science or information technology undergraduate program. It is a fundamental and powerful proof technique, but it is also one with which many students have great difficulty. We consider a simple technique to assist students in constructing inductive arguments for proofs of elementary inequalities. Student reaction at Bond University has been positive.

1. Introduction

Mathematical Induction (MI) is a standard tool of the trade for a mathematician or computer scientist. It is not uncommon in published proofs to see an "obvious" identity, justified by the statement "by induction, it therefore follows that . . . ". Details are, in many cases, not given, and a casual reader may form the impression that MI is a rather unimportant process, whereas in reality, it is a very powerful and widely-used proof technique. We are concerned here with aspects of teaching this technique to computer science (CS) or information technology (IT) students. MI is an important topic in a modern CS or IT undergraduate program, and classic computer science volumes such as Aho & Ullmann [1] and Graham *et al* [7] make it absolutely clear that induction is fundamental to those disciplines.

For CS/IT students, the value of induction is clear. It has strong links to recursive structures and algorithms, and provides the logical basis for their claimed correctness [5], [11]. Further, the definition of the syntax of programing languages is conventionally expressed in the recursive (inductive) Backus-Naur formalism (BNF) [1]. Such material is fundamental to the theory of languages and language translators [1], and is a very important component of the education of a CS/IT professional. Just as induction allows a proposition regarding an infinite collection to be proved, BNF allows description of an infinite set of strings (i.e., a language) by a finite, compact, inductive notation.

Any modern text on discrete mathematics will cover induction; for example [6], [8], [9], but perhaps the richest source of insight into the nature of the inductive proof technique is to be found in Polya's classic work [12]. It contains many interesting examples and is well-worth reading. A more modern work on proof techniques in general is that of Solow [13]. It is a slender volume, but well-written and very readable; unfortunately its coverage of induction is somewhat scant. A very good article on mathematical induction, targeted to high-school teachers, and appearing in the 1970s, is that of Výborný [15].

For CS/IT majors, induction finds its natural niche in a typical discrete mathematics subject, targeted to such students. We have said that it is a fundamental and powerful proof technique, but it is also one with which many students have great difficulty. At Warthog Delta '01 [14], I described my work at Bond University in teaching this topic to IT majors, and my attempts to give the students a framework for handling it. In that paper, I noted the requirement for *domain knowledge* and the apparently simple job of realizing when the proof was complete. True, there are examples of inductive proof where one suddenly seems to be *home free* and it all seems somehow too easy ("have we really proved anything?"). This occurs occasionally when the normally difficult part of the induction argument, the so-called *inductive step*, i.e., showing that $P(k+1)$ follows from $P(k)$, is almost trivial. However, a much more common situation for students learning the technique of induction is the problem of completing the inductive step in cases where a significant amount of algebra is involved, and especially where inequalities are present. In such cases, it may be very difficult, especially from a student viewpoint, to establish the truth of $P(k+1)$ from the assumed truth of $P(k)$.

Thus, we assert that, among the more difficult inductive proofs, at least at an elementary level, are those involving inequalities. It is well-known that inequalities are generally more difficult to deal with than equations. A sequence of estimates is usually required, and such a sequence requires a level of mathematical maturity which often appears to be lacking in significant numbers of CS/IT undergraduates. In this paper, a very simple procedure is proposed where inductive inequality proofs (at least the "elementary" ones) may be simplified.

2. Teaching of algorithms considered harmful?

Anderson *et al* [3] seem disappointed that students "appear to be carrying out an algorithmic procedure" when doing an inductive proof. Indeed, in some areas (including that of primary school arithmetic), the teaching and learning of algorithms has developed a bad name in recent years [10]. While some may claim that the rote learning of a procedure is better than nothing, we assert that a deep understanding of the structure of induction and a practical facility of applying it to what might otherwise be very difficult proofs must be the ideal. Yes, some procedural aspects of an inductive proof, as followed by many students, do certainly appear to be "algorithmic"; indeed, at least one discrete mathematics text [2] introduces induction as an algorithm. In any case, it should be acknowledged that the final inductive step, i.e., that of establishing $P(k+1)$ from $P(k)$, can be almost arbitrarily difficult, and in certain cases, essentially impossible. We argue here that any reasonable process (algorithmic or otherwise) that can smooth the way for students, thus allowing them to focus on this difficult, last step, is surely worth considering. In this paper, we present a rather traditional "algorithmic framework" for proceeding with an induction proof. This is certainly not new, but when combined with our strategy for proceeding with elementary inequality proofs, allows students to focus on the difficult part of the proof, i.e., the so-called "inductive step", by simplifying the inequality.

3. Classes of elementary inductive proofs

While our little taxonomy here is not exhaustive, elementary inductive proofs that CS/IT students are typically expected to master may be broadly classified as in the following subsections [6].

3.1. Divisibility

Problems in this category often take the form $a \mid ((a+1)^n - 1)$ for $n \geqslant 0$ and integer $a \geqslant 2$. This result may be shown by invoking the binomial theorem, but it is rather easier to do by induction. The problems in this general category are significantly easier than those in the categories below, and the inductive step merely requires knowledge of $x^{k+1} = x \times x^k$ and minimal further algebraic manipulation.

3.2. Summation

Induction is routinely used to establish such results as

$$\sum_{j=1}^{n} j^2 = \frac{n}{6}(n+1)(2n+1)$$

Algebra required here is the addition of a cubic and a quadratic, extraction of a common linear factor, simplification of the sum of quadratic and linear expressions, and finally, factorization of the resulting quadratic. These steps can pose significant difficulties for those with modest algebraic skills.

3.3. Solution of recurrence or difference equation

If a recurrence relation is solved by the method of iteration (guessing a pattern) [6, p475], or similar non-rigorous method, the students are then encouraged to verify any claimed solution by appeal to induction. For example, the Catalan recurrence $(n+1)\,x_n = (4n-2)\,x_{n-1}$, with $x_0 = 1$ has solution

$$x_n = \frac{(2n)!}{n!\,(n+1)!}$$

and this may be shown by induction.

3.4. Inequality

This category is of particular interest to us. It is potentially huge, so we confine ourselves to a few special cases. In each case, we consider inequalities of the form $f(n) > g(n)$ for $n \geqslant n_0$, where f and g are "simple" functions.

(a) We wish to show that the terms of a recursively-defined sequence exceed some (usually simple) function of n. For example, in the Warthog Delta paper [14], I considered the proposition that the Catalan numbers eventually exceed 3^n. This result is rather easy to prove and perfectly illustrates the present technique of Section 5. We supply a similar proof in the present Section 6 that the Catalan numbers beyond the zeroth are bounded above by 4^n.

(b) The second case concerns inequalities of the form $a^n > g(n)$, where $g(n)$ is a linear or quadratic function of n. In this class of problem, a is typically a positive integer. For example: $2^n > n^2$ for $n \geqslant 5$.

(c) The third common case takes the form $n! > a^n$, in which a is again a positive integer. For example: $n! > 10^n$ for $n \geqslant 25$.

4. Induction – a four step approach

At the Warthog Delta '01 meeting, I presented some notions for using the Excel spreadsheet to generate some more interesting inductive problems [14]. By using the spreadsheet, the students are able to invent their own hypotheses, to be later proved by induction. In that paper, I gave a four-step approach, which is amplified below. The usual two-step approach (first show true for $n = n_0$ and then show $P(k) \Rightarrow P(k+1)$) is expanded to four steps here. Suppose we wish to establish the proposition $P(n)$ for $n \geqslant n_0$.

1. Show true for $n = n_0$.

   This is usually trivial, but in some cases is not; these may benefit by use of Excel or similar computer software [14].

2. Suppose $k \geqslant n_0$ and that $P(k)$ is true.

   This step is mindless/algorithmic and requires no imagination whatsoever. From a computer science viewpoint, it is merely like macro processing or string substitution (we replace n by k in the proposition to be proved).

3. State GOAL: Given the information in step 2, to establish $P(k+1)$.

   Similar remarks as for step 2 apply here. Perhaps it is not completely mindless: usually one needs to replace k by $k+1$ but put the latter expression into parentheses. More macro processing here.

4. Establish GOAL by making use of the assumption in Step 2 and domain information.

   Aha! Now we need to put our thinking caps on. Somehow, $P(k+1)$ must be related to $P(k)$. As noted earlier, this can be of arbitrary difficulty.

5. Refinements for inequality proofs

We propose two extensions of the method just described, for the classes of inequality described in Section 3.4.

(a) At step 2 of the procedure described above, we introduce a slack variable (a term borrowed from linear programming and the simplex method). This converts an inequality to an equation. The rationale for this is that equations are more easily dealt with than inequalities. All we have to remember about the slack is that it is positive.

(b) At step 3, we convert the inequality to be proved in the inductive step to one in which the right side is zero. This has two principal advantages: cancellation often occurs, and our goal is always the same (to ultimately show that the left side is positive).

6. Examples of the method in action

1. As an example of type of problem outlined in category (a) of Section 3.4, we prove the claim $P(n) : 4^n > x_n$ for $n \geqslant 1$, where (x_n) is the sequence of Catalan numbers, which for $n \geqslant 1$ obey the recurrence $(n+1)\,x_n = (4n-2)\,x_{n-1}$ and begin with $x_0 = 1$.

   (a) Since $x_1 = 1$, it is clear that $P(1)$ is true.

   (b) Suppose $k \geqslant 1$ and that $4^k > x_k$. That is, there exists $\alpha > 0$ such that $4^k = x_k + \alpha$.

   (c) State GOAL: Given the information in (b), to show that $4^{k+1} > x_{k+1}$.

   (d) Establish GOAL by making use of the assumption in (b) and domain information. Consider $4^{k+1} - x_{k+1}$, and put $n = k+1$ in the defining recurrence for Catalan, to obtain $(k+2)x_{k+1} = (4k+2)x_k$. Then:

$$\begin{aligned} 4^{k+1} - x_{k+1} &= 4^{k+1} - \left(\frac{4k+2}{k+2}\right) x_k \\ &= 4\left(4^k\right) - \left(\frac{4k+2}{k+2}\right)\left(4^k - \alpha\right) \\ &= \left(\frac{6}{k+2}\right) 4^k + \left(\frac{4k+2}{k+2}\right) \alpha \\ &> 0 \end{aligned}$$

2. As an example of type of problem outlined in category (b) of Section 3.4, we prove the claim $2^n > 100n$ for $n \geqslant 10$. We have $2^k = 100k + \alpha$. For the inductive step, we must show $2^{k+1} - 100(k+1) > 0$. As always, the crucial step is to connect $P(k+1)$ with $P(k)$. Here, we need $2^{k+1} = 2 \times 2^k$. Thus, $2^{k+1} - 100(k+1) = 2\,(100k + \alpha) - 100(k+1) = 100k - 100 + 2\alpha$. We are not quite home free here, since we still have a negative quantity. However, since everything is linear now, we may write $100k - 100 + 2\alpha = 100\,(k-1) + 2\alpha$. It's all over now, since we know $k - 1 > 0$; in fact $k - 1 > 8$ since $k \geqslant 10$.

7. Student reaction

Discrete mathematics classes are generally quite small at Bond (often, 30 or less). In the January semester 2005, the class was only 10 students. Although it is clearly a very small group, their responses are nonetheless encouraging, with three quite positive comments, and none negative. I have had similar reactions in previous semesters, but January 2005 is the first time I have made any attempt to semi-formally poll the students. The three student responses follow.

1. *This is in reference to the "slack method of induction". I think that the method you taught us is better then the "uneven method" (method that has greater than or less than in it).*

The reason I believe this is because it creates a more algorithmic approach to the proof. Whereas in the original method a fair bit of interpretation is required in order to know how to rearrange the statement and when to introduce new statements (ie; add a statement then subtract a statement) the "slack method" creates an algorithm.

You simply have to follow the steps and you will achieve a desired outcome. Whether this is a good thing, (students may become "monkeys" in that they follow the algorithm but have no idea of what it is they are achieving) is debatable, however, the method itself is simpler, more clear, and seems to remove some ambiguity about what you should be doing (we are used to dealing with equations, we have far less experience with relations).

2. *Being new to the subject of discrete mathematics and having no prior experience dealing with Induction I would have to say:*

   The method of introducing a "slack" proved a lot easier to grasp. The reason being is that all I needed to know to follow the algorithm was, Algebra and substitution. The other method involved remembering the extra rules that the greater/less operators entail and was a little more abstract in my opinion.

3. *Sorry for taking so long to reply. I would just like to say that the I find your method on induction superior to the normal process.*

   The reasons being, firstly, that I found the normal method in which you add and subtract the same variables abstract. Your method in which you use a slack variable to substitute into the equation, is much more "algebraic" and it seems to make more sense.

   I also found your method was a lot shorter to implement. I am sure anyone who saw the different methods would also agree with me in respect to the length.

8. Conclusions

A method designed to assist students who struggle with certain classes of elementary inductive inequality proofs has been described. It appears to do so. The method consists of two main features: first, the conversion of the inductive assumption inequality to an equation by means of a "slack" variable, and second, the conversion of the final inequality to be established in the inductive step to one in which the right side is zero. Rationales for these steps are respectively that students with modest algebraic skills generally find equations easier to handle than inequalities, and that the expression of the inequality to be proved as a difference often leads to useful cancellation.

It is clear that some classes of inequality (even "elementary" ones such as $2^n > n^4$) are still likely to be difficult for many students learning induction, since significant algebra and estimation is involved in the final inductive step. However, for a reasonable class of problems, viz., those for which the algebraic effort is moderate, the proposed technique certainly appears to assist the students.

It was interesting to note the comments of one student regarding algorithms and monkeys. This was totally unsolicited and nothing remotely similar was said in class by anyone at all, including the student who sent me the "monkey" comment by email. My own view is that tertiary level CS/IT students definitely need to learn some algorithms!

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Opportunities For Success In Mathematics: What Do Students And Teachers Say?

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No matter how mathematics achievement and persistence are measured, some learners from disadvantaged communities lag behind their peers. Instances of success can be found, but disproportional poor matric (final year high school learners) results in mathematics remain the norm despite significant advances in mathematics education research. Yet, at the same time, there are examples of schools achieving excellent results. The purpose of this study is to advance the understanding of why some mathematics classrooms in disadvantaged communities work. This study involves qualitative and quantitative data gathered in schools with similar learner demographics and socioeconomic characteristics. Only focus groups with students and individual interviews with teachers will be reported. This investigation shows that there are some factors that appear to influence disadvantaged learners decisions to persist and achieve in mathematics in spite of their disadvantages.

1. Introduction

A thorough understanding of mathematics is an asset if not essential for applicants interested in obtaining employment in South Africa. According to Steen [1], mathematics does not only empower people with the capacity to control their lives but also offers science a firm foundation for effective theories and promises society a vigorous economy. At the most basic level, mathematics is a requirement for science, computer technology and engineering courses. Seen from a social perspective, mathematical competence is an essential component in preparing numerate citizens and is needed to ensure the continued production of highly skilled persons required by industry, science and technology.

Throughout the world a major difference between the advanced and the underdeveloped countries of today has been essential in their level of development in modern science and technology. There is thus a compelling need for South Africa as a developing country to keep up with new and emerging technologies. This paper will discuss a qualitative study used to trace some factors that facilitates achievements in mathematics in traditionally disadvantaged schools.

2. Related Literature Review

Literature on the academic achievements and success of historically disadvantaged learners in mathematics is mostly about their academic failure. Several studies conducted locally as well as internationally have highlighted certain shortcomings in the mathematical achievements of South African learners. In the Third International Mathematics and Science Study (TIMSS) [2] seventh-grade and eighth-grade learners in South Africa were ranked last in mathematics out of 41 countries. Regarding the performance of students in the South African grade twelve examinations, the Limpopo Province produced the poorest or second poorest matriculation results out of nine provinces for several years.

Since mathematics is a requirement for science, computer technology and engineering courses as well as for advanced mathematics courses, it has in fact become a barrier preventing many of these learners from pursuing careers related to these areas at tertiary institutions (universities or technikons). According to Kahn [3] the number of higher-grade mathematics passes for disadvantaged students are around 3000 per year. This current pool thus serves as the source for tertiary institutions graduates in science, engineering, technology, and mathematics professions among the disadvantaged population.

In view of the importance of mathematics in society, the preceding discussion raises serious concerns among educators and policy makers. Disadvantaged learners comprise the majority of high school learners in South Africa [4]. Hence, it will affect the quality and quantity of human resources of the nation as a whole. The under-performance of learners in mathematics in high schools is also a concern to the instructors at tertiary institutions. The most obvious reason why school mathematics education should matter to university instructors is that a continuing influx of mathematically incompetent students would lower standards in the university mathematics curriculum. Studies such as Peng and Hill [5] show that high school graduates with low achievement in science and mathematics, who continued their education after high school, were less likely than other students to register for science and mathematics-oriented fields at the university.

3. Relation with Other Studies

In an effort to identify the causes for low achievement in mathematics nationally and internationally, some researchers ([6], [7], [4], [8] & [9]) have suggested that achievement in mathematics in secondary schools is influenced by a number of variables. These variables include learners abilities, attitudes and perceptions, family and socio-economic status, parent and peer influences, school-related variables such as poor learning environment, learning cultures, past racial discrimination and low expectations by principals and teachers. Such factors alone cannot account for the lack of mathematics achievement and persistence differences among traditionally disadvantaged learners. According to Singh et al [10] many of these variables are home and families related and thus are difficult to change and outside the control of educators. Furthermore, the above picture conceals the outstanding performances in mathematics of some historically disadvantaged learners and from whom one would not expect much in the way of success [11]. In particular, these explanations fail to account for intra-group achievement differences and the success of some South African disadvantaged students in spite of these background factors. Some well-achieving disadvantaged learners come from the same communities and share similar socio-economic backgrounds, schools and classrooms.

4. Research Method

4.1. Participants in the study

The participants in this research were grade 12 teachers and learners from historically disadvantaged schools from similar socio-economic backgrounds. This study is specific to grade 12 learners from historically disadvantaged communities, in view of the fact that perceptions are not stable entities within cognitive structures, but are dynamic and context dependent. Secondly, the study is based on the perceptions of both learners and teachers from these communities, because teaching and learning processes jointly influence the learning outcomes. Gender was not considered. Ten rural schools in Limpopo (Vhembe District) participate in the study. All of them were government schools. Although this was a convenience sample, with schools selected on the basis of their accessibility and performance, all were located in Vhembe district, and represent high performing and low performing schools in mathematics. The department of education assisted in selecting the schools based on their performance in mathematics over the period of three years.

4.2. Research stages

The study was conducted in three phases involving qualitative and quantitative data in the form of:

- Three weeks of classroom observations and teachers interviews (phase one);
- Focus group interview sessions with learners (phase two).

- Questionnaires for both teachers and learners (phase three).

In this paper we report on phase two-the focus group sessions with learners and some teachers individual interviews. According to McMillan and Schumacher [12] focus groups may develop concepts or theoretical explanation of what was observed. Focus groups could either be used to triangulate with more traditional forms of interviewing, questionnaire and observation methods or a stand-alone method of inquiry when conducting educational research [13]. With regard to the group size researchers seem not to agree on the size suitable for interview. However, the outside limits appear to be no fewer than four and generally not more than twelve [12]. Considering that teachers and students were coming from different schools, the focus group was, as Bloor et al [14] put it:

> Small enough to ensure that everyone was able to be part of the discussion, and large enough a group that contributed to diversity in perspective

In this regard, three focus groups were conducted with students, each of the focus groups contained four to eight students and gender was not considered. The first focus group was conducted with the best students (all students were taking additional mathematics). The second focus group was conducted with average students and the last group with the below average students.

In order to obtain a sample with a range of mathematical skills, teachers were requested to select students according to their performances in the grade 11 mathematics final examination. One high-achieving learner score of at least 75% or learner who is doing additional mathematics (an advance mathematics subject in grade 12), one middle-achieving learner score between 40% and 60%, and one low-achieving learner score at most 40%) was selected. If there were no students who scored more than 75% in the schools chosen, then all three students were correspondingly chosen on the basis of their examination rankings as compared with other students in the class. Lastly, only students who were willing to participate in the study were chosen. In this regard, eighteen students (twelve males and six females) from ten schools participated in the focus group interviews. Interviews explored students experiences in the subject, focusing on issues observed in phase one and teachers individual interviews concerning interest and devotion, proficiency and self-concept, instructional methods, perceptions and attitudes and career choice. The summary of all the themes is given in table 1, however this paper will report on only two themes.

4.3. Analysis

The analysis of the interview data in this research involved a systematic approach for discovering and categorizing the ideas conveyed by the interviewee [15]. We used categorization to refer to noting the themes and patterns in the interview data. This was achieved by using Strauss and Corbin [16] open and axial coding techniques. The first phase of the analysis, referred to as open coding, involved a process of which the content of the interview was carefully searched for discrete in stances of learners expression of concept or idea. Once the main idea was identified, the identified concepts were grouped according to their characteristics. These items were later grouped into a single category by their common characteristics. Once the categories were identified, they were given a name to characterize their relationship to the main idea. As we developed the themes questions were constantly asked as to whether the information was relevant to the research question or not. For each theme developed a code was given a label. For example, for research question no 1, What in your opinion makes learners to succeed in mathematics? One of the themes the researcher found was interest and level of devotion. A code Q1A to represent this theme was use, which indicated that this was an answer to question 1 and that it was the first theme that each of the teachers seem to consider important. For each data set, the researcher labeled the narratives that referred to interest and level of devotion as Q1A. We considered the information from each learner and teacher vital even if the theme did not appear in each data set. A special care was taken not to lose the richness in the narrative data from the interviews by relating the theme to the context [13]. When coding the data there were more commonalities than differences in each group.

5. Data Results of the Focus Group Interviews with Teachers and Learners

Some of the questions in each of the interviews were: in your own opinion what are the factors that contribute to learners good performance in mathematics? Give the most important factors that contribute to learners poor performance in mathematics? How do you motivate your learners (Teachers)? What motivates you in mathematics (Learners)?

5.1. Examples of Themes

5.1.1. Theme 1: Interest and devotion

Most students and teachers considered students characteristics such as work ethics, laziness, time management and lack of discipline as causes of poor performance for learners. In this regard students from successful schools reported always working together with other students in their classes, mostly those who were motivated to do well in mathematics. These learners were always there even when normal teaching was over and they never missed any mathematics class. Learners and teachers made the following statements regarding interest and devotion.

> I believe in time and time only. I normally tell my learners that it is not math it is give me time subject because if you offer it enough time, it will be very simple. Students need to give this subject enough time and avoid bad social behaviours. (Teacher from low performing school)

> I used to prepare myself a few days or a week before the examination or test but this time I have realise that I cannot manage, there is a lot of work that we need to finish. I use to play a lot but this year I have changed. (Learner from high performing school)

> Most of the learners are lazy. When they are given homework to do, they arrive home and just put it aside and do not look at it. Learners are very fond of banking the work. You only look at it when there is an announcement of a test. (Learner from low performing school)

> Most of our learners do not know what they want to do after matric. In our school here, the year I have taken my learners in a trip to Gauteng to visit science laboratories and some other companies, I have attained best results in mathematics. When we came back from that trip everyone was encouraged to work very hard, because they were shown the goods brought by being a mathematician. (Teacher from high performing school)

> I think this is the subjects that require someone to give himself more time to study and serious dedication. Not only relying on the teacher to give you information, because the teacher cannot do everything for you, teachers also have their own personal issues to deal with. (Learner from high performing school)

> It is advisable to love and respect your teacher in order to succeed in any subject, and every mathematics problem must be approached, do not tell yourself that this sum is difficult. (Learner from high performing school)

5.1.2. Theme 2: Proficiency and self-concept

In interviews some learners frequently reported doing only the assigned homework exercises; for the most part problems in the homework exercises were given based on previous examination questions but for most learners from high achieving schools it was different. Students from high performing schools were very positive about themselves. The following comments reflect some of this aspect:

> One thing that motivates me is that math is interesting. I want to do something that is not common, something that other people are afraid of, something unique which is not done by many people, approaching life differently (Learner from high achieving school).

> When I practice math, I usually do those exercises that are challenging, the once that are indicated by a star in some books. The difficult once makes those which are not marked by star to be known even if you havent done them .We practice both previous examination questions and textbook exercises. (Learner from high performing school)

> I think the most important thing is dedication on your part. Some of the topics you can be able to teach yourself and understand. It is also better that you try to be ahead of your teacher every time so that when the teacher comes, you will only be adding on what you already know. (Learner from high performing school)

> Most of the students here at this school have told themselves that mathematics is difficult then every thing we do in mathematics the person doesnt care. When they are preparing to write a test they know that they are going to get a zero, they do not practice anymore. They are discouraged even when the lesson is on, you may find that some are asleep not listening at all. The problem is that there are some students who spread bad news that mathematics is difficult no matter what you do you will not pass it. (Learner from a low performing school)

6. Discussion

Motivation refers to factors that influence our initiation, drive, and persistence of behavior [17]. Self motivation was the item mostly mentioned by both students and teachers to influence mathematics success. Interview responses suggested that learners motivation was largely directed towards career choice, homework completion and examination success. Students reported that they were doing the subject because it is a requirement for the career they would like to follow. However, there was a positive attitude expressed when students felt that there would be tangible benefits resulting from performing well in mathematics such as financial assistance to their career choice. When comparing successful and unsuccessful students, successful students put more emphasis than teachers to those factors directly within their control, such as class attendance, active participation and homework exercises completion, whereas unsuccessful students placed more importance than teachers on the instructional methods and teacher personality. Overall, it appears that teachers in this study were more inclined than learners to attribute learners poor performance either to factors which were related to learners background or behavior. Teachers from high performing schools convey high expectations for students backed up with support services, caring and devoted teachers and a positive learning experience. For example, organize special classes for underperforming students and recruiting guest speakers to encourage their students. Although teachers endorsed various explanations for factors that facilitate achievement in mathematics, our results suggest that the most frequently endorsed factors were related to learners characteristics. In particular, differences in students motivational levels, discipline and work ethics. Students from high performing schools knew about the minimum symbols required by the institutions they want to register at. Most of them didnt receive the information from their teachers or parents instead they got the information from friends who already graduated or from recruiting universities or technikons.

7. Conclusion

Prior research on mathematics learning has made it very clear that disadvantaged learners do not achieve to their potential in mathematics due to many factors that are understandable. The outcomes of this research have particular relevance to the disadvantaged learners who are still under-represented in mathematics related careers. By understanding what factors facilitate achievement in mathematics of students from historically disadvantaged communities; it may be possible to assist other schools and students from similar communities to better achieve in mathematics.

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Table 1: Summary of themes

Themes	Categories
1 Interest and devotion	• Participated in a mathematics or science tour/excursion? • Teacher respect • Attend extra classes • Remain after school doing math • Homework completion • Regular practice
2 Proficiency and self concept	• Active participation in the class • Working ahead of your teacher • Look forward to mathematics classes • Difficulty of the subject • Doing something different • Worked on challenging exercises
3 Instructional methods	• Class discussion • Too many in the class • Work with another learners • Teacher give different examples • Chance to explain responses
4 Perceptions and attitudes	• Supportive school environment • Desire to be rated with the best • Attempt challenging exercises • Good teachers in lower grades • Expected to do well • Friendly teachers
5 Career choice	• Usefulness in future career

Mathematics Education in a South African University of Technology: A Changing Environment

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There are major changes taking place in higher education in South Africa and specifically in the newly formed universities of technology. With the adoption of The National Qualifications Framework, higher education institutions have committed themselves to providing quality learning and to the promotion of the development of a nation committed to life-long learning, all within a rapidly changing environment. In the Discussion Document to stimulate discussion and research, The International Commission on Mathematics Instruction (ICMI) list five changes as profoundly affecting the teaching of mathematics at tertiary level, all five applicable to the situation in South Africa at present. This paper focuses on the universities of technology in South Africa and report on how these changes manifest themselves in this environment and how these universities are coping with the changes.

1. Introduction

In the discussion document of The International Commission on Mathematical Instruction [1] five changes are mentioned as profoundly affecting the teaching of mathematics at university level. (i) The increase in the number of students who are now attending tertiary institutions; (ii) major pedagogical and curriculum changes that have taken place at pre-university level; (iii) increasing differences between secondary and tertiary mathematics regarding the purposes, goals, teaching approaches and methods; (iv) rapid development of technology and (v) demands on universities to be publicly accountable.

These changes have been forced upon the newly formed universities of technology in South Africa, partly because of the acceptance of the National Qualifications Framework [2] and partly because of a changing political dispensation. The victory over apartheid in 1994 set policy makers in all spheres of public life the mammoth task of overhauling the social, political, economical and cultural institutions of South Africa to bring them in line with the imperatives of a new democratic order. The National Plan outlines a framework for the transformation of the higher education system as a whole. The question addressed in this report is:

How is the changing environment of higher education in South Africa, and specifically the newly formed concept of universities of technology, addressing these changes as mentioned by the ICMI study?

2. What is a University of Technology in South Africa?

Previously referred to as technikons, universities of technology in South Africa offer career-orientated educational programmes designed to meet the needs of industry and commerce in a hi-tech global economic environment. Their approach to education is practical and outcomes-based, with the result that graduates are immediately employable and productive.

Such institutions exist in many countries, perhaps under a different name, viz. Universities of Applied Sciences (Germany), Universities of Technology (Australia and Hungary) and Institutes of Technology (USA and Australia). This concept is however, essentially new in the minds of most South Africans.

Universities of technology are distinguished from the other universities, not by the quality of their educational product, but rather by their focus. According to the Committee of Technikon Principals (CTP), this focus aims to "provide and promote, in conjunction with the private and public sectors, quality career and technology education and research for the development needs of a transforming South Africa and a changing world." [3]. Key elements of education at a university of technology include the application of technological knowledge; the training of technicians and technologists; a focus on applied research; direct interaction with employment providers; cost-effective and quality career-orientated education; multidisciplinary subject packages; outcomes-based, demand-driven curricula; and emphasis on immediate and productive employability.

3. Changes and challenges for a university of technology

With the South African situation as a backdrop, the five changes profoundly affecting higher education in mathematics, as set out in the ICMI study quoted above are discussed on how it has impacted on the universities of technology.

3.1. The increase in the number of students who are now attending tertiary institutions

There is an increased need in higher education brought about in part by the explosion in worldwide telecommunication. The rapid technological advances and the need for increased skills have put extreme pressure on educational systems worldwide. With the birth of South Africa's new democracy and an education policy to suit the majority and not only a privileged few, both conventional universities and universities of technology are dealing with a much larger influx of students.

Student enrolments have grown from 473 000 in 1993 to 675 128 in 2002, a growth rate of 18%. Whereas African students constituted 40% of the student body in 1999, in 2002 they made 60% of the total enrolment [4].

These students come from very diverse backgrounds, schools and cultures. South Africa has eleven official languages, but at universities of technology the language of instruction is English, and has been for at least the last 10 years. Many other institutions are still grappling with this transition. It simply means that most students are studying in their second and sometimes even in their third language. In a study done at the University of Pretoria to determine the influence of studying mathematics in a second language it was generally found that proficiency in everyday English is not necessarily predictive of mathematics achievement [5].

More and more courses at universities of technology contain some degree of mathematical skills. The mathematics required are often dealt with in other departments, for example, the statistics required for marketing courses will be offered by the Management Sciences and Operational Research is offered by the Information Technology Department. According to Lynn Arthur Steen [6] we have a paradox: As widespread use makes postsecondary mathematics increasingly essential for all students, mathematics departments find themselves playing a diminishing role in the mathematical education of postsecondary students.

Although significantly more students are now entering university, an increasingly smaller percentage of students are opting for studies that require substantial amounts of mathematics. In "Is mathematics running out of numbers?" the authors investigated the trends in numbers of students majoring in mathematics at South African universities and compared it to international trends [7]. They show an alarming decrease in numbers of mathematics majors but, on the other hand, mathematics departments are faced with the influx of students who do not intend to follow a career in mathematics and whose preparation, background knowledge and even attitudes are quite different from a few years ago.

3.2. Major pedagogical and curriculum changes that have taken place at pre-university level

South African schools are facing major changes, such as mathematical literacy for all learners not doing mathematics as a grade 12 subject being introduced as from the beginning of 2006. The principle is: mathematics for all, rather than mathematics for the few. The South African

Mathematical Society welcomes this decision as a step in the right direction [8].

The new syllabus for pre-university learners will deal with more relevant issues, such as statistics and problem solving and "contributes to personal development through an enjoyment experience in a deeper understanding and successful application of its knowledge and skills" [8]. Mathematical Literacy provides learners with an awareness and understanding of the role that mathematics has in the real world. The South African Mathematical Society (SAMS) welcomes the attempt to ensure that learners who attain the proposed outcomes in mathematics are well prepared for the challenges of tertiary studies in the natural sciences, engineering and technology. There are, however, many concerns that will have to be faced in the long term.

One of the concerns is that an already demoralized, under-qualified, shrinking body of teachers will be unable to cope with the new syllabus, so instead of the envisaged increase in mathematical literacy, the country will instead suffer a decrease [8]. Many teachers, especially in rural areas, are already under-equipped to deal with elementary mathematics and they will now be expected to cope with the demands of the new syllabus.

Proper implementation of the new syllabus is vital and many South Africans are concerned that the Department of Education is not doing enough to prepare schools and teachers. Also, a large number of students at universities of technology gained entrance into programs, such as engineering, by obtaining mathematics on a standard grade level. Whilst entrance at conventional universities always required a higher grade pass in grade 12 mathematics, large numbers will be lost if the requirement of higher grade is expected at universities of technology. On the other hand, mathematical literacy may not be on a sufficiently high level to allow access into areas such as health sciences, biological sciences or engineering.

3.3. Increasing differences between secondary and tertiary mathematics regarding the purposes, goals, teaching approaches and methods

The challenge here is not so much the differences between secondary and tertiary mathematics than the fact that the South African educational system is being overhauled from primary school level to tertiary level as part of the rejuvenation of post-apartheid institutions, and to bring the country in line with current trends in international education. The South African Government has set very rapid time frames for the transformation of education. Primary and secondary school learning programs must be transformed to Outcomes-Based Education (OBE) by 2005. Tertiary level undergraduate programs must be transformed by June 2001 in order to lead to recognised professional and academic qualifications.

The language associated with OBE is complex, confusing and new and South Africans are still getting used to the introduction of terms such as Outcomes Based Education (OBE), Continuous Assessment (CASS) and summative and formative assessment, to name but a few. OBE is a system of education where everything that is taught has a specific purpose. The curriculum and all assessment are developed around these desired outcomes [9]. Assessment is ongoing and not exam driven. The aim with OBE is to bring about a mind shift away from an authoritarian mode of teaching to a co-operative mode of learning. The new model makes the educator a facilitator, and the learner an active participant in an interactive learning partnership. In the old model it was expected of pupils to unquestioningly memorise curriculum content without necessarily understanding the significance or relevance of information to be mastered. The new mode requires of the educator to facilitate the development of both critical and practical skills.

Students generally welcome OBE, but it has increased the workload of lecturing staff, who already have many demands on them to do research and increase their qualifications. The system has been implemented in schools, but unfortunately with mixed emotions and results.

However, the principles behind OBE are sound and a sign that society is reaching for other definitions of knowledge and reasoning. If South Africa is to take up its position in the global village, it needs to embrace the new vocabulary of competence, skills and outcomes. Countries in Europe, the Pacific Rim, Australasia and North America have either adopted or moved in the direction of a national qualifications framework, underwritten by a commitment to outcomes based education [10].

3.4. Rapid development of technology

The rapid development of technology manifests itself in an increasing use of computers and calculators in mathematics instruction. This raises questions as to, especially at a university of technology, how much of our teaching and learning should make use of and rely on software packages that are already available.

From a practical point of view, the introduction of technology into everyday learning is expensive and out of reach of most South Africans. In a country where there are still schools struggling to supply their learners with textbooks it is hardly relevant if you are keeping up with latest developments on the computer scene. Private universities and colleges are finding it easy to keep up with the latest on computer ware, but they are only for the privileged few, since class fees are exorbitant.

All students enrolling at a university of technology will soon be required to be computer and internet literate. Many other disciplines are using software to enrich students, but only in some mathematics departments is software being used in the teaching of mathematics, for example at Tshwane University of Technology in the Computer Systems Engineering Course.

3.5. Demands on universities to be publicly accountable

In most universities of technology mathematics is the subject that serves the most diverse needs. The department teaching mathematics must meet the needs of a diverse group of students including agriculture, engineering, business and biochemistry. Lynn Arthur Steen says that: "thus the spotlight of educational improvement often falls first and brightest on mathematics" [11].

The primary purpose of mathematics programs in universities of technology should be to help students learn whatever mathematics they need, both for their immediate career goals and as a preparation for life-long learning. "As we move further into the age of knowledge, the workforce will require more sophisticated education and training to sustain competitiveness. Criteria such as relevance of knowledge and applicability of skills are increasingly becoming more important determinants of employability"[12].

Experiential Learning or work-integrated learning (WIL) was always regarded as one of the key strengths of the old technikons. Now, universities of technology are involved in collaborative industry-directed research programmes and this involvement is in turn reflected in curriculum design. The ability of graduates to 'hit the ground running' and immediately begin to be economically productive is a key objective.

According to Prof R du Pre: [12] "Universities of technology are seen as institutions with a greater commitment of service to, and uplifting of, the community than has previously been the case with higher education institutions of South Africa. Due to the focus on relevancy and transferring applicable skills, many students are able to render a valuable contribution to community training and upliftment".

4. Conclusions

As the world changes rapidly and higher education grows explosively, universities seem to evolve at a leisurely pace. Courses, curricula, and exams remain steeped in tradition, some centuries old, while autonomy and academic freedom rule in the classroom. Few institutions of higher education readily embrace a culture of assessment that is required to ensure relevance and effectiveness of their curriculum [6]. This is perhaps not true for the universities of technology of South Africa, which have come a long way in a short time.

In many aspects the universities of technology have addressed the challenges set by the National Plan for Higher Education [2]. Perhaps the most difficult challenge has been the large influx of students. With the constraint of resources and finance, state-subsidised institutions, such as technological universities, often struggle to keep up with the latest in the world of technology. All students must be computer literate and large numbers of students must be accommodated, some of these students having had no previous experience.

Outcomes Based Education has been introduced into universities of technology, and although they have experienced many teething problems, lecturers and students are adapting to the new system. The challenge to the mathematics departments is to keep the syllabus relevant so that a student learns whatever mathematics they need.

Generally, much has been done for the skills level, competency and employability of university of technology students. Constant upgrading is provided through short courses, taking the institution into the workplace. Communication with industry has become increasingly important to ensure that prospective employees receive a relevant education.

Lastly, the introduction of mathematical literacy is causing some concern, but fortunately the universities have 3 years until the first grade 12s finish under the new system. Issues that still need to be addressed are:

- What should the enrolment criteria be?
- How will it affect our professional standards?
- Will it enhance the interest in mathematics?
- Will the system give a much-needed boost to the number of students opting for studies that needs some mathematical background?

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Why using a CAS is important in teaching beginning differential equations

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Computer algebra systems (CAS), such as Derive, Mathematica and Maple, have changed the way we are teaching beginning differential equations. Many courses now have a computer laboratory component and rather than teaching specific solution techniques for classes of equations from a menu of types, the emphasis is upon systems and nonlinearity. In this article a survey of investigations by the authors and their students and colleagues is given. These investigations give rise in a natural way to many research problems suitable for students in first and second courses on ordinary differential equations The idea is that a CAS is not simply a fancy calculator making classical exercises and computations easier, but rather that a CAS can lead one to do new and deeper mathematics with very little effort.

1. Introduction

Solving an initial value problem (IVP) numerically using any one of the new algorithms in an ode solver package or from one of the computer algebra systems (CAS) is almost effortless and the numerical solution is generally quite accurate. Thus when teaching differential equations, obtaining a solution to an IVP is not much of a problem and consequently we may concentrate our attention on the qualitative aspects of the equation and the model it represents. This point of view is in stark contrast to the classical approach of teaching solution techniques for equations from a lengthy menu of types. For nonlinear equations we seldom have a closed form solution in any event and numerical solutions are a necessity. Students do not need to be experts in numerical analysis to be proficient in solving equations with a CAS. This means that qualitative questions can be explored in greater depth and it is this type of thinking that students need. Our students in these beginning courses come from a variety of majors, not just mathematics, and it is this qualitative reasoning that will find application in their respective fields of study. The graphics capabilities alone warrant use of a CAS; we use primarily *Mathematica* [27] but any of the popular packages will suffice.

There are two fundamental questions that should be explored. Consider a second order initial value problem (IVP) of the form

$$\begin{gathered} \ddot{x} + f(\dot{x}, x) = g(t) \\ x(0) = a, \dot{x}(0) = b \end{gathered} \tag{1}$$

where t is the independent variable (usually thought of as time),x is the dependent variable usually thought of as displacement, and the dots over x represent differentiation with respect to t as is customary. (I) For what values of initial conditions $x(0) = a$ and $\dot{x}(0) = b$ is the solution to the IVP (1) bounded, say for $0 \leqslant t \leqslant 200$, and (II) when bounded, when is the solution periodic? We concentrate on second order equations as these are the type most often encountered in beginning courses; for a third order model see [24] and for a fourth order model arising from coupled second order equations see [22]. Neither of these questions is easy to answer except in the case of the equation being linear and the forcing function $g(t)$ relatively simple (sinusoidal).

The interest in bounded solutions is obvious, an engineer would like to know if the model will blow up or burn out. The question of periodicity is of interest from the general observation that any nonlinearity in an otherwise linear (bounded) model creates a nonsinusoidal periodic response.

2. The energy approach

Spring models are common fare in most beginning texts, and the undamped undriven model leads to an IVP of the form

$$m\ddot{x} - f(x) = 0$$
$$x(0) = a, \dot{x}(0) = b$$

where m represents the mass of a weight fastened to the bottom of a spring suspended from the ceiling. This simple model, even when $f(x)$ is linear, embodies interesting motions and leads to more complicated models and motions. Phase plane analysis can give a global qualitative view of the possible motions represented by the model equation. In order to obtain a phase plane portrait of the model, we apply the energy approach well known to physicists. The idea is to consider the equation obtained by multiplying through by $\dot{x}$

$$m\ddot{x}\dot{x} - f(x)\dot{x} = 0$$

and integrate, obtaining

$$\frac{m\dot{x}^2}{2} - \int f(x)\dot{x}dt = C$$

where the first term $m\dot{x}^2/2$ represents the kinetic energy and the second term potential energy of the model; C is a constant representing the total energy of the IVP. Letting $\dot{x} = y$, we write

$$E(x, y) = \frac{my^2}{2} - \int f(x)\dot{x}dt$$

and call the surface represented by this *energy function* $E(x, y)$ the *energy surface*. Contours in this surface represent phase plane trajectories for the model.

Using the contour plot capabilities of a CAS can rapidly produce phase plane portraits. Thus it becomes relatively easy to investigate nonlinear polynomial type restoring forces $f(x)$ in the spring model. This approach is discussed in considerable more detail in [17] and [18] where a number of student investigations are suggested. Answers to the two fundamental questions concerning boundedness and periodicity become readily apparent from the phase portrait.

3. Numerical solutions

As mentioned in the introduction, generally speaking, the suite of ode solvers found in a CAS perform quite well and are accurate. Thus solving a particular IVP numerically might be considered a solved problem and hence an instructor can feel justified in having a focus on the qualitative aspects of the equation and model it represents. However it is a nontrivial problem to decide just how accurate a numerical solution is. Is it accurate to 3 decimal places or 6 decimal places and if so over what time interval does it have these accuracies? There are simple models that can give one numerical fits, see [19] for example.

A student does not have to be an expert in numerical analysis in order to exploit the numerical capabilities of a CAS, but the student should be aware of what can happen and be on the look-out for problems. Much of this is discussed in the Proceedings of Remarkable Delta:03 [20]. On the other hand, students also need to have some idea of what to expect, rather surprising results can be obtained when just plotting points, see [15].

4. Fourier series

Arguably, Fourier series is one of the most important applied mathematics topics and it holds a central position in analysis. Indeed the history of modern analysis shows that the need to explain the effectiveness of Fourier series drove the subject for over 50 years. For example, Fourier series are indispensable when studying partial differential equations and Sturm-Liouville theory. A series of articles [2] - [7], has been trying to popularize the use of Fourier series in beginning differential equations courses for two major reasons. First, Fourier series provide a very natural way to solve the periodically forced harmonic oscillator equation and these solutions provide more qualitative information than do the solutions obtained by the classical approach of using the Laplace transform. Second, many students have a beginning course in differential equations as their terminal course in mathematics and would otherwise never be exposed to this useful and important topic. A CAS makes computations with and plotting of Fourier series approximations almost effortless.

Furthermore, if a solution to an IVP is known to be (or suspected to be) periodic, then a Fourier series approximation is straightforward to generate from the numerical solution. This approach is discussed in [21]. Coefficients can be generated and then an accurate closed form solution obtained. For example, the elliptic sine function $sn(u,k)$ is the solution to the IVP

$$\begin{gathered}\ddot{x} + (1+k^2)x - 2k^2x^3 = 0 \\ x(0) = 0, \dot{x}(0) = 1\end{gathered}$$

For a given value of the parameter k, an accurate numerical solution to this IVP yields the period and hence frequency of the solution. From this the first few Fourier coefficients can be determined and a closed form solution obtained accurate to 10 decimal places or more is therby obtained. The elliptic cosine and elliptic denominator functions are solutions to other similar IVPs and can be handled in the same way. For more details on this see [6]. One advantage in doing so is that the closed form truncated Fourier series solution is completely determined by its coefficients and thus is platform independent and software package independent provided one can evaluate sines and cosines.

Fourier series solutions to the periodically forced harmonic oscillator have all the expected convergence properties. One replaces the periodic forcing function by its Fourier series and solves the harmonic oscillator problem term-by-term. This produces a uniformly convergent Fourier series solution with many nice properties. More details are found in [2], [3], [4], and [5].

5. Periodic solutions

The question of when a differential equation has a periodic solution is difficult one and only in a few cases can direct methods be used to detect the presence of a period solution. Thus the efforts of many great analysts have centered on this question. If one has an IVP whose solution is periodic, then Fourier series coefficients can be numerically generated and a highly accurate truncated Fourier series solution determined. This form of solution has the advantage that it can be substituted back into the equation and a residual is obtained that yields accuracy and accuracy interval information. The problem, of course, is knowing the initial conditions that lead to the periodic solution.

One deep method for finding the initial conditions for a given period is the homotopy method [23]. This method is very accurate and applies to a wide variety of equational forms. However implementation requires some substantial programming and it is somewhat easier and perhaps more appropriate and instructive for a beginning student to use other approximation techniques.

5.0.1. Van der Pol plane analysis

The Dutch electrical engineer Balthazar van der Pol (1889 - 1959) developed a technique for finding approximate initial conditions yielding periodic solutions of a given frequency. Once such approximate initial conditions are obtained, tweaking them by trial-and-error can produce the desired initial conditions.

Suppose we are interested in periodic solutions of period P to an equation of the form

$$\ddot{x} + x = f(x, \dot{x}, t)$$

where $f(x, \dot{x}, t)$ is periodic in t of period $P = 2\pi/\omega$. The idea is to assume that the periodic solution has the first order Fourier series approximation

$$x(t) = a(t)\cos\omega t + b(t)\sin(\omega t)$$

where $a(t)$ and $b(t)$ are twice differentiable, and to substitute this form into the equation and see what the constraints are that force the solution to be of the correct form. We will illustrate this (with most all calculational details omitted) with the Duffing equation

$$\ddot{x} + x + \varepsilon x^3 = F\cos\omega t$$

which we take as representative of the nonlinear equations a student is likely to encounter. Duffing's equation is only mildly nonlinear and embodies many types of solutions: unbounded, bounded, periodic, and almost periodic.

Substituting

$$x(t) = a(t)\cos\omega t + b(t)\sin(\omega t)$$

into the Duffing equation, differentiating, using a trigonometric identity for the cubic terms and neglecting terms of high frequency, we obtain

$$\begin{aligned} &\left[\ddot{a} + 2\omega\dot{b} - a\left\{(\omega^2-1) - \tfrac{3\varepsilon}{4}(a^2+b^2)\right\}\right]\cos\omega t \\ +&\left[\ddot{b} + 2\omega\dot{a} - b\left\{(\omega^2-1) - \tfrac{3\varepsilon}{4}(a^2+b^2)\right\}\right]\sin\omega t = F\cos\omega t \end{aligned}$$

Matching coefficients and simplifying, it follows that

$$\begin{aligned} \ddot{a} &= -2\omega\dot{b} + a\left\{(\omega^2-1) - \tfrac{3\varepsilon}{4}(a^2+b^2)\right\} + F \\ \ddot{b} &= 2\omega\dot{a} + b\left\{(\omega^2-1) - \tfrac{3\varepsilon}{4}(a^2+b^2)\right\} \end{aligned}$$

This coupled second order system can be turned into a coupled first order system by introducing auxiliary variables $\dot{a} = \alpha$ and $\dot{b} = \beta$ which leads to the 4x4 system

$$\begin{aligned} \dot{a} &= \alpha \\ \dot{\alpha} &= -2\omega\beta + a\left\{(\omega^2-1) - \tfrac{3\varepsilon}{4}(a^2+b^2)\right\} + F \\ \dot{b} &= \beta \\ \dot{\beta} &= 2\omega\alpha + b\left\{(\omega^2-1) - \tfrac{3\varepsilon}{4}(a^2+b^2)\right\} \end{aligned}$$

Critical values to this system of first order equations are of the form $(a, 0, b, 0)$ where a and b are solutions to the pair of equations

$$\begin{aligned} a\left\{(\omega^2-1) - \tfrac{3\varepsilon}{4}(a^2+b^2)\right\} + F = 0 \\ b\left\{(\omega^2-1) - \tfrac{3\varepsilon}{4}(a^2+b^2)\right\} = 0 \end{aligned}$$

Since F is assumed nonzero it follows that $b = 0$ and critical values are of the form $(a, 0, 0, 0)$ where a is a root of the cubic

$$\frac{3\varepsilon}{4}a^3 + a(1-\omega^2) = F$$

Thus a critical value $(a, 0, 0, 0)$ leads to a set of initial conditions $x(0) = a, \dot{x}(0) = 0$ for Duffing's equation whose solution may be nearly periodic.

The study of the critical values and the ab-plane , called the second order van der Pol plane, can lead to information concerning boundedness of solutions as well as periodic solutions (harmonic and subharmonic solutions). For more details on this approach see [8], [9], and [10].

6. Resonance

Linear resonance is usually discussed in beginning texts by considering the harmonic oscillator equation

$$\ddot{x} + \lambda^2 x = F(t)$$

where $F(t) = A\cos\omega t$ and/or $F(t) = A\sin\omega t$. If $F(t)$ is periodic of frequency $\omega/2\pi$ and $\omega = \lambda$, the the solution to the equation is oscillatory and has an amplitude that grows linearly in time without bound. In nonlinear equations solutions can be unbounded without being oscillatory or can be oscillatory and bounded for quite some time before abruptly becoming unbounded. For Duffing's equation, Davis [1] discussed resonance and the jump phenomenon, and in [10] this idea was reinvestigated and explained using second order van der Pol plane analysis.

Another type of resonance can be investigated as well. Consider the Duffing equation

$$\ddot{x} + x + \varepsilon x^3 = F\cos\omega t$$

with initial conditions fixed, $x(0) = a, \dot{x}(0) = b$. We can think of the left-hand side of the equation as modeling a spring or electronic circuit and the right-hand side representing an external force acting on the system, say a signal generator in the electronic circuit case. For ε fixed and fixed initial conditions, and for a given forcing frequency ω, starting with a small amplitude value of F all responses are bounded. As F is increased suddenly and with obvious warning the response will become unbounded. There is a clear threshold boundary called the *stability boundary* between bounded responses of the system and unbounded responses in the ωF-plane. This boundary has many interesting features among which are a *low resonance frequency* and a higher *jump frequency*. A typical stability boundary is shown in Figure 1.

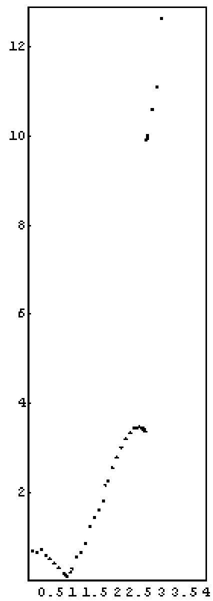

Figure 1: Astability boundary in the ω F-plane.

The low resonance frequency yields ω_L global and local minimum point (ω_L, F_L) for the stability boundary and F_L is very close to being zero. This means that at this forcing frequency almost no forcing amplitude will produce a bounded response. For example, with $x(0) = \dot{x}(0) = 0$, and $\varepsilon = -1/6$, we have $\omega_L = 0.8852$ and $F_L = 0.077$.

Another interesting feature of the stability boundary in the ωF-plane is a jump discontinuity occurring at what we call the jump frequency ω_J. Again for example with $x(0) = \dot{x}(0) = 0$, and $\varepsilon = -1/6$, we have $\omega_J = 2.67$ and $F_J = 9.901$. For $\omega = 2.66$, the point of the stability boundary is $(2.66, 3.360)$, so one has a jump of nearly three times the magnitude at ω_J.

Of course the low resonance frequency and the jump resonance frequency are functions of the initial conditions and the value of ε. A research problem is to find an analytic expression for this relationship or at least a first order approximation to this relationship. More research problems, student problems and small group investigations are suggested in [12] and [16].

7. Examples

In this section we draw attention to specific differential equations and models that students might find interesting to investigate.

7.1. van der Pol's equation

This equation is found in almost every beginning text and it is a fundamental example having a limit cycle.

$$\ddot{x} + \varepsilon(1 - x^2)\dot{x} + x = 0$$

But the forced van der Pol equation

$$\ddot{x} + \varepsilon(1 - x^2)\dot{x} + x = F\cos\omega t$$

exhibits even more interesting responses and some seemingly chaotic responses that are not due to the numerical solver. For example, for $\varepsilon = 5, F = 1$, and $\omega = 8/10$, there is a limit cycle that encircles the origin three times in one period. See Figure 2 for a trajectory starting at the origin.

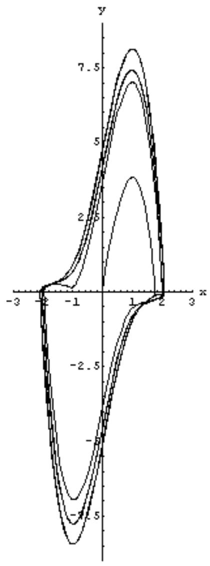

Figure 2: Forced van der Pol equation trajectory.

For $\varepsilon = 5, F = 2$ and $\omega = 2$, trajectories appear to have unstable qualities. Numerical investigations for various values of ε and forcing frequencies and amplitudes are appropriate student projects. Also there is the important notion of *frequency entrainment* which would make a good topic for a small group investigation.

7.2. Duffing's spring equation

We have already remarked on the interesting responses of one of the simplest forced nonlinear equations

$$\ddot{x} + k\dot{x} + x + \varepsilon x^3 = F\cos\omega t$$

This model with viscous damping ($k > 0$) for both positive and negative ε exhibits an interesting limit cycle feature and many subharmonic solutions exist (periodic solutions whose period is an integral multiple of the period of the forcing). The stability boundary (for resonance) in the phase plane is quite interesting as well and there are a number of student investigations suggested in [13] and [14].

7.3. The pendulum equation

The classical pendulum equation

$$\ddot{x} + \sin x = 0$$

is a beautiful example to investigate in a beginning class that has a laboratory component. One can compare the numerical solutions of the approximations

$$\ddot{x} + x - \tfrac{1}{6}x^3 = 0$$
$$\ddot{x} + x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 = 0$$

and one can compare these numerical solutions with that of the exact analytical solution which involves the elliptic sine function. Details of some of this can be found in [11].

7.4. A radiative reaction model

Almost all models investigated in a beginning course in differential equations are either first order or of second order based upon Newton's law. The only higher order model seen is that of a bending beam which is of fourth order and is usually relegated to an exercise. Thus a third order example based on early work of H. A. Lorentz and P. A.. Dirac on the clastic electrodynamical equations of motion for a radiating particle becomes of interest particularly because the model is linear.

A fundamental phenomenon involving charged particles is that electromagnetic radiation may be emitted when the particles accelerate. This is the basis for all modern communications systems. For example, radiation at a desired frequency is emitted from a radio antenna by the accelerating charge in the antenna circuit. Radiation at microwave frequencies in a microwave oven is produced by accelerating electrons in a Klystron tube.

This radiative model equation is

$$\ddot{x} - a\dddot{x} + \lambda^2 x = f(t)$$

where a and λ are positive constants and $f(t)$ may be zero or perhaps a sinusoidal forcing term. The time variable can be rescaled so that the coefficient a of the third derivative becomes 1. Now the interesting problem is to find the constraints that yield a bounded solution. Details can be found in [24].

7.5. Coupled springs

Basic spring models are very useful in a beginning course. The linear model for a weight attached to a spring suspended from the ceiling is particularly nice as one introduces many concepts: Hooke's Law, Newton's :Law, periodic solutions, beats, damped oscillatory motion, critically damped motion, and linear resonance to mention just a few. Multiple spring models appear to have fallen out of favor with textbook authors for these models have either been omitted entirely or relegated to the exercises. However, by investigating two springs and two weights suspended in series from the ceiling gives rise to a pair of coupled second order equations which then can be uncoupled and a fourth order model obtained. In this case one gets to investigate when the weights move in phase with each other or out of phase with each other and the motions obtained are much more interesting and varied than those for the single spring and weight. These models permit discussion of many concepts including accuracy of numerical algorithms, continuous dependence of solutions on the parameters and initial conditions, periodicity and beats, limit cycles, etc. See [22] for more details on this type of spring model.

Another spring model that is necessarily nonlinear is that of two springs and one weight. One spring is attached to the ceiling and one is attached to the floor directly beneath the other and the free ends of the springs are attached to the weight. Motion is constrained to be planar. The algebra in deriving this model is a bit more complicated than the previous model but not beyond beginning students. Moreover, simple physical models are easy to build. Once this two vertical spring one weight model is developed, it is a simple matter to attach two more horizontal springs (fixed to the left-hand wall and the right-hand wall and attached to the weight) so that we have four springs and one weight. The resultant motions of both models are much more interesting than those motions for a single spring. These models are explored in [26].

7.6. A magnet spring model

One's physical intuition obtained by "playing with magnets" is reinforced by a simple linear spring model with an inverse quartic forcing law. The autonomous nature of the resultant spring model equation permits the energy approach to be used to full advantage. Students get exposed to a different type of nonlinear forcing for the harmonic oscillator equation which has a physical basis.

Assume for simplicity that the spring is linear and the spring equation has the form

$$\ddot{x} + x = g(x)$$

where we have normalized the mass of the weight to be 1, $-x$ is the linear restoring force and $g(x)$ is an external force. Usually one has a positive coefficient of x, say λ^2, but a rescaling of the time variable ($\tau = \lambda t$) permits, without loss of generality, the coefficient of x to be assumed to be 1. Note that this is an autonomous equation without the time variable t appearing explicitly and thus the energy approach discussed above can completely determine the range of resultant motions.

Again for simplicity, two magnets will be assumed to be identical and their interaction to be that of point dipoles. A derivation of the force between two such point dipoles depends upon being able to describe the dipole field and its gradient (force). In this case the force between the dipoles is proportional to $\pm 1/z^4$ where z is the distance between them. The plus sign or minus sign is taken according to the north-south pole orientation of the magnets. A complete derivation of this can be found in [28, chapter 5]. We shall assume that the constant of proportionality is 1 so that the force between them is proportional to one over the fourth power of the distance between them.

We attach one end of the spring to the ceiling and attach one magnet (weight) at the bottom of the spring and allow the system to come to rest in equilibrium. Deflections from equilibrium in the vertical will be measured by $x(t)$ as usual with deflections below equilibrium being positive and deflections above negative. The center of gravity of the weight (magnet) is from where all measurements are taken. When a second magnet is placed on the floor directly beneath the first suspended magnet, the first magnet is given an initial displacement and initial velocity and the external forces acting on the system are due to the magnetic interactions between the two magnets.If L is the distance from equilibrium to the top of the stationary magnet on the floor, then the magnetic force becomes

$$\frac{\pm 1}{(L-x)^4}$$

where signs are assigned according to the orientations of the poles of the magnets. There are two cases depending upon the orientation of the suspended magnet over the fixed magnet, an attracting case and a repelling case

If the suspended magnet has its north pole on the top and its south pole on the bottom and similarly for the fixed magnet, then the south pole will be attracted to the north pole of the fixed magnet and the attractive force will be downward and hence positive. In this attracting case, the model will be

$$\ddot{x} + x = \frac{+1}{(L-x)^4}$$

The energy function associated with this attracting model is

$$E_A(x, y) = \frac{y^2}{2} + \frac{x^2}{2} + \frac{1}{3(L-x)^3}$$

If the poles of the top magnet are reversed so that the north pole of the top magnet is on the bottom, then as the top magnet gets closer to the bottom magnet, the force is repelling. The forces change sign from the cases studied above. The model becomes

$$\ddot{x} + x = \frac{-1}{(L-x)^4}.$$

The energy surface for this repelling case is

$$E_R(x, y) = \frac{y^2}{2} + \frac{x^2}{2} + \frac{-1}{3(L-x)^3}$$

Students can turn the model equations into first order systems, calculate and classify critical values, and use contour plots of the two energy surface functions to investigate the two phase planes. These plots then are useful as the student's experience with magnets is reflected in the resultant phase portraits. For more details and phase plots see [25].

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A real-life application of the absolute value function in the teaching of beginning differential equations

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The absolute value function and its derivatives and optimization techniques are some of the cornerstone concepts of undergraduate calculus. An understandable, real-life application of these concepts is dealt with when we derive a model of a motor-vehicle wheel striking a speed bump. This model turns out to be nonlinear before an application of the absolute value function and its derivative is considered. The model in question is then transformed into an initial value problem (IVP) for ordinary differential equations (ODEs). By examining this model and its solution, students are exposed to a useful application of both the absolute value function and optimization techniques. They are also introduced to the underdamped, overdamped and critically damped situations for spring-damper pairs. A computer algebra system is used to do the necessary calculations and to produce the graphics.

1. Introduction

While considering a strategy to design tyres to minimize the formation of dirt road corrugations (see [4] and [1]), the authors and their students considered various drive-train models (see [5]). For instance, when a motor-vehicle wheel hits a speed bump on a stretch of otherwise smooth horizontal road, the vehicle body (which has large inertia) will remain in a fixed position above the road for a relatively long period of time, say, T. The wheel, however, receives an upward velocity, say, v_1, causing it to leave the road. The spring-damper pair of the suspension system will cause the wheel to make contact with the road again after a relatively short period of time, say, $\tau_1 < T$. The tyre, being elastic, might bounce off the road again after another short period of time, say, τ_2, with $\tau_1 + \tau_2 < T$. In fact, the entire process might repeat a number of times until the damper reduces the motion to a negligible amount and $\tau_1 + \tau_2 + \tau_3 + \cdots < T$. As a first approach to modelling this situation, assume that the wheel is perfectly elastic and that if it were to bounce off the road, it would do so with a velocity equal in size but in the opposite direction to that with which it hit the road. To keep things a simple as possible, assume that the shock absorber provides viscous damping and that the spring is linear (it obeys Hooke's law).

Consider the following diagram of the situation at hand:

Keeping in mind that the road acts as the position of rest of the wheel mass m, we analyze the forces involved in a standard fashion (see, for instance, [6]) and conclude that the model to solve is:

$$\left.\begin{array}{rclr} m\ddot{x} + \phi\dot{x} + kx &=& 0 & x > 0 \\ \dot{x}^{+} &=& -R\dot{x}^{-} & x = 0 \\ x(0) &=& 0 & \\ \dot{x}(0) &=& v_1 > 0 & \end{array}\right\} \qquad (1)$$

where $\dot{x}^{+}$ indicates velocity in the positive x direction, $\dot{x}^{-}$ indicates velocity in the negative x direction and R is the coefficient of restitution of the tyre with respect to the road. As a first approximation we choose $R = 1$. Because $m \neq 0$, we can write this ODE in the customary

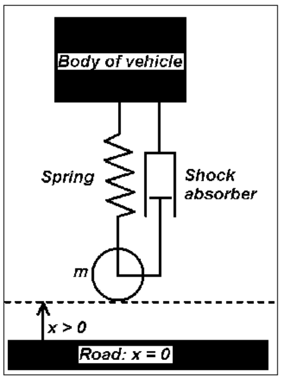

Figure 1: The tyre, vehicle body, spring-damper pair and the road.

form:

$$\left.\begin{array}{rcll} \ddot{x} + 2\lambda\dot{x} + \omega^2 x & = & 0 & x > 0 \\ \dot{x}^+ & = & -\dot{x}^- & x = 0 \\ x(0) & = & 0 & \\ \dot{x}(0) & = & v_1 > 0 & \end{array}\right\} \tag{2}$$

where

$$\lambda = \frac{\phi}{2m} \tag{3}$$

and

$$\omega = \sqrt{\frac{k}{m}} \tag{4}$$

This model is considered to be nonlinear because it cannot be modelled by one differential equation. However, we can transform this nonlinear problem into a linear problem by using the absolute value function. Let

$$x = |z| \tag{5}$$

Consequently,

$$x = 0 \iff z = 0 \tag{6}$$

It is well known that for $z \neq 0$:

$$\frac{d}{dz}|z| = \frac{|z|}{z} \tag{7}$$

Hence for $x > 0$ (and so $z \neq 0$) the chain rule yields

$$\dot{x} = \frac{d\,|z|}{dz}\frac{dz}{dt} = \frac{|z|}{z}\dot{z} \tag{8}$$

and the triple product rule yields

$$\ddot{x} = \frac{d}{dt}\left(|z|\,\dot{z}z^{-1}\right) = \frac{|z|}{z}\ddot{z} \tag{9}$$

Consequently, the ODE transforms into:

$$\frac{|z|}{z}\ddot{z} + 2\lambda\frac{|z|}{z}\dot{z} + \omega^2|z| = 0 \tag{10}$$

Multiplying through by z and dividing by $|z|$, we obtain:

$$\ddot{z} + 2\lambda\dot{z} + \omega^2 z = 0 \quad z \neq 0 \tag{11}$$

We proceed to solve the IVP

$$\left.\begin{array}{lcll} \ddot{z} + 2\lambda\dot{z} + \omega^2 z & = & 0 & z \in \mathbb{R} \\ z(0) & = & 0 & \\ \dot{z}(0) & = & \nu_1 > 0 & \end{array}\right\} \tag{12}$$

and then examine the result for $z \neq 0$. The characteristic equation:

$$m^2 + 2\lambda m + \omega^2 = 0 \tag{13}$$

of (12) has solution

$$m = -\lambda \pm \sqrt{\lambda^2 - \omega^2} \tag{14}$$

Obviously these roots can be complex or real, and there are three well-known cases to examine.

Case 1: underdamping

If the spring-damper pair in the suspension system is **underdamped**, that is, if $\lambda < \omega$, then (14) yields a pair of complex conjugate solutions and the general solution to (12) is

$$z_{ud} = e^{-\lambda t}\left(A \sin \omega t + B \cos \omega t\right) \tag{15}$$

Applying the initial condition $z(0) = 0$ and $\dot{z}(0) = \nu_1$ yields $B = 0$ and $A = \frac{\nu_1}{\omega}$ respectively. Hence the solution reduces to

$$z_{ud} = \frac{\nu_1}{\omega} e^{-\lambda t} \sin \omega t \tag{16}$$

which represents damped oscillatory motion.

If a speed bump is considered as an inclined plane which rises 3 cm over a horizontal distance of 12 cm, then the upward velocity of the wheel of the car travelling down the road at 60 km . h^{-1} is

$$\nu_1 \approx 15\,\mathrm{km\,.\,h^{-1}} \approx 4\,\mathrm{m\,.\,s^{-1}} \tag{17}$$

Consulting [2], typical values for k and m for a pickup truck are

$$k = 4 \times 10^4\,\mathrm{N\,.\,m^{-1}} \tag{18}$$

$$m = 30\,\mathrm{kg} \tag{19}$$

Hence, using a CAS or a hand-held calculator we determine that

$$\omega = \sqrt{\frac{k}{m}} \approx 37\,\mathrm{s^{-1}} \tag{20}$$

In order to illustrate the underdamped situation, assume that

$$\lambda = 10\,\mathrm{s^{-1}} \tag{21}$$

A typical graph of z_{ud} follows in Figure 2, using a CAS:

Hence for $z_{ud} \neq 0$, a graph of what we can expect the wheel to do follows from:

$$x = |z_{ud}| \tag{22}$$

The tyre will not be in firm contact with the road, which could be dangerous.

1.1. Case 2: critical damping

If the spring-damper pair in the suspension system is **critically damped**, that is, if $\lambda = \omega$, then (14) yields

$$z_{cr} = (A + Bt)\, e^{-\lambda t} \tag{23}$$

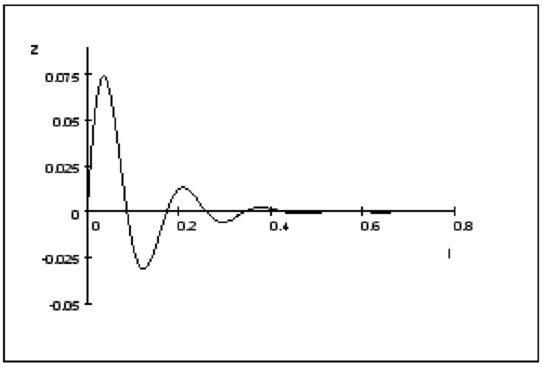

Figure 2: z for the underdamped case.

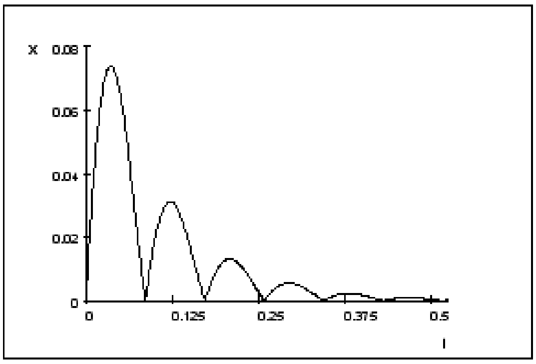

Figure 3: Displacement x of the wheel for the underdamped case.

Applying the initial condition $z(0) = 0$ and $\dot{z}(0) = v_1$ yields $A = 0$ and $B = v_1$ respectively. Hence, the solution reduces to

$$z_{cr} = v_1 t e^{-\lambda t} > 0 \tag{24}$$

Thus $\quad z_{cr} = x_{cr}$. A typical graph of the motion of the tyre x_{cr} follows with:

$$\lambda = 37\,\text{s}^{-1} \tag{25}$$

The vertical velocity of the tyre rapidly returns to zero and the tyre thus quickly regains full contact with the road. A critically damped spring-damper pair is much safer than an underdamped pair.

1.2. Case 3: overdamping

If the spring-damper pair in the suspension system is overdamped, that is, if $\lambda > \omega$, then (14) yields

$$z_{od} = Ae^{m_1 t} + Be^{m_2 t} \tag{26}$$

where

$$m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2} \tag{27}$$

$$m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2} \tag{28}$$

with

$$m_2 < m_1 < 0 \tag{29}$$

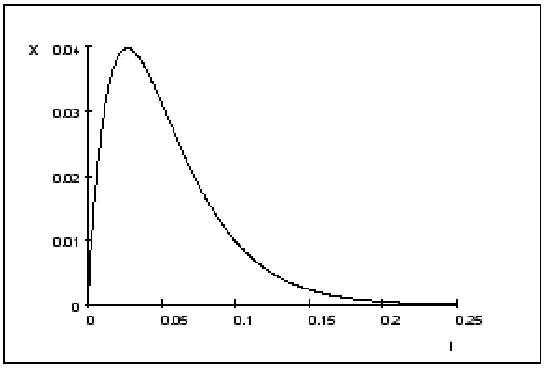

Figure 4: Displacement x of the wheel for the critically damped case.

Applying the initial condition $z(0) = 0$ and $\dot{z}(0) = v_1$ yields

$$A + B = 0 \tag{30}$$

$$Am_1 + Bm_2 = v_1 \tag{31}$$

Hence, using Cramer's rule we obtain:

$$A = -\frac{v_1}{m_2 - m_1} = \frac{v_1}{2\sqrt{\lambda^2 - \omega^2}} > 0 \tag{32}$$

$$B = \frac{v_1}{m_2 - m_1} = -\frac{v_1}{2\sqrt{\lambda^2 - \omega^2}} < 0 \tag{33}$$

Consequently

$$z_{od} = Ae^{m_1 t} + Be^{m_2 t} > 0 \tag{34}$$

and so

$$z_{od} = x_{od} \tag{35}$$

In [2], using a shock absorber in good condition, a typical value for ϕ was determined to be

$$\phi = 2 \times 10^4 \ \text{kg}\,.\,\text{s}^{-1} \tag{36}$$

Consequently, a typical graph of the motion of the wheel follows with

$$\lambda = \frac{\phi}{2m} \approx 333\,\text{s}^{-1} \tag{37}$$

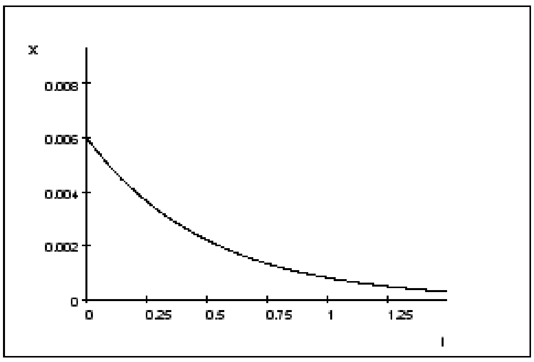

Figure 5: Displacement x of the wheel for the overdamped case.

The small maximum vertical displacement (6 mm) rapidly reduces to zero (for all practical purposes). Hence the tyre rapidly regains full contact with the road. As for the case of critical damping, an overdamped spring-damper pair is much safer than an underdamped pair.

2. Student problems

1. Show that there is only one root, one maximum point and one inflection point for the solutions (23)and (34) . Hence the positive x-axis is a horizontal asymptote. Consequently the tyre cannot bounce off the road again after the initial lift-off from the speed bump (see [3]).
2. How would one go about solving (1) if $R < 1$? In [2] it was estimated that $R = 0.911$ for a pickup truck tyre.

3. Conclusion

Ask any tyre expert why it is necessary to have good shock absorbers on a motor-vehicle and the standard reply is ”to keep the tyres on the road”. The model above illustrates the wisdom of this reply! The solution of this model illustrates the application of many basic things we teach in calculus (usually without emphasizing applications) such as:

1. the absolute value function
2. the first and second derivative of the absolute value function
3. the chain rule
4. the triple product rule
5. maximum and inflection points
6. non-linear ODEs
7. the under-damped, critically-damped and overdamped situations for a linear oscillator.

Acknowledgments

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Filling a tank: The mathematics of a dipstick

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A cylindrical tank, lying on its side or tilted somewhat, is partially filled with fluid. If we know the depth of the fluid, the question is how much fluid is in the tank. This question is addressed beginning with the simplest case where the tank is a cylinder with plane vertical ends, moving to a case where the ends are spherical, then to the case where the joints are smoothed by adding a torus section between the cylindrical and spherical sections and lastly to the case where the tank is elevated at the one end. This example shows the relevance of a wide scope of mathematics, from high school trigonometry to multivariate calculus and even some transformation geometry.

1. Introduction

The undergraduate mathematics curriculum is often confronted with the question of its relevance to practical experience. While a person exposed to higher levels of mathematics or other sciences cannot doubt the importance of a subject such as calculus to everyday human life, convincing freshman students that this is the case is not an easy task. The problem exposed and solved in this paper can be used for this purpose. The authors were approached by a manufacturing company with a problem of calculating the volume of the fluid in a cylindrical tank in terms of the depth of the fluid in the tank. The seemingly simple practical question lead to a series of mathematical calculations at different levels of complication, depending on the shape of the ends of the tank. The problem divides itself into four cases. The first and simplest case occurs when the ends of the cylindrical tank are vertical planes (the tank is a cylinder). Elementary trigonometry can be used to find the volume of fluid in terms of the depth of the fluid. The level of difficulty increases as the problem expands. For the second case the cylindrical tank, still lying on its side, has spherical ends on either side so that part of a sphere is attached left and right to give it a rounded appearance (the radius of the sphere is at least that of the cylinder.) The spherical ends do not join smoothly to the cylinder. In this case trigonometry just will not do the job and it is necessary to calculate the volume of the fluid in terms of a double integral. The problem becomes even more interesting when a smoothing section is added between the cylinder and the spherical section on either side, a practice that is common for tanks used for both transport and storage of fluids. This is the third case. Careful thinking brings the insight that part of the shell of a torus is inserted between the cylinder and the spherical ends. The fourth case is of a tilted tank with one end elevated. Again the problem is to find the volume of the fluid as a function of the depth of the fluid in the tank. This leads to more complicated applications of multivariate calculus. Furthermore, the complicated formulas lead naturally to the idea of using software for evaluating the volume. Computer programs using numerical integration were developed and applied. In this way the answer to a simple question related to volume takes us from school mathematics to advanced calculus and numerical methods showing the practical relevance of the studied techniques. Although formulas for calculating the volume of similar tanks are available, e.g. [2], [3], [4], [5] or [6], the considered problem is to a certain extent a new application of calculus techniques since these do not include the third and fourth cases.

2. A general setting

We turn to the mathematics of the problem and start by defining variables for the third case, which includes the first and second cases as special cases. Consider a partially filled tank, consisting of three types of sections, a cylindrical section, two torus sections and two spherical sections. All of these can be seen as resulting from rotating one or more curves about the (say) y-axis. Assume that the y-axis of the coordinate system is the axis of symmetry of the tank. Then the cylindrical section is obtained by rotating the straight line $x = r_c$, $z = 0$ about the y-axis, the spherical sections at the ends of the tank are obtained by rotating an arc of a circle with radius r_s and centre on the y-axis about the y-axis and the torus sections result from rotating a (smaller) circular arc with radius r_t and centre away from the y-axis, again about the y-axis. The problem is to find the volume of the fluid in the tank at depth a.

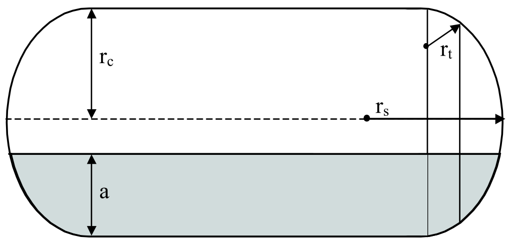

Figure 1: Vertical cross-section along the axis of symmetry of a horizontal tank (the yz-plane).

Let $x = f(y)$ be a function that is defined and continuous for $y \in [\alpha, \beta] \subset \mathbb{R}$. The surface obtained by rotating the curve $x = f(y)$, $z = 0$ about y-axis and the planes $y = \alpha$, $y = \beta$, encloses a volume

$$V_0 = \pi \int_{\alpha}^{\beta} f^2(y)\,dy, \tag{1}$$

a well-known formula, found in any standard text on Calculus, e.g. [1]. We consider the situation where this volume is partially filled with fluid. The problem is to find the volume of the fluid as a function of its depth.

Our approach is to calculate the volume by integrating the areas of the horizontal cross-sections of the fluid with respect to the depth. Apart from producing easily computable formulas, this approach has the additional advantage of providing further insight into the problem. The horizontal cross-section represents the open surface of the fluid if the volume is filled to that level and the area of the open surface is the rate of change of the filled volume (as function of the depth) at that particular level.

We consider the situation where the tank lies horizontally as in Figure 1. We assume that the xy-plane of the coordinate system is horizontal, the y-axis being the axis of symmetry of the tank, as mentioned before.

For technical convenience the level to which the tank is filled will be represented as a positive or negative deviation h from the xy-plane that cuts the tank in half. With a formula for the volume of fluid in terms of this deviation, one can easily adapt this formula to a function of the vertical distance from the lowest point of the volume, i.e. its depth and so $h = a - r_c$.

Note that formula (1) is derived by integrating the area of a cross-section of the volume perpendicular to the $y-$axis with respect to y. While, in principle, this approach is possible for calculating the considered volume of fluid, its implementation leads to problems with generalizations such as in the fourth case where the tank is tilted.

The surface of revolution generated by rotating the graph of $x = f(y)$, $z = 0$ about the y-axis has the equation

$$x^2 + z^2 = f^2(y). \tag{2}$$

The intersection of this surface of revolution with the plane $z = h$ is given by $x^2 + h^2 = f^2(y)$. Therefore, the intersection of the plane $z = h$ with the solid of revolution is the region bounded by the curves $x = -\sqrt{f^2(y) - h^2}$, $x = \sqrt{f^2(y) - h^2}$, $y = \alpha$ and $y = \beta$. This area can be calculated as

$$A(h) := 2\int_{I(h)} \sqrt{f^2(y) - h^2}\ dy, \tag{3}$$

where the set $I(h)$ is defined by $I(h) = [\alpha, \beta] \cap \{y : f(y) \geqslant |h|\}$. Note that

- for $|h| \leqslant M = \max_{y\in[a,\beta]} f(y)$ we have $I(h) \neq \varnothing$;
- for $|h| \leqslant m = \min_{y\in[\alpha,\beta]} f(y)$ we have $I(h) = [\alpha, \beta]$.

Also note that if f does not have a local minimum on (α, β) then $I(h)$ is either an interval or an empty set. In industrial constructions of tanks, the function $f(y)$ is typically concave downwards (convex). Hence it has no local minimum on (α, β). In what follows we assume that $f(y)$ is concave downwards (convex). Then the equation $f(y) = |h|$ has at most two roots y_1 and y_2 ($y_1 < y_2$) in the interval (α, β). In the case of one root we assume that $y_1 = y_2$. For $h \in [-M, M]$ let

$$\underline{y}(h) := \begin{cases} \alpha & \text{if} \quad f(\alpha) \geqslant |h| \\ y_1 & \text{if} \quad f(\alpha) < |h| \end{cases} \quad \text{and } \overline{y}(h) := \begin{cases} \beta & \text{if} \quad f(\beta) \geqslant |h| \\ y_2 & \text{if} \quad f(\beta) < |h| \end{cases} .$$

In terms of this notation the area (3) can be written in the form

$$A(h) = 2\int_{\underline{y}(h)}^{\overline{y}(h)} \sqrt{f^2(y) - h^2}\ dy.$$

The volume filled when the fluid reaches the level $z = h$ can be obtained by integrating $A(z)$ from $-M$ to h, i.e.

$$V(h) := \int_{-M}^{h} A(z)dz = 2\int_{-M}^{h}\int_{\underline{y}(z)}^{\overline{y}(z)} \sqrt{f^2(y) - z^2}\ dydz. \tag{4}$$

3. Filling the tank

In this section we use the general formula (4) to calculate the volume of fluid in the different sections of the tank as illustrated in Figure 1.

3.1. Filling the cylinder

Using highschool trig it is easy to see that the volume $V_c(h)$ of the fluid in a cylinder with radius r_c and length L at depth $a = r_c + h$ is given by:

$$V_c(h) = L\left(r_c^2 \arccos\left(\frac{-h}{r_c}\right) + h\sqrt{r_c^2 - h^2}\right)$$

3.2. Filling a section of a sphere

Consider a sphere with equation $x^2 + y^2 + z^2 = r_s^2$ which is cut off by the plane $y = d$, $0 < d < r_s$. The volume of this section is bounded by the plane $y = d$ (see Figure 2) and the surface generated by revolving the arc $x = \sqrt{r_s^2 - y^2}$, $d \leqslant y \leqslant r_s$, about the y-axis. In terms of the notation adopted in Section 2 we have$f(y) = \sqrt{r_s^2 - y^2}$, $m = \min_{y\in[d,r_s]} f(y) = 0$, $M = \max_{y\in[d,r_s]} f(y) = f(d)$ and $\underline{y}(z) = d$, $\overline{y}(z) = \sqrt{r_s^2 - z^2}$ for $|z| \leqslant M$. The volume $V_s(h)$ of fluid at level $z = h$ can be represented by using formula (4) as follows

$$V_s(h) = 2\int_{-M}^{h}\int_{d}^{\sqrt{r_s^2-z^2}} \sqrt{r_s^2 - y^2 - z^2} dydz \quad , \quad |h| \leqslant M$$

3.3. Filling a volume bounded by a torus surface

Rotation of the circle with centre $(0, l)$ and radius r_t about the y-axis generates a torus. We consider the volume enclosed by a part of its surface, namely the volume obtained by rotating the arc $x = l + \sqrt{r_t^2 - y^2}$, $y \in [0, b]$, $b \leqslant r_t$, about the y-axis. The parameters are shown in Figure 2. In terms of the notation adopted in Section 2, we have$f(y) = l + \sqrt{r_t^2 - y^2}$, $m = \min_{y \in [0,b]} f(y) = f(b) = l + \sqrt{r_t^2 - b^2}$, $M = \max_{y \in [0,b]} f(y) = f(0) = l + r_t$ and

$$\underline{y}(z) = 0 \quad \text{for } |z| \leqslant M\,, \quad \overline{y}(z) = \begin{cases} b & \text{if} \quad |z| \leqslant m \\ \sqrt{r_t^2 - (|z| - l)^2} & \text{if} \quad m \leqslant |z| \leqslant M \end{cases}.$$

Using formula (4) we see that the volume $V_t(h)$ of the fluid at level $z = h$ is given by

$$V_t(h) = 2 \int_{-M}^{h} \int_{0}^{\overline{y}(z)} \sqrt{\left(l + \sqrt{r_t^2 - y^2}\right)^2 - z^2}\, dy dz \quad , \quad |h| \leqslant M\,.$$

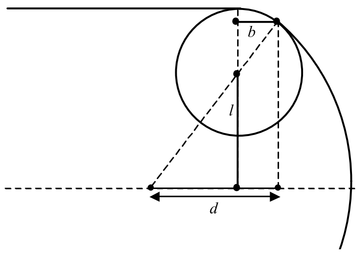

Figure 2. The meaning of the parameters

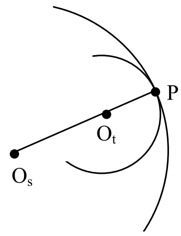

Figure 3. Smooth joints

4. Joining the sections smoothly

For smoothness the curves in the vertical cross-section of the tank (see Figure 1) join in such a way that the resulting function is differentiable at the joints. Using the notation introduced in Section 2, it is clear that in order for the joint between the torus section and the spherical section to be smooth, the centre O_s of the circle defining the sphere, the centre O_t of the small circle defining the torus and the point P where the torus joins the sphere should be collinear, see Figure 3, the radii should satisfy $r_t \leqslant r_c \leqslant r_s$. The formulas for the volumes of the different sections of the tank as derived in Section 3 can then be used to calculate these volumes. From Figure 2 the following expressions for the parameters in the formulas can be derived:

$$l = r_c - r_t\,, \quad b = r_t \sqrt{1 - \frac{(r_c - r_t)^2}{(r_s - r_t)^2}}\,, \quad d = r_s \sqrt{1 - \frac{(r_c - r_t)^2}{(r_s - r_t)^2}}$$

5. Filling a tilted tank

"Tilting" the tank means rotation of the tank at some angle. We consider rotation ("tilting") about the x-axis. For technical reasons we assume that the coordinate system remains aligned with the tank after it has been tilted. The tilt angle ω is therefore the angle between the horizontal and the new y-axis as in Figure 4. A horizontal plane (the surface of the fluid) has an equation of the form

$$-y \sin \omega + z \cos \omega = h \tag{5}$$

where h is the orientated distance from the origin to the plane. We consider the filled volume as a function of the parameter h, representing the level to which the tank is filled. Obviously, the maximum and the minimum values of h for which the plane (5) intersects the body of revolution will depend on the angle of rotation ω. Let us denote these values by M_ω and m_ω, respectively. Following the approach in Section 2 we derive analogues for formulas (3) and (4) for the situation considered here. The equation of the surface of revolution is given by (2).

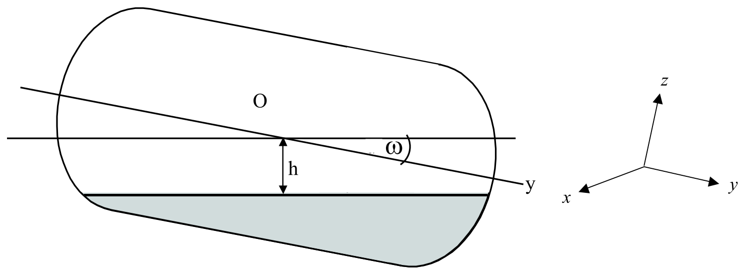

Figure 4: The tilted tank.

Therefore, its intersection with the plane (5) has the equation $x^2 + (y\tan\omega + h\sec\omega)^2 = f^2(y)$. The intersection of the plane (5) with the body of revolution is the area bounded by the curves

$$\begin{aligned} x &= -\sqrt{f^2(y) - (y\tan\omega + h\sec\omega)^2},\ x = \sqrt{f^2(y) - (y\tan\omega + h\sec\omega)^2}, \\ y &= \alpha \ \text{ and } \ y = \beta. \end{aligned} \tag{6}$$

This area can be calculated as $A(h) = 2\int_{I(h)} \sqrt{f^2(y) - (y\tan\omega + h\sec\omega)^2}dy$, where the set $I(h)$ is defined by $I(h) = [\alpha,\beta] \cap \{y : f(y) \geqslant |y\tan\omega + h\sec\omega|\}$. Under the assumption that the function f is concave downwards (made in Section 2) the function $g(y) = f(y) - |y\tan\omega + h\sec\omega|$ is also concave downwards. Therefore, for every h, the equation $f(y) - |y\tan\omega + h\sec\omega| = 0$ has at most two roots y_1 and y_2 $(y_1 < y_2)$ in the interval (α,β). In the case of one root we assume $y_1 = y_2$. For $h \in [m_\omega, M_\omega]$ denote

$$\underline{y}(h) = \begin{cases} \alpha & \text{if} \quad f(\alpha) - |\alpha\tan\omega + h\sec\omega| \geqslant 0 \\ y_1 & \text{if} \quad f(\alpha) - |\alpha\tan\omega + h\sec\omega| < 0 \end{cases}, \ \overline{y}(h) = \begin{cases} \beta & \text{if} \quad f(\beta) - |\beta\tan\omega + h\sec\omega| \geqslant 0 \\ y_2 & \text{if} \quad f(\beta) - |\beta\tan\omega + h\sec\omega| < 0 \end{cases}$$

In terms of this notation the area bounded by the curves (6) can be written in the form (similar to (3))

$$A(h) = 2\int_{\underline{y}(h)}^{\overline{y}(h)} \sqrt{f^2(y) - (y\tan\omega + h\sec\omega)^2}dy$$

The volume filled when the fluid reaches the level of the plane (5) is

$$V(h) = 2\int_{m_\omega}^{h}\int_{\underline{y}(z)}^{\overline{y}(z)} \sqrt{f^2(y) - (y\tan\omega + z\sec\omega)^2}dy\ dz\,,\ \ h \in [m_\omega, M_\omega]$$

For particular shapes explicit expressions for $\underline{y}(h)$ and $\overline{y}(h)$ can be derived in a way similar to sections 3.1 to 3.3. We omit these technical details.

6. Implementing the formulas

Constructing a solution in terms of double integrals may sooth the mathematician's mind but is not in an acceptable format for the manufacturer. The integrals need to be processed and ideally a formula in closed form in terms of the parameters should be produced. The integral formulas, derived in Section 3, look ominously full of square roots and attempts to derive a closed form by using a computer algebra system such as Mathematica fail miserably, as suspected, with a twenty line formula filling the screen, complex parts and all. Yet with specific values for the measurements the package works well and the values obtained seem reasonable. It is clear, however, that a formula in closed form is not to be found and that numerical integration is the way to go. Using a multivariate Simpson Algorithm, a software programme in Matlab is developed, for no other reason than the authors' programming fluency in this software package. The values obtained pass all checks. Running the programme, sketching graphs, checking and rechecking give the necessary assurance that the integrals are correct. To illustrate the procedure, we consider an example.

Example. Consider a tank with dimensions $L = 5$, $r_c = 4$, $r_s = 6$ and $r_t = 1.5$. If the tank is horizontal, the depth of the fluid in the tank can vary between 0 and 8 units. The volume of the fluid in the tank is plotted as a function of the depth of the fluid in the tank. Figure 5 represents the values obtained in the untilted case for the volume of the cylindrical section, the torus sections and the spherical sections as well as the total volume.

The tank is now tilted and Figure 6 illustrates the volume of the fluid in the tank as a function of the depth of the fluid for three different values of the tilt angle ω for the tank with the same dimensions. Note that while the total volume remains the same for the different values of ω, the total depth (the vertical distance between the lowest and highest points in the tank) changes with ω.

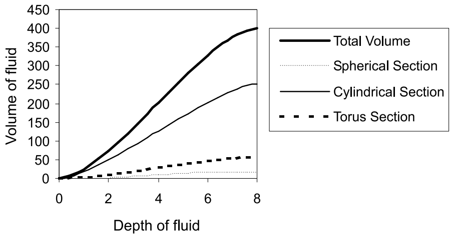

Figure 5. Horizontal tank: Volume vs. depth

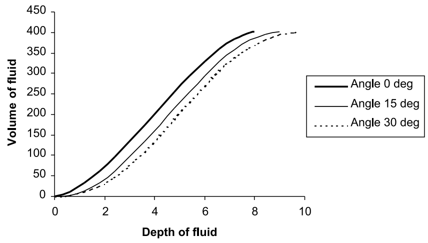

Figure 6. Tilted tank: Volume vs. depth

7. Conclusion

In a world without mathematics, sometimes called the real world, this problem is tackled differently. A yardstick is let down into the tank and a measured volume of fluid is let into the tank. The volume is ticked off on the yardstick at the level of the fluid. This process is repeated with the same amount of fluid added again and again and the result is a yardstick with volumes ticked off on it. This is not accurate, fairly primitive and not applicable in the design stage but of practical value for existing tanks. In a sense it is the reverse of the mathematical way. The depth is first determined as a function of the volume and then the depth is used to measure the volume.

One could argue that doing the mathematics leads to integrals that have little value for the non-mathematically minded and the people who need to do the job often fall into this category. It is therefore essential to take the problem to the next stage where software is produced and parameters can be keyed in to determine the volume of the fluid. This is an approach far more suited to our technology orientated society and even accessible to the worker. Thus, by using technology, the mathematical formulation of a problem can be packaged as an accurate and useful tool.

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