6th Southern Hemisphere Conference on Mathematics and Statistics Teaching and Learning

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Foreword

This conference is the sixth in a series of conferences on undergraduate teaching and learning in mathematics and statistics. It is the first time a DELTA conference is held in South America, and so it marks a milestone for the DELTA community.

The DELTA conference series started in Brisbane, Australia in 1997. The DELTA community grew since then to include members from all parts of the world. This was reflected in an internationalisation of the DELTA conferences which have since 1997 been held in various countries in the Southern Hemisphere: near the Great Barrier Reef in 1999, at the Kruger National Park (South Africa) in 2001, in Queenstown (New Zealand) in 2003 and in Fraser Island (Queensland) in 2005. With Calafate DELTA 07, all continents in the Southern Hemisphere have now been covered, giving true meaning to the title of these conferences.

Continuing with the original Delta concept to encompass the constant change that takes place in undergraduate education, the theme of Delta'07 is "Vision and Change for a new century". The program covers almost the whole spectrum of current issues related to providing a meaningful undergraduate mathematics education tailored to the needs of the 21st century, and we hope that, following the DELTA tradition, this conference makes a significant contribution to mathematics and statistics teaching and learning at undergraduate level.

The conference has two publications: a special issue of the *International Journal of Mathematical Education in Science and Technology*, and this *Proceedings* book which includes refereed and non-refereed papers. The refereed papers published in this book have undergone a rigorous blind reviewing by at least two peers from an international team of referees.

We hope that you will find many opportunities to interact with the delegates from all continents, to discuss ideas and to form new ones. We also wish you a pleasant stay in the surrounding of magnificent sceneries of Patagonia.

Finally, we would like to thank the reviewers and the International Committee for their selfless and valuable contribution to putting together the program for another DELTA chapter.

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Visual Proofs: Images for Understanding

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The aim of this paper is to describe how proofs without words give opportunities for a better understanding of mathematics in the classroom.

Keywords: Visual thinking, proofs, mathematical visualization
2000 Mathematical Subject Classification: 97U80, 97C90

1 Introduction

For many years the so-called "proofs without words" have been published in *Mathematics Magazine* and *The College Mathematics Journal* as well as in other publications and webs (see e.g. [1], [2]). This author and Roger B. Nelsen have been making for a long time research seminars on how the use of these visual proofs could play a role in a better understanding of mathematical concepts. As a result of our collaboration we have been publishing also in [3] a full description of how to produce these visual proofs for the basic topics arising in undergraduate mathematics.

This paper sums up some conclusions on the problems and benefits of using visual proofs in the teaching of mathematics.

2 On mathematical images and visual thinking

Mathematics has always combined words, numbers, diagrams, real objects hands-on-materials, etc. for its presentation ([4]). For teaching purposes, images help to develop visual thinking so they open new possibilities for a better understanding of geometrical, arithmetical, algebraic or functional developments. For example, for George Pólya mathematical problem solving was often best done beginning with a visual representation.

For either learning mathematics or for doing mathematical research, it is clear that attention must be devoted to the development of *visual thinking*. In the heuristic of mathematical discovery, internal visualization plays a major role and, in some cases, this process may be the keystone of new research.

According to Rudolf Arnheim, visual thinking is "an active exploration, selection, grasping of essentials, simplification, abstraction, analysis and synthesis, completion, correction, comparison, problem solving, as well as combining, separating, putting in context,…", i.e., visual thinking is a powerful tool which can be applied in many situations.

In [5] we find Senechal's definition of *visualization and its relationship to the general framework of visual thinking: visualization is any process producing images (pictures, objects, graphs, diagrams,. . .) in the service of developing visual thinking.*

3 Visualization in the classroom

Historically, visualization in the classroom has occurred with pencil on paper or with chalk on the blackboard. While this practice may be changing with more and more students having access to computers and graphing calculators, the traditional methods may never completely disappear.

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Technology opens new possibilities to visual experiences, from the modest superposition of transparencies on an overhead projector to the latest software (Cabri II™, Cabri 3D™, Geometer's Sketchpad®, Cinderella, Mathematica®, Maple®, Derive™, Matlab®, Geobra) which can be used to make precise drawings by computer and to project them on a screen. Internet resources also provide a large collection of high quality pictures which can be used in the classroom.

Beyond the tools used for visualization, from ordinary chalk to the latest software, visualization in the classroom has its own pedagogical values [6].

At the outset, visualization may be a tool to develop intuition, to start solving a problem or a natural way to identify concepts. But it also deserves a central role in the important task of creating proofs.

Proof, for the mathematician, is an essential component of research, but proofs in the classrooms may have the added value of *explaining* the properties under consideration. Recently G. Hanna and H. N. Jahnke [7] wrote the following:

> Clearly, an explanatory proof in school mathematics, as in any other context, must be one that not only demonstrates the truth of its assertions, but also helps one understand why the assertions are true. The aim of such a proof is always to bring to light underlying relationships that place its assertions in a broader mathematical context. In the classroom, however, an explanatory proof must rely upon the more limited mathematical knowledge of students and make use of the properties of objects best known to them.

Proofs and the act of proving have also been shown to be of great pedagogical value insofar as they aid students to gain a better understanding of mathematics. Therefore, the key issues are how to construct appropriate exercises involving proof in the classroom at all levels (for appropriate topics) and how to avoid unnecessary formalism and rigidity of presentation that may squelch learners' interest in mathematics.

4 Proofs without words

To produce proofs based upon visual descriptions we can combine hands-on materials and drawings.

To experiment with hands-on materials opens new possibilities for observing how mathematical facts appear in reality.

Following De Villiers (see [3]) we can consider experimentation as comprising non-deductive methods including intuitive, inductive or analogical reasoning. Its important aspects are:

- *conjecturing* (looking for an inductive pattern, generalization,. . . ;
- *verification* (obtaining with certainty the truth or validity of a statement or conjecture);
- *global refutation* (disproving a false statement by generating a counterexample);
- *heuristic refutation* (reformulating, refining or polishing a statement by means of local counterexamples);
- *understanding* (the meaning of a proposition, concept or definition or assisting with the discovery of a proof).

These considerations make it clear that experimentation is important in mathematics and that it plays a significant role in learning.

For example in the following pictures we can appreciate how soap films describe minimal surfaces (figure 1); how different conic sections appear as sections of cylinders and cones (figure 2); how Galton's machine shows the binomial distribution (figure 3) or how a parallel line with a given slope in the plane may form an interesting helix in a cylinder (figure 4).

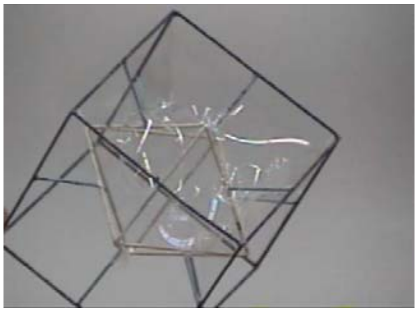

Figure 1

Figure 2

Figure 3

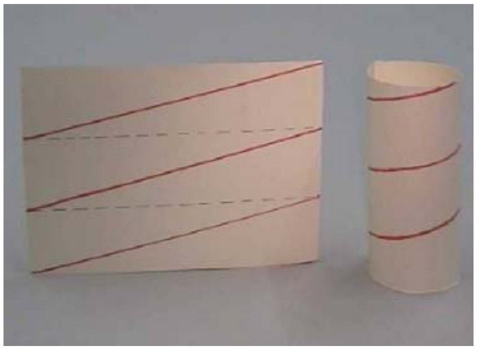

Figure 4

Concerning drawings we have identified in [3] some fruitful methods for producing visual proofs:

- Representing numbers by graphical elements
- Representing numbers by lengths of segments
- Representing numbers by areas of plane figures
- Representing numbers by value of objects
- Identifying key elements in a figure
- Employing isometries for transforming a figure
- Using similarities for transforming a figure
- Using area-preserving transformations
- Making planar development of 3D figures

- Creating tilings with double geometrical patterns
- Combining several copies of a figure
- Making sequential frames
- Making dissections and moving puzzles
- Moving frames along functional graphs
- Using interactive generation of figures
- Introducing colors in tilings
- Visualizing by inclusions of one figure into another

and of course, using hands-on materials and combining all the above resources.

As examples we give here four interesting images

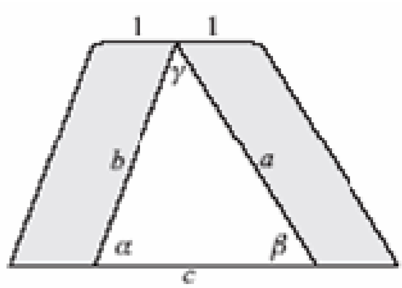

Figure 5

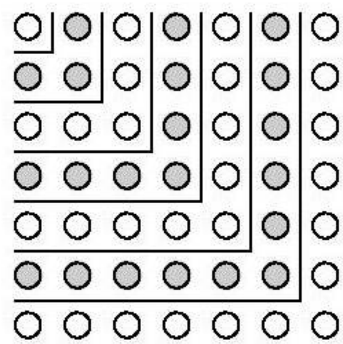

Figure 6

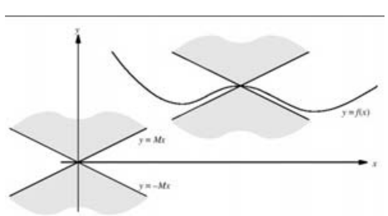

Figure 7

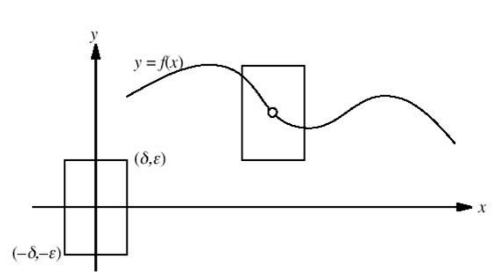

Figure 8

Figures 5 and 6 show, respectively, the sinus theorem $1{\cdot}b\sin\alpha = 1{\cdot}a{\cdot}\sin\beta$ and the classical Fubini's principle on how to count in two ways (1+3+5… or n^2). In figure 7 and 8 we follow Miguel de Guzmán to present how to see the Lipschitz condition and the uniform continuity of a real function.

5 A teaching research

This author has been applying this visual approach to proofs with his undergraduate students of mathematics and architecture and has been giving seminars to teachers in various places of Spain and Argentina. In general, at the beginning of applying visualization students and teachers are surprised because they have been trained for following formal proofs, most of them involving algebraic manipulations. But after a series of good examples they begin to find their own ways of finding proofs. A deeper understanding of properties has been appreciated. And this has contributed to a more creative approach for learning and teaching mathematics.

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Modelling and applications: approaches, processes and competencies in mathematics courses for engineering students

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In this work, we will present theoretical results related to the processes and competencies of engineering students at university level when they attempt to solve a mathematical problem associated to a real situation. Four different approaches to mathematical modelling and applications will be considered within the context of the mathematical classroom.
Benefits and drawbacks will be mentioned in reference to the processes that take place and the competencies that the students are able to develop with each approach.

Keywords: Mathematical modelling, application, processes, competencies, motivating example, difficulties.

1. Introduction

The teaching of applications and modelling in mathematics courses may be oriented to consider applications and modelling as a means to facilitate and support the students' learning of mathematics as a subject, and conversely, to consider the learning of mathematics as a means to further a basic competency for applying mathematics and building mathematical models '[1]'.

According to [1], in modelling activities, the related question is which mathematics will be an aid for the posed real world problem. Instead, in applications, the question is which real world problem will be solved with the taught mathematics. In the former, the model is constructed and the mathematical techniques have to be identified. In the latter, the model is given and the mathematical techniques are previously designated.

A key question in the pedagogical sense is to make clear and explicit which are the different sorts of knowledge and related abilities that each of these activities may require. This question is a particularly appropriate when teaching mathematics for engineers, as the competency to build models and mathematical models of a given situation, to interpret them and to use already existing ones is expected together with a high competency in mathematical knowledge.

Thus, a first step in the analysis of these issues is the identification of processes and competencies required to engage satisfactorily in these activities.

The purpose of this work is to present theoretical results which are the product of the reflection on the results of engineering students´ performance when facing different modelling activities and problem solving within the mathematical classroom in two different approaches:

a) when different degree of information, guidance and assistance were provided in order to solve a same problematic situation '[2], [4]' and

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b) when different sort of activities requiring a variety of strategies and approaches related to organization and structuring of the model, strategies and manipulation of data were presented '[3], [5]'.

These research studies '[2] - [5]' were carried out with students of different levels (novice, intermediate and advanced) and permitted us the identification and analysis of students' processes, required competencies and difficulties in three stages of the modelling activities: structuring the model, formulation of a strategy and resolution and interpretation of the results.

This description will be presented in section 2. In section 3, we will discriminate and analyse which of these processes and competencies are involved in different activities developed in the courses like modelling real situation, mathematical modelling and applications as examples and as motivational devices.

In section 4 we will present a summary and final remarks.

2 Processes and competencies in different stages of the modelling activities

2.1 Stage I – structuring the real model

2.1.1 Processes

Understand the given problematic situation.

Generally, modelling situations are non-standard problems for the students. In order to solve them, previous knowledge must be restructured. Understanding involves identification, discrimination, generalization and synthesis, perceiving the links between apparently isolated facts, properties, relationships, etc., organizing them in a consistent whole.

Formulate the mental model, which structures the contextual situation.

Three processes may be considered when thinking about novel situations: reasoning by similarity, by mental simulation and by formal reasoning.

Reasoning by similarity: the given situation is considered similar up to some extent with a previous known situation. These similarities could be superficial features of the problem or more abstract relational characteristics. This approach ranges from the direct retrieval of a situation at one extreme to the analogical reasoning at the other, passing through processes like generalizing, evoking and reasoning by means of a paradigmatic example.

Mental simulation: the problem is analysed by imagining the consequences of an action. *Formal reasoning*: a formal symbolic system is used to approach the problem.

Generally, several of these processes are activated simultaneously or sequentially in an intuitive and not controlled way.

2.1.2 Competencies

Relational understanding. It consists on the correct interpretation of the involved concepts and in their connection and structural organization.

The relational understanding leads to the formulation of a mental model of the real situation within the context in which the given situation is mentally represented.

Several of our studies have shown that the mental model that a student produces is the simplest one that presents certain degree of coherence with the given information.

Accessibility (the ease with which particular mental contents come to mind), availability and handling of previous knowledge.

Creativity. Creativity has been observed as a process external to mathematical theories and to science theories. It is required for the design of strategies (and Mathematical Models) since this design may demand the student to create new ideas and to relate old ideas in a new way, extending the context in a way that is different from the context that was known before.

Ability to check up the feasibility of the proposed model. The student should develop some criterions (based on related knowledge and/or common sense) that allow him to evaluate if the approach to the problem may generate a feasible strategy of resolution.

2.2 Stage II – Elaboration of a strategy and resolution

2.2.1 Processes

a) Real context

Select operative representations within a contextual framework. That is to say, representations that permit to reason, solve and interpret the results within the chosen working context.

The representational resources allow the student to structure the information and economize mental resources and reasoning. Though the representations may or may not be done in a mathematical context, our main interest was focused on the mathematizing processes.

b) Mathematical context (Mathematization)

Construct a mapping between a non mathematical system of representation and the mathematics that the student associates with it.

Select operative representations within the mathematical framework (variables, tables, diagrams, graphics, formulas, etc.)

Use mathematical techniques. The elaboration of the mathematical strategy usually requires the extension of knowledge, whether procedural or/and conceptual '[7]' to a novel domain.

The conceptual knowledge is characterized by being rich is relationships among information units. When a relationship is built at a higher level of abstraction than the information that is being connected, this relationship is produced in a reflective level.

Procedural knowledge is characterized by the sequential relation of procedures or sub-procedures. It encompasses two parts: a) formal language or representational symbolic

system and b) algorithms or rules to complete the mathematical tasks.'[7]'.

2.2.2 Competencies

Creativity in the combination of mathematical concepts and representations.

Human beings have the tendency to organize and integrate intuitive, logical and analytical cognitions within a coherent and efficient structure. With experience and instruction, strong and stable beliefs are established. These beliefs act at the moment of choosing the strategy for the resolution of a problem '[6]'.

Accessibility and availability of mathematical knowledge: concepts, mathematical techniques of resolution, procedures and sequences of procedures.

Flexibility to develop different types of reasoning and representational changes.

Several representations of the same concept may be integrated making the abstraction process possible. Experts deliberately retrieve different known representations and change from one to the other seeking for the most suitable one, while students generally carry out this process intuitively and in a not controlled way.

2.3 Stage III – Interpretation of results

2.3.1 Processes

Evaluate the effectiveness of their own proposed model and the strategies and representations that have been selected. This evaluation involves certain verifications such as:

Whether all the conditions and restrictions of the situation have been accounted for, whether the conclusions or results really respond or solve the problem and whether the strategies or methods are feasible.

Select communicable symbolic representations to elaborate the discourse. The use of discursive elements in mathematics is influenced not only by the representational system that the student uses, but also by the student's suppositions about what is expected from him in terms of formalization.

2.3.2 Competencies

Use criteria to evaluate the model. That is to say, the student should be able to analyze the possibilities and restrictions of the proposed model.

Be familiar with and able to use a range of discursive resources.

Most of the mention processes and the competencies are required when modeling a real world problems. However, within the mathematical classroom, these processes and related competencies may vary according to the degree of information and guidance provided to the students and also according to the sort of problem that they have to face. In the next section we will discuss and exemplify processes and competencies in different situations: modeling and mathematical modeling with different guidance, and applications used as example and as motivational device.

3. Modelling and applications in courses

3.1 Real world modelling

We will focus on those activities of building the real model, which admit a mathematical translation. These activities seem to be the richest ones in terms of processes and related competencies, and in this sense, they are important in the formation of engineers as they encourage the ability to extend different pieces of knowledge to other contexts, which is essential for the development of their professional life.

However, when a free modelling activity is presented to the students within a mathematics class with predetermined teaching and learning goals, it may happen that the students'formulation of the real model involve a solution with little interest for those goals.

For instance, a group of (40) novice engineering students were asked *"How would you find out a mountain's height"?.* The students could use any knowledge, technique or device to solve it. 30% of them solved it in a variety of a non-mathematical context. For example, a student proposed:*"I would use a thermometer. I know that each 100 meters of height the temperature decreases 1 degree. So, I would check the temperature on the base of the mountain, and then I would climb, and would take the temperature on the peak of the mountain. In this way, I could know approximately the height of the mountain"*. '[3]'

Other students proposed to reach the peak with a helicopter and to read the altimeter, or to use a scale evaluating the differences of weight of the same person in the base and the peak.

Though the students evidenced great creativity in the attempt to solve the problem and some of these solutions, correctly posed and developed could lead to a successful solution, they would not be interesting if the goal is working in trigonometry. Only 6% of the students proposed this approach. Moreover, the feasibility of the methods – like hiring a helicopter to measure any mountain's height, or climbing a mountain carrying a scale to weigh a person- which is a relevant issue in engineering studies, should be discussed.

Sometimes, the formulation of the real model requires a specialized non-mathematical knowledge (physics, chemistry) that is not always available to the students.

For instance, a group of 35 students were asked to give a method to find out whether a buried tank that contained a salted solution had a leakage, knowing that no direct way of measuring the volume was available, but the injection and removal of some liquid and also the measuring of the corresponding salt concentration was possible. '[2]'

Again, in this case, the students proposed barely feasible empirical non-mathematical methods, (like a direct measure of volume -17%- for example*, "empty the tank, fill it again and measure the resulting volume"*). Only with extra suggestions, could they attempt a mathematical model of the situation. Yet 25% of the students considered that the salt concentration was constant, and they had difficulties when working with this notion.

The main processes and related competencies involved in these activities are the mentioned in Stages I and II a) and summarized in Table 1: Understanding, (which is the problem), mental modelling: (the mental representation of the real situation and the steps he or she is going to follow in the real context looking for the solution), selection of operative representations in context (context variables like measures of concentration, of temperature, of angles etc.)

3.2 Mathematical Modelling

In mathematical modelling activities, the real model is given or suggested in a context.

The student should do the mapping of the real model components with the mathematical concepts and should design of a resolution strategy, selecting the mathematical representations to work (a coordinate system, derivatives for rates of change).

If the model requires other mathematical resources which exceed the course contents, or if the mathematical topics turn to be too complex, the solving process could need to be guided or suggested.

We have considered a subtle difference between the expressions *guided or assisted* and *provided by instruction*. While the term *guide* suggests a provision of information in steps, after which the student has to elaborate the mathematical model, the term *provided by instruction* entails a different process: the mapping between the real and the mathematical contexts results as an expected consequence of the mathematical content recently taught.

According to the guidance and assistance, the processes and competencies involved in this sort of activities are the ones mentioned in Stage II, and summarized in Table 1.

For instance, in the mentioned problem of the buried tank, different groups of students were given different sort of guidance. One of the groups was asked to model mathematically the cases of non-leakage and leakage with the single information that it was possible to inject and remove solution with a fixed concentration at a constant rate into the tank. In this case, many difficulties were found at the mathematization stage. 60 % of the students unsuccessfully used very simple mathematical concepts, showing a strong tendency to linear models. Instead, another group could solve it when the formulation of a differential equation was guided (*Guided Model*).

In another experience, a group of (80) students taking a course of Several Variables Calculus was suggested to construct a method for edge detection of digital images. Both definitions, image as a matrix of pixels and edge point, were provided. The mathematical model was neither guided nor suggested (*Unguided Model*). However, the solution was accessible for these students. 90% of them successfully solved it using discrete directional derivatives in the gradient direction to detect the edge points. '[4]'

3.3 Applications as examples

A main difference between a guided mathematical model of a real situation and an application presented like an example is that in the first one the student is assisted to construct a model. In the application, instead, several of the processes are developed through the instructional activity. As a consequence, the processes and competencies required by these activities are different. They are summarized in Table1.

For instance, in a third course of Mathematical Analysis (40 students), after teaching the Distribution theory, the students were asked to use distributions to solve the following problem: "*Find the force acting on a ball that is moving on a straight line on a flat horizontal surface without friction which is hit by a hammer*."

60% of these students solved the problem using distributions reaching the following final result: $F = k\delta$ where δ is the Delta functional. 32.5% could not implement mathematical techniques, 7.5 % made no attempts to solve the problem. '[5]'

This problem, solved by the students as an application example, was useful to detect whether they could identify the physics unit impulse with the defined delta functional.

In the applications presented as examples, the formulation of the real model is assisted, and the formulation of the mathematical model with the different processes involved (selecting operative representations in context and in mathematics), are provided by instruction. The corresponding mathematical knowledge must be available to the students. An application as an example to be solved by the students is used to show that the mathematical content recently taught may be transferred to another context, enhancing the meaning of mathematical concepts. In this way, mathematics proves to be a useful tool. An application as the mentioned problem could be used in another didactic way as the motivation to introduce a specific topic or theory.

3.4 Applications as motivation

A real mathematizable problem may be presented as an illustration of a mathematical theory recently taught, or also as a motivating introduction of this new content.

For example, the mentioned problem about hitting a ball with a hammer may be used as a motivation to introduce the theory of Distributions to extend the notion of function.

In an application used in this way, it is expected that the students discuss the formulation of the real model. As the mathematical knowledge is not available yet, the different mathematical models presented by the students during the debate are usually incomplete and unsuitable, producing the cognitive conflict, which will generate the need for the development of the new theory.

For instance, in a class of 30 students where this problem was used as a motivating example, the students associated the acceleration with an impulse, some of them knew from physics about "*a function that represents the impulse* : $\delta(t)=\begin{cases}0 & t\neq 0\\ \infty & t=0\end{cases}$ *and* $\int_{-\infty}^{+\infty}\delta(t)dt=1$ " while others defined it as equal to 1 at $t=0$. Contradictions with the theory of Riemann integrable functions were remarked at that moment, thus generating the cognitive conflict.

Another student proposed a function whose graphic is bell-shaped explaining *:"..., it does not happen that suddenly in one instant the ball changes from velocity zero to a velocity v because there does not exist an impulse lasting only one instant, but it is an interval (a very small one) where there is a large variation of velocity."*

This idea was shared by many other students of the class and was used by the teacher as a starting point to introduce the regular sequences approach to slow growth distributions.

The formulation of the real model and an attempt to construct a mapping between the real and the mathematics contexts, were present during the discussion. The criterion of feasibility was also encouraged at that moment. This resulting debate aimed to make the students recognize the need of an effective mathematical theory, in this case, the distribution theory.

In the same way, the problem of determining the instantaneous velocity, or how to draw

a tangent line to a function graphic curve, motivates the introduction of the concept of derivative in elementary calculus courses, and Fourier Series may be introduced after an incomplete attempt to solve the Heat Equation.

3.5 Summary

Table 1 presents the processes and competencies for each of the activities corresponding to the four different approaches: An Unguided Modelling Activity, a Guided Modelling activity, an Application as an Example to illustrate a mathematical topic (solved by the students), and an Application as a Motivating Example.

4 Final remarks

In this work, we have exposed processes and competencies related to different stages of the modelling activity in order to construct a mathematical model of a real situation.

A desirable modellig activity in mathematics courses might be to carry out the whole process of modelling ranging from the real world situation to the mathematical model and the corresponding resolution, however this kind of activities are time-consuming and the modelling activity is not always workable at the mathematics course. In addition, if the real contexts or the mathematical resources to be used are too complex or sophisticated, guided steps for resolution will be possibly required (Guided Modelling Activity). The processes and competencies developed will vary according to the provided guidance or assistance and/or according to the sort of proposed activity. The possible difficulties in the development of each of the activities must be also considered.

Stage	Processes	Activities					Competencies
		Modelling	Math modelling		Applications		
			U	G	Example	Motivation	
I	Understanding	**X**	*	*	*	**X**	- Relational understanding - Accessibility and availability of knowledge - Creativity - Monitoring
	Mental modelling	**X**	*	*	*	**X**	
IIa)	Selection o f operative representations in context	**X**	**X**	*	◊	◊	- Creativity - Accessibility and Availability of math knowledge - Representational Flexibility
IIb)	Mapping	**X**	**X**	*	◊	To be taught	
	Selection of Mathematical representations	**X**	**X**	**X**	◊	To be taught	
	Implementation of mathematical techniques	**X**	**X**	**X**	**X**	To be taught	
III	Evaluation of effectiveness	**X**	**X**	**X**	---	----	- Criterion of feasibility - Discursive resources
	Elaboration of discourse	**X**	**X**	**X**	----	-----	

Table 1. Processes and competencies in different stages for different sort of activities
U:Unguided, **G**: Guided, **X**: By himself, ***** : Assisted , ◊ : Provided by instr

An observed difficulty of these activities '[2], [4]' is that the students might miss the whole process and become distracted by the details of each step. The interpretation of the model will be poor and the development of the competencies will be affected. Adding an exploratory complementary work might enrich and balance this dissociating effect.

Applications of mathematics deal with mathematical aspects of reality and with mathematical models which are already built. '[1]'.

While deciding the sort of activity to be performed in class related to a real problem, an important question emerges at this point: if the teacher knows the model that fits the real situation in advance, may the problem still be considered a modeling activity or should it be considered an application given as an example of the use of the mathematics that was taught during the course? The answer has to do precisely with the processes that are involved in the activity and the competencies that are required to solve the situation.

Two different approaches for applications have been chosen: as a post-teaching example and as motivating example previous to the teaching of a mathematical content, that, as stated, require different sorts of knowledge and related abilities.

These approaches for applications are also used as didactical resources connecting students with some classical models, which they may explore as an advisable task, even though these models are not available to be constructed by them.

It is important to notice that a mathematical model used as a didactical resource must be adequate to promote favorable learning instances. It is convenient to look for models that correspond to real situations, that are attractive, accessible for teachers and learners, ready for simulation if possible, and that require little extra mathematical knowledge. If this is the case, the same problem might be used in the different approaches for applications or mathematical modeling activities.

Further research is needed to obtain adequate real problems fulfilling these conditions.

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Learning style-based solid of revolution calculation through MASTERMATIC, an intelligent tutorial system

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Abstract

This work describes the design and development of an intelligent tutorial system as a supporting tool to the teaching of solid of revolution volume calculation, which is one topic included in the Integral Calculus subject matter syllabus for students of engineering at Universidad Metropolitana.

The proposed instructional design considers stimuli associated with the prevailing learning style of the users [1], and combines both behavioral and constructivist approaches within a "learning-by-doing" [2] context.

A developmental methodology of educational application for multimedia [3] environments was chosen in order to define and produce the tool, incorporating typical schemes and details of intelligent tutorial systems [4], as well as elements of object oriented software development which altogether enrich and strengthen methodological designs where *a priori*-established objective achievement controlling rules of the multimedia application are structured.

As a result, MASTERMATIC—a teaching resource for the study of a mathematical application [5]—was developed. It integrates the constructivist approach, prevailing learning styles [6], intelligent tutorial systems and multimedia methodology. It is a dynamic, user-friendly and flexible application oriented towards handling, visualizing and understanding one of the most-widely studied and required-by engineering students [7] integral calculus applications.

Key words: intelligent tutorial systems, learning styles, educational software design and development.

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Engineering student competence development through mathematics teaching

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Abstract

The terms Knowledge Society and Knowledge Management are source of new challenges for universities which, at the present time, are considered as permanent learning oriented organizations [1]. New tendencies in education aim at competence development training to satisfy the needs of the XXI century professional [2]. According to this approach, competences lead the sense of the learning process, surpassing the gap between knowledge acquisition and the capacity to apply it [3].

In this sense, the need to reorganize existing training programs arises - taking into consideration all competence building processes [4] - with the purpose of creating curricular designs not only to learn but to continue learning as well, and to fulfil professional profiles required by both context and society [5].

On the other hand, one of the most spread general tendencies in the field of mathematics teaching today is based on the transmission of all discipline-related thinking processes in order to go far beyond mere content transference [6], [7].

Under the above mentioned framework, this research proposes a qualitative approach methodology [8] through which a diagnosis is made [9] in order to establish and isolate the generic competences and the specific contents that – derived from the teaching of mathematics - must be fostered and worked to consolidate a new way to train engineers at Universidad Metropolitana.

As a result, six subject matters were chosen to be used as support platform to develop seven identified generic competences, in a systematic, explicit and ordered way.

Keywords: Generic Competences, Cognitive Processes, Mathematical Competence.

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A report on the use of tablet technology and screen recording software in tertiary mathematics courses

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Recent developments in technology have allowed lecturers to experiment with new ways of presenting and recording lectures in several mathematics and statistics courses at The University of Auckland. Lectures have been delivered using a tablet computer, with all the activity on the screen captured as a digital recording along with audio-narration of the lecturer's commentary. This paper describes the thoughts, challenges and experiences of the staff involved in establishing this project, and reports on the findings from a number of sources of student-feedback with respect to the use of this technology. Whilst there have been some teething problems in the initial stages, our overall impression is that this technology greatly enhances students' learning experiences, and we should continue to develop the potential of this technology.

1. Introduction

Although tablet technology itself has been available since the early 1990's, Loch [1] notes that very few tablet PCs have been sold into academia, with an even smaller degree of use in teaching. Two papers presented at the Delta'05 conference [1, 2] reported on the author's use of tablet technology in teaching undergraduate mathematics. Recent software developments enabling a simple mechanism for recording lectures enabled this use. With the exception of the MathOnline project at The University of Colorado however [3], it appears that in most cases this technology has been used for static recording of lectures in pdf format without a video or audio component.

At the University of Auckland, there has been an increasing use of tablet technology in teaching since the start of 2006. Lectures delivered via the tablet have been dynamically recorded, capturing both the live writing and images shown to the class, and an audio of the lecturer's voice. The files were then placed either on individual course websites, or on Cecil, the university's learning management system, allowing students the opportunity to review any part of a lecture repeatedly at their leisure. What started in late 2005 as an innovative solution to a problem encountered by one of the lecturers with usual lecture-delivery methods rapidly expanded to be a key means of delivery in a number of courses in both mathematics and statistics. This paper will detail the history behind this project's inception and describe the technology, software and processes used by our mathematics courses to deliver, record and post the lectures. Another report has been separately presented on the use of this technology at the University of Auckland in statistics [4]. Experiences of teaching staff and students are described using departmental reports and feedback from student surveys and journal excerpts. Practical issues of implementation are considered, along with a discussion of potential pedagogical benefits and disadvantages. Finally, we discuss possibilities for future development, including the use of tablets in video-conferencing between research groups.

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2. Background

As is frequent with new developments, the use of tablet technology at Auckland was more a result of providential circumstances, than any awareness of its similar use elsewhere. Even after the initial decision to proceed, the project was more an organic response to local conditions than a reflection at that time of pedagogical or technological perspectives supported by literature studies. Associate Professor Paul Bonnington first became familiar with the technology through his role as Associate Dean to the Faculty of Science, responsible for Information Technology. Bonnington had long encountered difficulties with standard lecture presentation. He is left-handed and finds using either blackboards (or whiteboards) and overhead projectors problematic, as his hand either blocks or rubs out writing. Although PowerPoint seemingly offers a potential solution to this, it does not provide the spontaneity of a normal lecture and it is very difficult to reproduce the natural unfolding of a mathematical problem that takes place on a blackboard or OHP. PowerPoint slides that do attempt to reproduce this process are usually extremely time consuming to produce. Bonnington's initial idea was to use the tablet as a sophisticated OHP that would enable him to write out problems without the usual left-handed difficulties. At the same time, he realised that it would be simple to capture the lectures and make them available as a static pdf file, which would have the benefit of limiting the amount of notes students would have to take, without the need for him to prepare comprehensive notes or hand-outs in advance. Although he was aware that it was possible to capture the lectures dynamically, in this initial stage he did not realise that this was pedagogically desirable.

In early 2006, Bonnington met with members of the department's Mathematics Education Unit, and discussed his proposals and sought their advice on how to proceed from an educational perspective. They enthusiastically endorsed his ideas and strongly recommended that he should indeed record the lectures dynamically. Mike Thomas noted at this meeting that if possible Bonnington should capture both the written and audio parts of the lecture, arguing that this technology offered both the technological benefit of being able to link together multiple-representations in real time as well as the chance to review the links repeatedly! Bonnington then explored "lecture-recording" on the web to see what was being done elsewhere. There were many examples, particularly in distance learning, but most made use of video camera recordings. For example, the University of Western Australia delivered pre-recorded lectures on the web as opposed to CDs or cassettes, while the Department of Statistics at Auckland University provided a comprehensive set of resources for its large first-year undergraduate statistics course including narrated PowerPoint lectures, and small movies on particular concepts, made available to students on a CD [4]. Bonnington found little evidence of live recording of lectures, but more importantly, what was available was largely by way of video camera recording which generated large unwieldy files. These files are usually too large to load on a memory stick and too big to be easily downloaded over the web. Hence they usually require students to view them from a computer lab, or by purchasing a CD, which contradicts the spontaneous nature of live lecture recording. Knowledge of the MathOnline project at The University of Colorado [3] only became available later when examining the outcomes of Auckland's developments.

Bonnington first trailed the use of the tablet in delivering and recording lectures in his third-year course *Combinatorial Computing* in the first semester of 2006. This course for approximately 30 students seemed ideal, as it has a strong focus on graphical representation, covering aspects of the representation and generation of discrete mathematical structures, searching and sorting methods, graph algorithms, block designs, coding theory, and computational complexity. Despite some early teething problems (e.g. no sound in the second lecture), Bonnington was extremely excited and satisfied with the results of this trial, and despite no promises being made, the students also seemed highly supportive. Bonnington reported back on this trial to the mathematics and other departments in a seminar at the end of semester one 2006 (July), with the result that the Statistics department immediately adopted its use in semester two for their very large first-year course, which often has upward of 500 students in one lecture stream [4]. Another first-year mathematics course for mathematics majors also experimented with the technology in semester two 2006, and extended this use in semester one 2007, along with two other first-year courses and a second-year course.

The next section will detail the nature of the technology used, and how it was practically implemented. It will also discuss some of the educational issues that have arisen from this project with an examination of the literature.

3. Screen recording[1]

After making the decision to proceed with the initial trial at the start of 2006, choices had to be made about which particular technology to use. There was also a growing realisation of the educational and pedagogical issues involved. The main motivation had by then moved from a simple solution to the left-handed lecturing difficulties, to a strong conviction in the value of enabling students to review any part of a lecture as many times as they wish. The rationale behind this came from a feeling that 'much of the information presented in lectures is lost, and this is particularly true in the Mathematical Sciences'. In this section we will describe the technology we used and its implementation, as well as discussing some of the theoretical issues that have emerged from this project. The chief goals of the project were to give students the opportunity to re-visit lectures, both audially and visually, and to produce these resources with minimal effort and cost.

3.1 *Technological Perspectives*

Screen-recording packages capture the visual display used by the lecturer, and may include audio recording of the lecture. Over the 2005/2006 summer break, Bonnington evaluated several screen-recording packages, finally settling on a combination of an HP Tablet[2] to write on, and a software package called BB-Flashback[3], which enabled the recording and editing of onscreen activity with associated audio. Both the tablet and the software are relatively inexpensive, although one of the lecturers noted in their report that each lecturer or course really needs their own tablet, as sharing a tablet caused timetabling difficulties, which means additional expense. The tablet is also easily transportable, although of course this effect is lessened if a data-projector has to be carried as well. At Auckland, all main teaching rooms are equipped with a data-projector, but this was an issue for a small graduate course when Bonnington later lectured in a room without one.

For courses that involve writing on overhead transparencies, 'PDF Annotator'

http://www.ograhl.com/en/pdfannotator/

enables the capture of annotations made by the lecturer on the pdf version of the lecture notes. PDF Annotator allows the lecturer to write on any pdf document using a variety of pens and highlighters. In most cases, writing is done on the tablet (which takes a little practice) and Microsoft Office was installed on the tablet to enable Word files and PowerPoint shows to be displayed.

Although products enabling screen recording have been available for some time (e.g. Lotus Screencam in 1993[5]), these products produced large files and were limited in their editing facilities. Recent technological advances support smaller, more compact file formats such as Macromedia Flash[6] and have enhanced editing capabilities. These files are widely viewable on most web-browsers, Macintosh and Linux machines. Delivery of screen recordings in most courses in this trial was supported by Cecil[7], the university's own platform for internet delivery of resources, administration and communication, although some staff used their own course's website and employed other software products to transfer documents to the website. Cecil can be accessed from anywhere in the world, and students are familiar with Cecil which they use for routine management of their learning across all courses at the university. As a result of the students' exposure to Cecil, they are able to find each day's lecture easily. They can either play the lecture in its entirety or they can download the lecture to a memory stick for playing on their home computers without relying on internet access. The replay format allows them to find particular parts of the lecture that they wish to review quickly and easily, another advantage over other recording systems such as videos or CDs. The exporting facility of the BB-Flashback software allows for more than one format, such as the ubiquitous Adobe Flash format and self-contained Windows executable files.

Thus in addition to many potential educational benefits discussed in the next section, the practical advantages of modern screen recording technology over previous methods of recording are that it is much cheaper to deploy, it is more easily transportable, and the resultant files are much smaller and easier to work with. By way of comparison, a 1-hour lecture using a video camera would produce a file of around 100Mb in size, the equivalent screen recordings are about 14Mb. An entire semester's worth of lectures fits on one CD.

3.2 *Educational Perspectives*

The goals of the Auckland project assume that lectures will (and should) continue as a basis for primary delivery of mathematics courses, and that providing screen recordings of lectures will enhance learning. Both these assumptions should be examined critically. Lectures have come under increasing scrutiny and criticism, with suggestions from some quarters that we should abandon lectures in favour of the convenience and availability of technology [5]. Against this, we need to balance the value of such technology in learning, as one professor argues, 'just because I am competent with technology… (does not mean) my students would magically learn better…the criterion for bringing technology into my courses should always be: will this enable me to pose questions that better engage my students, spark their curiosity, and push them to think critically and, ultimately, to learn?' [6].

[1] Screen recording http://en.wikipedia.org/wiki/Screencast

[2] HP tablet http://www.hp.com

[3] BB-flashback http://www.bbsoftware.co.uk/

[4] http://www.ograhl.com/en/pdfannotator/

[5] Lotus screencam http://www-306.ibm.com/software/lotus/

[6] Macromedia Flash http://www.macromedia.com/software/flash/

[7] Cecil http://www.cecil.edu/html/about.htm

In [5], Cretchley argues powerfully for the continuation of lectures. She notes that 'there is substantial and documented evidence across a wealth of educational literature that teacher-centred learning has a strong positive effect on student performance' [p. 43]. Lectures provide an important source of socialisation and sense of community and purpose for students; they can inspire and motivate, and provide a natural environment to establish complex links between the different representations of mathematical concepts [5, 7, 8]. Despite the widespread availability of online materials, and many courses providing extensive online resources including PowerPoint shows of lectures and in some cases pre-recorded lectures, the majority of students still attributed a large percentage of their learning to lecture attendance [9]. Further, Cretchley [5] found in her study, that in contrast to some reports, lecture attendance has not greatly diminished.

However, given that lectures remain an important component of student learning, there is a significant problem that much of the information presented in lectures is often lost. Very often, the full explanation for steps in a calculation or logical deduction is presented to the student verbally, or written on an OHP or blackboard. Inevitably, some students cannot reconstruct the explanation from the written notes and confusion, misunderstandings, and even frustrations can result [2]. Lectures present a great amount of complex information, usually in an equally complex variety of modes and representations. Students need to make sense of multiple representations and confront cognitive conflicts within the unique discourse of mathematics, whilst simultaneously taking notes [7, 8, 10, 11, 12]. Ball, Bass and Hill [13] argue that advanced mathematics is compressed into abstract symbolic forms and teachers need to unpack this mathematics so that what they present to the students is level-appropriate and accessible, a view supported by Tall [10] when he notes that:

> ‘Advanced mathematics, by its very nature, includes concepts which are subtly at variance with naïve experience. Such ideas require an immense personal reconstruction to build the cognitive apparatus to handle them effectively. It involves a struggle […] and a direct confrontation with inevitable conflicts, which require resolution and reconstruction’.

These demands are compounded greatly both for what Tall [14] describes as *Natural* learners, and for students with English as an Additional Language (EAL) [11]. The latter study found that EAL students experience a 10% disadvantage in overall performance through lack of textual understanding. They conclude with respect to undergraduate mathematics, that

> there is some evidence that there is a fundamental change in the nature of the discourse: not only do the normal features continue to get more complex, but also the use of mathematical discourse changes in several ways…The roles of definitions, axioms and theorems in mathematical argumentation are subtly indicated in their linguistic expression. General English is used in increasingly creative ways to describe the increasingly sophisticated nuances of mathematical concepts [11].

Compounding this may be the fact that many lecturers are unaware of much of these complexities, exhibiting what Nathan and Petrosino [15] term an *expert blind spot:*

> …educators with advanced subject-matter knowledge of a scholarly discipline tend to use the powerful organizing principles, formalisms, and methods of analysis that serve as the foundation of that discipline as guiding principles for their students conceptual development and instruction, rather than being guided by the knowledge of the learning needs and developmental profiles of novices. [15, p. 906]

Not surprisingly many students struggle with this complex combination of demands, and can experience what is often referred to as cognitive overload. Far from decreasing this overload, the plethora of resources and technological aids commonly now made available to students in their courses may indeed add to it, as students struggle to identify what is important and what is not. Using lectures as the principal delivery method, scheduled at fixed times, does also not take into consideration individual learning styles and preferences. Cretchley asserts that an ‘awareness of tertiary students’ changing goals, needs, preferences and perceptions is vital if educationalists are to respond quickly and appropriately’ [5, p. 42]. D’Arcy-Warmington describes many ways of learning, and observes that ‘university tends to use principally linguistic and logical-mathematical modes in teaching, thus missing the opportunity to relate to all of the learning styles and hence all students’ [16, p. 175]. The importance in recognising and catering for individual styles is supported by Oates et al [17] who conclude that:

> The most comprehensive conclusion that can be reached from this study is in the area of individual student preferences for different styles of learning … It is clear from the findings of this study that the standard lecture delivery method is not catering for the needs and/or the preferences of the majority of our students … there is a clear indication that we should include a significant amount of (different learning) opportunities within our courses for students. [17, p. 738]

Recording lectures using tablets and screen recording may be seen to address many of the issues identified in the preceding discussion about lectures and individual learning styles and preferences. In addition to stimulating interest and meeting changing student expectations [2, 5,18], tablet technology provides several advantages over previous methods of recording and suggests a range of potential pedagogical benefits. In their report, the Statistics team note that when students revisit lecture material that they have found difficult, there is an 'association between the concepts delivered in the lecture and the reinforcement that they receive from the screen recording of the same lecture. This is an important difference between screen recording and pre-recorded lectures' [4]. Further, given that human contact is a significant feature of lectures for students [5, 9], the direct association of the recordings with a specific lecture may well add a human dimension to the use of technology that is not possible with pre-recorded lectures or PowerPoint shows [19]. Bonnington supports this and notes the potential for greater benefit from re-watching earlier lectures towards the end of the course, as it is very likely that what was said in earlier lectures would be more meaningful to students later in the course. This is not unlike re-watching a murder story once one is aware of the identity of the villain – the clues and connections are far more obvious! He sees another advantage over video recordings in that the screen recording focuses entirely on the written work (with audio-support), avoiding the potential distraction of what he describes as a 'talking-head'.

The tablets also provide an excellent mechanism for realising the benefits that technology provides for linking multiple representations of mathematical concepts [2, 8, 18]. Not only does the tablet provide a versatile platform for computer-aided learning (e.g. Matlab, Maple), but also instruction in the use of these can be demonstrated in lectures using the tablet, which students can later review. Although not as good as the students interacting directly with the technology (as would occur in a teaching laboratory), it is a big improvement on passive demonstration without the opportunity for later interaction offered by the recordings [1]. In this respect, the tablet offers an opportunity to integrate the many aspects of the lecture process, perhaps lessening the potential for cognitive overload [2, 18]. In addition, the way that the tablet allows students to both follow and later review the spontaneous development of a mathematical problem is a significant advantage over such non-interactive programs as PowerPoint [1].

> Research has shown that stepping through examples can improve classroom dynamics, boost students' confidence levels, and promote the understanding of mathematical concepts and function, and advance problem-solving ability [2].

Professionally, recording the lectures clearly offers teaching staff previously unavailable opportunities for critical reflection. Ensuring that mathematically well-qualified teachers see the importance of unpacking their subject matter knowledge is an important goal for professional development [13]. However, the Statistics team at Auckland warn that there could well be concerns amongst staff about their lectures being critically reviewed by other teachers [4].

The recordings allow for different modes of learning, as students can interact with the recordings in different ways [1, 16]. EAL students can review the lectures giving them an opportunity to pick up contextual nuances that they may have missed in the complexity of the lecture, while allowing students to replay the conceptual areas of the lecture they found difficult [11, 12, 14, 15] benefits all, but particularly low-achieving students. There is at least one serious risk with the recording of lectures for all students, which is highly likely to impact more on students from these two groups, and those with poor study schemas. While the recordings allow students who miss the occasional lecture to catch-up, there is the suggestion that some students may choose not to attend lectures and rely on the recordings. Cretchley [5] found significant performance benefits for students attending lectures, and D'Arcy-Warmington [16] warns that students risk missing visual cues by not attending lectures.

> Research has shown that gestures may be the window to an individual's thinking…Basic gestures and movements can make an impact on information moving to the working memory and consequently being memorable…Simple gestures and body language can convey up to 80% of information and once recognised by the educator can be used to improve teaching at all levels

This problem may be further compounded if lecturers use other modes of lecture display, such as a document camera and the overhead projector which will not be visible on the screen recording.

The preceding discussion has highlighted the pedagogical and technological issues surrounding the use of tablet technology and screen recording of lectures. Next, we describe the experiences of staff and students in using this technology in the project at Auckland University, and consider these findings in light of the issues identified in this discussion.

4. Teachers' Experiences (extracted from departmental reports)

As might be expected, lecturer's experiences varied somewhat with individual teaching styles, and staff in the earlier trials experienced more teething problems than those who adopted its use later. All staff expressed some frustrations with the use of the tablet, although many of these occurred in the early stages of the project, and may be considered as singular problems associated with the experimental nature of the trial, e.g. difficulties with sharing a tablet, not having available software, inadequate IT support when performing unfamiliar tasks such as saving and uploading files, and problems inter-facing with the lecture-room systems. The great majority of the remaining issues were of a practical physical or technological nature associated with the equipment, e.g. difficulties with using the pen and reading the buttons, the small size of the screen which for some staff made writing and viewing material very tricky, and a common concern that being connected to the tablet via the microphone restricted and inhibited their teaching. One lecturer observed that he had a distinctive dramatic style involving much arm-waving and walking around to emphasise points, and being tied to the tablet severely limited this. While it must surely be possible, as noted in one of the lecturer's reports, to overcome the physical element of this problem using a wireless microphone (this is already described in [3]), there is still the problem as noted in the earlier discussion of educational issues using the tablet that students reviewing the lectures miss the imagery and visual cues associated with the teacher's actions. However it is quite possible that many of these actions are more for dramatic effect or entertainment value than for any special learning benefit. Also, as with many of the other issues, this did get easier as the staff got more familiar with the technology. One lecturer described how as she became more familiar with using the tablet, she adopted the pen as her standard 'gesticulator' while another noted in his report that:

> It took me a little while to actually get used to giving a lecture on the tablet - I am used to walking around and pointing a bit more. I started off doing that, and then realised that this was not being captured, so had to discipline myself to pointing to everything using the actual pointer on the laptop.

Time factors received considerable mention. All staff cited the need for extra preparation time as material ideally should be ready on a pdf file (or Word or PowerPoint) as opposed to being hand-written. One lecturer noted that 'extra preparation is required as all prepared notes need to be in 24 font size with large enough spaces to write in during the lecture'. However, this must be measured against preparation time for previous methods, as it is surely no longer common practice for lecturers just to write notes from memory on a blackboard or OHP. As the statistics team noted in their report [4],

> The investment in time and resources to produce screen [recordings] of each lecture is much less than that to generate narrated PowerPoint slides, with the advantage that updates and amendments to each course can be incorporated with minimum additional effort.

Extra time was also required after the lecture converting the files for student use which initially took approximately 40 minutes. Lectures need to be saved in 3 different forms - the dynamic forms in swf for Mac and Linux users and execute for PC users and the pdf version of the final version of the slides for all students to use as lecture notes. The properties of the files needed to be checked before they were loaded onto Cecil. This process is much simpler now that a link has been set up between Aitken and Cecil.

Further problems were encountered with using other forms of media in lectures. In addition to the pedagogical problem that using other equipment (e.g. document cameras, OHPs) means that aspects of the lecture are not available on the screen recordings, there were also physical and technical difficulties. For example, it was not possible on Auckland's e-lectern system to use the document camera (for example for a graphics calculator or to show manipulatives) and guest computer (the tablet) simultaneously, and switching back and forth caused technical difficulties when it was done too quickly. However these problems will be addressed as Auckland University is committed to the eLearning environment and the lecturers will make increasing use of computer-based or scanning facilities. It would be helpful to have two screens available to allow for dual use, with one lecturer observing that being able to 'use both the tablet and the document camera together would enhance the teaching and students would realize they are missing things if they do not attend'. Staff in the Statistics department experienced some other difficulties of a technical nature. They discovered their Dell laptops have poor microphone inputs, and they encountered sound synchronisation problems if recordings were edited extensively.

Bonnington noted in hindsight that one important 'mistake' with the trial was that there was insufficient training offered, although the way in which this project grew as an organic response meant that staff were not initially aware of what training might be needed. Clearly some of the difficulties encountered could have been avoided with proper training, and one of the reports accentuates the need for initial training in several areas as described in the following list:

1. the necessary folders to set up,
2. use of the software: BB flashback, PDF annotator.
3. connections required in the lecture theatre,
4. checking systems are working before starting the lecture,
5. where and how to save the files created,
6. how to create exe., swf., pdf., files with html files for each lecture,
7. how to set up folders in the link and transfer the files each day etc.

Bonnington agrees that the method needs full documentation and training support, but he believes that the total training time required (unguided) to become familiar with the technology is not great, about 2-4 hours being sufficient, with about half the time for the software, and the remainder for the tablet.

Other concerns included apprehension about being recorded, especially since BB Flashback records the lectures very clearly, mistakes and all! One lecturer stated that ‘I never actually heard any of my own lectures (and don't really want to)’. Such a reluctance or lack of time to do so unfortunately negates the potential professional development benefits of critical reflection offered by this technology suggested in 3.2. Lecturers also need to remember to repeat any questions from a student before answering it so the recorded answers make sense, since only the lecturers voice, not the students, is recorded; this is good teaching practice as often students do not hear the initial question. In larger courses with several teaching staff teaching streams at different time, equity issues for students arise if not all staff use the tablet and ideally all staff should do so. However, this did not seem a big issue in one course where this was not possible. While the students who were not taught using the tablet did indicate a sense of being disadvantaged, many of them also noted that they made use of the lectures recorded for other streams, even if they were not delivered by the same teacher! In this respect, one of the lecturer’s reports stated:

> I think it is nearly as good as attending lectures. It was notable that students from … further afield [*extra mural students who did not attend lectures*] felt that they had participated – they felt they knew the lecturers (and were very familiar with me!), had been able to enjoy the jokes and to hear the lecturers answer other students’ questions. At least one reproduced the motions that I felt may have been lost in teaching determinants. On a blackboard, one uses an arm to cover up a row or column, … on the tablet, I used the highlighter pen to cover them, then deleted that highlighting before highlighting another row or column. Last week I watched a student from Hawera covering a row with his finger or pen as a matter of course.

Bonnington noted that lecture attendance was certainly down on previous years, some days less than a third of the class attended. However, there was a core group of regulars, and overall he felt that performance and understanding of all students was up on previous years. Another class had attendances in the order of 60%, which did not seem greatly different to what was usual. The lecturer of this class did however notice that there was a tendency for some students to rely more on the lectures and not read the manual or text, and therefore to miss out on material not covered in the lectures. One of the lecturers worried that the less industrious students used the recorded lectures as an excuse not to turn up. This was particularly noticeable in the 8 am stream, with numbers dropping from about 70 to 25. Similarly, another lecturer observed that while the recorded lectures allowed students choices that could be beneficial, some students chose not to attend lectures with the intention of viewing them later, unfortunately later never arrived! Although there is a realistic concern that some students are especially vulnerable to missing material covered in lectures, we have no knowledge of the extent to which they viewed the recorded lectures, or any way of measuring what effect this may have on their results. As was discussed in section 3.2, this may simply reflect their learning preferences, or at least their individual circumstances.

On the positive side for staff, there is the suggestion that the nature of the technology encourages good presentation practice, as lecturers are forced into considering issues previously possibly ignored, e.g. care needs to be taken with preparation of slides and writing during lectures, and repeating student’s questions certainly ensures that the question has been correctly interpreted. There is an opportunity for classroom lectures to become more like tutorials, with lecturers highlighting particular points, and leaving students to review lectures recorded in previous courses themselves. In addition, although this has yet to be verified by way of any statistical study, there was evidence in particular exam questions of increased understanding of lecture material, especially those questions with a direct connection to what was presented in lectures. It certainly seemed that performance on such questions was above that of previous years. Bonnington noted as well that student evaluation scores of his lecturing increased dramatically in many categories. For example, the mean in the *effective provision of resources* category increased from 7.67 to 9.17 (calculated on a Likert Scale of responses ranging from Strongly Disagree to Strongly Agree), while *stimulation of interest* increased from 7.21 to 8.96. There is also an indication that the recorded lectures encourage independence in students. One course-coordinator observed that ‘a lot fewer students are now knocking on my door to get notes or ask questions. They are traditionally a very demanding group who need a lot of help, but are now using Cecil and downloading the notes themselves’. Although certainly not sufficient justification for introducing the technology, the same lecturer noted the positive spin-off that ‘students see us as being up to date with the latest technology available and hence have a very positive image of our Mathematics Department’.

The following excerpt from a lecturer’s departmental report after their semester one 2007 teaching provides a useful and balanced insight into their experiences with the tablet and recorded lectures:

> I felt that, after some practice, I was lecturing at much the same standard as usual, with the advantage of standing facing the students all the time (with the tablet on the lectern), rather than having to turn from the board or raise my head from the document camera. And I did not have to rub out the blackboard, while still having the same freedom to write given by a blackboard. The pages are a little narrow, but a bigger tablet will not translate into a bigger screen for the students, and I have become accustomed to the narrower space. I think I also lectured somewhat faster, having the security of knowing that most students printed the lecture notes or looked at these later if they missed something important.

In summary, the overall impression from staff was that while using the tablet could be frustrating and there is a definite need to resolve several issues, the generally positive impressions of students (as is described in the next section), along with many individual benefits they experienced in their teaching certainly warrant continuing to develop its use.

5. Student Feedback

All the studies reviewed in the earlier discussion [1, 2, 3, 4], and all of the classes using the tablet technology in the Auckland project reported largely positive feedback from students, although there is as yet no concrete data that demonstrate improved learning. In the first trial at Auckland, no formal evaluation was conducted, but feedback was sought via email. The 30 students in this third-year class (with almost 2/3 EAL) were extremely supportive and pleasantly surprised with how effectively the new technology was implemented. Early in the trial, one student emailed totally unsolicited comments saying ‘I’m just emailing to say how excited I am about this new system and am already finding it useful….’ and ‘…I hope once everyone is used to the system each lecture will be recorded and uploaded from start to finish. It's absolutely fantastic - keep it up’. Positive comments received after the course included ‘This is my first A+ in maths…it was all because of the recordings’, ‘I like the way that we can download the whole lecture and listen to it again, especially that since English is my 2nd language’ and ‘I think all courses in the University should follow the same route and style of teaching’. The feedback did suggest that some students felt uncomfortable asking questions when the lecture is recorded, even though their voices could not be heard on the recordings, and a worry that the recordings may lead to fewer formal lectures and a lack of social interaction, especially for students not attending lectures in favour of the recording.

Statistics [4] gained a surprisingly large 680 responses to a voluntary request for feedback at the end of their first screen recording trial in semester two, 2006. 53% of students said they found the screen recording useful at revision time, while 58% said they used screen recordings to play (and replay) any concepts with which they had difficulty. 66% of respondents who had missed some lectures reported that they used the screen recordings to catch up and to hear virtually exactly what they would have heard had they attended class. While most feedback was positive, they received 12 complaints about slow download speeds, and a few comments about poor sound quality, technical issues which they hope to address. They note with concern that 24% of respondents appeared to use the screen recordings instead of attending class, warning that this is an unintentional outcome when they state that 'we do not wish to discourage students from attending class when there is a very real benefit in them doing so' [4]. They also describe strong support for one innovative use of the technology when they provided students with a screen recording showing a step-by-step run through of a previous semester test by one of the lecturers as an aid to test revision. However, a mathematics lecturer reports that when she tried to emulate this her recording was 'very flat, as I needed the adrenaline that flows when lecturing'.

Both first-year courses using the screen recording in semester one 2007 at Auckland reported highly positive feedback from students both in weekly reflections and in open responses to the formal evaluation. The surveys include the opportunity for students to write open-ended comments about "What was most helpful for your learning?", and "What improvements would you like to see?" As is common with open-ended items, most students choose to write nothing here. Despite this, the recorded lectures received several comments including 'recordings and Cecil are excellent', 'love the idea that lectures are online', and "I have found the recordings of the lectures, the diagrams and the examples useful in helping me learn maths better'. The only comment about improvements mentioned the quality of the recordings, but even then the student noted that this was IT-related. The principal lecturer of the other first year course at Auckland observed that:

> The recorded lectures are certainly extremely popular. In the recent…course evaluation, they got more mentions in answer to "what has helped me to learn in this course" than even the tutorials, which are usually mentioned most. Many students who attended the lectures made a practice of downloading the notes later in the day and reviewing them to check their understanding of the lecture. Where they did not understand, they then listened to the appropriate section of the lecture again. Students are now using the recordings again as part of their exam revision.

The formal university student evaluation of one first-year course also sought students' response to the statement "The availability of recorded lectures in this course helped me learn". One lecture stream had a response of 8.26 (44 responses from 81 enrolments, maximum possible is 10, calculated on a Likert Scale of responses ranging from Strongly Disagree to Strongly Agree), with 82% of students indicating "Agree" (A) or "Strongly Agree" (SA) to this statement; others were similar. There was a surprising result from another stream of the same course, taught at a different campus, where the tablet was not used in lectures. Many of these students clearly accessed the online lectures from the other stream, and agreed that this helped them a lot, even though the recordings were of a different lecturer to their own (30% A+SA, 40% Neutral). Comments included 'the video lectures being available (helped me learn). I could go over them more slowly', and '…the video lectures on-line helped me, when you don't understand the lecture you can look at it again'.

The reference to "video' in these comments is interesting, as it indicates that when the advantage of the association of the playback with previous lecture attendance is removed, the distinction between screen recording and pre-recorded lecture using video-technology is not clear. This is probably similar for distance students: one of the lecturers in the Auckland trial and the MathsOnline Project [3] noted their students found recorded lectures very beneficial. While playback of the screen recordings and downloading the smaller sized files is certainly easier, perhaps the quality concerns may be greater on the screen recordings (e.g. handwriting on the tablet) than a professionally produced studio video-recording?

Similar responses to those of the Auckland students were found in the studies reviewed earlier. In [1], 92 % of the 65 participating students agreed that "its great to have the computer-generated lecture notes on the web-site, while 100 % of the 35 respondents to a survey in [2] (out of 65 in the course) agreed that they found the tablet PC a reasonably useful learning resource. In a survey of students in the MathOnline project, 104 students stated they preferred the use of the tablet over the blackboard, although another 12 did prefer the blackboard, believing the technology was "distracting" [3].The same survey posed the scenario to students that if they had free access to all the software and hardware used in the course over the internet, with full recording of the lectures using screen recordings, would they still choose to regularly attend lectures? 106 students said yes, 8 said no. Most of the "yes" responses cited student interaction and social factors as their single most important reason, but of course whether all these students would in fact attend is not proven! All three studies noted factors such as increased visibility, the ability to integrate other technologies (e.g. graphics packages), and the ability to review worked examples step by step as significant benefits of the tablet and screen recording technology.

Loch [1] does however suggest that it is possible that much of the positive student feedback reflects only students' perception of the delivery of course material, as opposed to any discernible learning benefits. Even if this were so, most teachers would agree that positive student perceptions usually translate into better learning outcomes, and in any case, it is clear from the results shown here that there is strong student support for continued development and use of this technology.

6. Summary and Future Directions

This paper has given an extensive report on the history and current status of the use of tablet technology to deliver and record live lectures in mathematics courses at The University of Auckland. It has provided evidence that strongly endorses the continued use of this technology. It is well supported by both staff and students, with many potential benefits identified. We anticipate that many of the difficulties encountered in the trial, especially those of a practical or technological nature can be addressed with little difficulty in future, and that concerns about increased demands on teaching staff in particular will diminish as experience and the availability of reusable resources increase. Some reservations remain about lecture attendance in particular, but this study provides a firm basis for designing a suitable study that can ask appropriate questions to examine this and other pedagogical issues more critically in future.

There is potential for many additional benefits in distance learning and other remote educational areas. One obvious benefit of the technology suggested by the authors of [3] is in facilitating group work between distance learners, who can interact via the internet using tablets. All those involved in the study see definite advantages in off-campus students being able to see and hear what is going on in lectures. The small size of the files means that users with poor Internet connections can access them from a central source on, for example a memory stick or CD. As noted earlier, a whole semester's worth of lectures can be stored on one CD providing access to students from more disadvantaged areas. One lecturer observes that 'I think it should be possible to grow the extramural numbers using this system, as it no longer restricts the course to those able to understand well mainly from the written word'. Another lecturer sees definite potential for schools involved in a new outreach initiative with Auckland University. The technology is relatively inexpensive, and could facilitate interaction between mathematics teachers in schools and research mathematicians at the university, allowing teachers to postulate questions and work through and record lecturers' responses.

Another area that has already been tested with great success at Auckland is the use of tablets as a virtual whiteboard in video-conferencing. Auckland has been developing the use of AccessGrid technology in videoconferencing between research groups. On several occasions, participants in mathematics conferences have used the tablet as a means of postulating and commenting on each other's ideas. Examples can be sketched on the tablet at one end, and researchers at the other can interact directly with what's been written, at the same time watching and listening to the participants as they carry out their actions. The new facility to save the resulting discussion and review it later is regarded by those who have used it as a very useful development. Screen recordings with audio commentary could also be very effectively used in research interviews instead of just audiotapes, allowing the links between what's spoken and what's written to be more easily established and transcribed. This is especially important in mathematics education research interviews, where examples are often used to illustrate what is being discussed.

A suitable conclusion to this discussion is provided by one lecturer in the Auckland trial, who at the end of her 2007 semester one report frankly observes that 'when everything goes well the tablet is a great idea, but it has limitations and (it) needs to be improved to be really effective'.

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A look about number theory

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Abstract

Young people enjoy the recent influx of new technologies everyday in both communication worlds with Internet, mobile phones and digital libraries and recreational worlds with digital cameras, CD, DVD, MP3, MP4, and iPods. In fact they probably cannot imagine life without them. They are completely ignorant of the mathematical theory that lies behind these technologies, Elementary Theory of Numbers. Educators should seize this prospect to use this enthusiasm and readymade motivation for these technologies to connect mathematical theory with relevant 'close to their hearts' applications. This is a golden opportunity of teaching a branch of mathematics that is often only briefly mentioned if at all on the curriculum. Mathematics and in particular the Elementary Theory of Numbers are the foundations of most information communication technologies. Compression of Data, the Code and the Cryptography are obtained by the application of concepts and methods of Elementary Theory of Numbers. This paper will show the current work using these connections carried out in the Facultad Regional Paraná de la Universidad Tecnológica Nacional, which is part of a Project of Investigation Is this a particular project?

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Supporting students by knowing them: timely and personal interventions through an electronic learning log

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Abstract

Student Progress Files and their incorporated element of PDP (Personal Development Planning) have become an established part of the HE scene. The Mathematical Sciences are not immune from this development. It is widely recognised that for students in higher education, personal development planning is a useful and necessary activity to help them to become more effective and independent learners.

There are two particularly interesting features. One is that any work on a progress file should be embedded in the student's main area of study, and that therefore subject academics need to be involved; the other is that the Progress File should include a means by which students can monitor, build and reflect upon their personal development. This paper describes the implementation of the student portfolio element - delivered and managed by a web-based system developed at SHU. At SHU progress files in Mathematics are assessed at regular intervals through the course, and some early results of this process are reported.

Assessment of the PF seems to be critically important in getting students to engage with the process. Embedding this within a particular module seems to be working well, and students accept that the skills being assessed and the whole reflective process are of importance, both to them as mathematicians (where communication skills are traditionally of secondary importance) and as future employees.

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Investigation of completion rates of engineering students

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The 2002 first year engineering cohort completion rates were investigated to the end of 2006. About 20% of these students actually graduated from engineering during this time and approximately 60% had discontinued the course altogether. A support program (QUTMAC) is run along side the mathematics units in the first year. Students who used this service were twice as likely to complete their course compared to those who did not.

Keywords: Engineering, Completion rates, mathematics support, retention.

1 Introduction

The issue of completion rates and retention of students at tertiary level has been on going for decades. As Tomkinson et al [1] says of the UK 'With wastage rates in science and engineering often in excess of 20%, for many of us the issue of student retention is of future viability. For others the main issue is of the human cost of so many students missing out on an opportunity'. In the United States colleges of engineering are finding that they lose up to 50% of engineering students due mostly to the challenges of the first two years, with women more likely to stay within the course than men [2]. There is a tension between government pressure to allow more students the opportunity of tertiary education and the reduction in proportions of quality students who are really attracted to the university courses. To accommodate these tensions many institutions have started programs of learning support at either the local course level or on some case university wide learning support centres or as in the case of the UK Centres of Excellence across institutions and funded by government, eg Sigma: The UK's Centre for Excellence in Teaching and Learning in Mathematics and Statistics Support, see [3].

All students who are enrolled in an Engineering Degree are required to take a unit of Engineering Mathematics in the first Semester. These units are designed to help the students with the transition from high school mathematics to the mathematics that they will need in their future units of engineering. In Queensland the secondary curriculum allows students to do up to two subjects in mathematics. For engineering students the appropriate subjects are Mathematics B and Mathematics C. About half of the students entering engineering have completed both Maths B and Maths C, the other students have completed Maths B only or enter with some other assumed mathematics background.

Students with Maths B only, do the Engineering Maths 1B (MAB180), while those who have both Maths B and C enter the Engineering Maths 1A (MAB131). The MAB131 is more in depth mathematically than MAB180 as it takes into account the prior experience these students bring having done Maths C in school.

This paper considers those students who entered engineering in Semester 1 2002 and explores what has happened to them in their course since then. Students that are in a regular full time degree program should have completed their engineering degree by the end of 2005. Students in a full time double degree program should have finished at the end of 2006. Some

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students decide in the first semester that engineering was not the most appropriate career choice for them and discontinue the program. Others discontinue at later stages of the program, while some students change to another type of engineering speciality.

It was in 2002 that the Queensland University of Technology's Maths Access Centre (QUTMAC) commenced operation, see [4]. With a modest annual budget and extensive in-kind support from the School of Mathematical Sciences, the Centre has had far-reaching impact on student learning at QUT. Despite its youth and limited budget it is rapidly becoming a leading model of university-wide support in mathematics and statistics learning, helping undergraduates, postgraduates and staff across disciplines, and with established UK linkages.

Engineering courses are highly vulnerable to diversities or weaknesses in mathematics backgrounds because these courses require the widest variety of both specific and generic mathematical skills as quickly as possible. Mathematical thinking is the lifeblood of engineering, feeding its full range of skills, from the most creative to the most technological and theoretical. This causes a raft of difficulties for engineering teaching staff and curriculum designers. Many engineers, both academic and professional, are aware of the many roles of mathematical thinking in engineering, but for many others, mathematical thinking has become so much a part of them that they have forgotten how they acquired it. The increasing diversity of mathematical abilities and backgrounds amongst engineering students, and the mathematical needs of modern engineering within course structures that tend to have the least flexibility where flexibility is most needed, is a formidable combination of challenges for all staff involved in teaching and supporting engineering students. The QUTMAC has a deep understanding of all these challenges for engineering students and staff, and its data and analysis provide much valuable information, see [5] and [6].

Operational objectives of the Centre include:

support for skills and understanding, and in developing student confidence and lifelong learning across all mathematics and statistics service and core units

provision and fostering of an environment of partnership and openness in mathematical learning – within and between all student cohorts and staff

development of diagnostic testing and associated support strategies in any unit in which student difficulties in basic mathematics are causing problems, or have the potential to cause significant difficulties in later units

consultation, collaboration, advice and support for staff on learning and teaching matters involving quantitative skills

data collection and analysis on quantitative aspects of learning and teaching

pursuit of scholarship of teaching in tertiary learning that involves mathematical and statistical thinking across disciplines.

Components of the QUTMAC's program include:

- weekly student-driven, unit-specific support sessions
- a drop-in centre/student work area with extensive specific-purpose paper resources, wireless facilities, and a schedule of duty tutors
- sessions on mathematical problem-tackling, including test/exam preparation
- roles in mentored tutor training

- development and implementation of diagnostic tests and associated student support in units in Science, Nursing, Engineering and Information Technology courses
- data analysis of student performance and progression with respect to a range of possible predictors, and associated advice and strategies for staff and management
- statistical thinking symposia for postgraduate students across all disciplines
- development and implementation of data collection and analysis strategies for monitoring and evaluation of QUTMAC programs

The features that are relevant to this paper are those aspects that relate to engineering programs, that is the weekly support session and the exam preparation workshops.

Weekly support sessions: These are unit-specific, optional but scheduled sessions driven by student questions and requests, that focus on building students' confidence, self-help and study skills, and on tackling their holes and weaknesses in the underpinning mathematical concepts and skills needed for current and future learning. The QUTMAC does not provide units for students without the official assumed knowledge for their program of study because such prerequisites are available in units in which students can enroll. The purpose of the QUTMAC's support sessions is to help students who officially have the prerequisite background for their program. There are many reasons why students with the official assumed background find that their skills and operational knowledge are insufficient; these range from inadequate identification of skills and knowledge that are assumed to the many problems and challenges described in the introductory overview.

Although they are optional, the support sessions are scheduled to ensure that students' timetables allow their inclusion in appropriate programs of study. The nature of the sessions varies from unit to unit, and the sessions adapt in response to students' needs. The principles, however, are always the same - to provide a supportive, friendly, open environment in which no question or difficulty is too small, and which provides the utmost encouragement for students to own their learning and to turn weaknesses into opportunities to learn and to grow.

Up to three weekly support sessions were provided in each engineering mathematics unit MAB180 and MAB131.

Exam preparation workshops: These are specific to first year Engineering Mathematics units and a small number of other units. They are held at key stages during the semester and aim to help students develop study and problem-solving skills. For engineering students, similarly to the support sessions, these have been held since 2002 and have become increasingly popular, often requiring extra repeated sessions to meet demand. These typically would last for one day with 3 two hour sessions. Again these are student driven with the tutor responding to the needs of the students.

2 The Data

The information on these students was provided by QUT Student records System 'Calista' by the Senior Client Services Officer, Student Systems – SBS. The records indicated all students that had discontinued the course including those who may have changed courses within the engineering faculty. A search was done on the individual records of all students who were recorded as discontinued to determine whether they had changed to other courses within engineering, or changed to other courses within the university, or had left the university. Table 1 provides a summary of these findings including the number of discontinued students who did the engineering mathematics units MAB180 or MAB131 and also indicates students that discontinued during the mathematics units.

Table 1: Destination of discontinued students							
	MAB180			**MAB131**			**All Eng. Students**
	Completed MAB180	Discont-inued MAB180	Total Discont-inued	Completed MAB131	Discont-inued MAB131	Total Discont-inued	Discont-inued
Left QUT	77	48	125	63	22	85	210
Diff Eng course	12	2	14	18	2	20	34
Diff Course QUT	23	10	33	23	2	25	58
total	112	60	172	104	26	130	302

Some students have left QUT completely (Left QUT) (42%), others have changed to a different strand within engineering (Diff Eng course) (7%), while others have remained at QUT but changed to a non-engineering degree (Diff Course QUT) (11%). The discontinued MAB180, MAB131 columns indicate students who officially left the course <u>and</u> the Engineering mathematics unit before the end of the semester 1 2002.

Taking into account the above information, Table 2 provides data on the number of students entering the engineering degree program and the numbers and percentages of those who completed or discontinued by the end of 2006. About 15% of students are still enrolled and this includes those who are part time, or have failed some units and need to repeat them. At the start of the course only about 10% enroll as part time and during the course an unknown number change to part time and some discontinue and return later.

It is of concern to note that nearly 50% of students discontinue their engineering program, this is to be compared with the 20% retention rate in the UK [1]. This is higher for those who did MAB180 (58%) while for those who did MAB131 it is close to 42%.

Another interesting result from this data is the low number of students that complete their degree within the 5 years. Only 22% of MAB180 students have completed their 4 year degree in 5 years, while for those who enroll in MAB131 it is higher at about 36%.

It is of interest to know if students' grades in Engineering Mathematics have some effect on their completion or discontinuation. Table 3 provides information on Engineering grades and completion and discontinuation by the end of 2006. The scale of grade is 1 to 7 with 7 being the highest and greater than 4 a pass, 3 is a conceded pass. A grade of W or K indicates either a withdrawal from or incompletion of the unit.

Table 2. Enrolments and Completion or discontinuation of course					
Unit	**Enrolled 2002**	**Completed before the end of 2006**	**% completed their course by the end of 2006**	**Discontinued by end of 2006**	**% discontinued**
MAB180 (single degree)	254	58	22.83%	148	58.27%
MAB180 (double degree)	22	5	22.73%	8	36.36%
MAB180 (all)	276	63	22.83%	156	56.52%
MAB131 (single degree)	215	80	37.21%	92	42.79%
MAB131 (double degree)	46	14	30.43%	18	39.13%
MAB131 (all)	261	94	36.02%	110	42.15%
All Engineering	537	157	29.24%	266	49.53%

Table 3: By grade in mathematics unit completed/discontinued (Includes double degree)						
Grade	MAB180 2002			MAB131 2002		
	enrolled	% Comp	% Discont	enrolled	% Comp	% Discont
W,K	68	4.41%	92.65%	43	20.93%	55.81%
1	14	0.00%	100.00%	11	0.00%	81.82%
2	21	0.00%	76.19%	30	3.33%	93.33%
3	13	7.69%	84.62%	25	32.00%	48.00%
4	53	15.09%	58.49%	43	27.91%	53.49%
5	39	41.03%	38.46%	53	41.51%	37.74%
6	39	46.15%	33.33%	25	76.00%	20.00%
7	29	58.62%	31.03%	31	74.19%	16.13%
all	276	22.83%	56.52%	261	36.02%	42.15%%

Of the students who received a grade of 7, the percentage of MAB180 who discontinued is nearly twice that of MAB131. If a student does not pass MAB180 it is unlikely that they will complete the course within 5 years, and most of them will discontinue the course. It is only slightly better for those students who fail MAB131.

2.1 Data from the QUTMAC

The purpose of this section of the paper is to see if those students who used the QUTMAC programs were more likely or not to complete or discontinue their course as compared to the total cohort described above.

MAB180

Seventy engineering students in MAB180 used at least one component of the QUTMAC programs in semester 1 2002. Of these 70, 32 used the maths support sessions and 63 used the exam workshops, and 25 used both. That is, 7 students who used the support session never used the Exam workshops and 38 students who came to the exam workshops never used the support sessions. At the end of 2006, 30 of these 70 students had completed their engineering course, 6 were still enrolled, 18 have left QUT, 6 transferred to other QUT courses, and 3 transferred to other Engineering courses.

MAB131

One hundred and one engineering students used at least one component of the QUTMAC program. Of these 101 students 88 used the maths support sessions, 95 used the exam workshops and 82 used both. That is, 6 students who used the support session never used the Exam workshops and 13 students who came to the exam workshops never used the support sessions. Nine of the 101 students are still enrolled at the end of 2006, and 55 had completed their engineering course. 14 have left QUT, 5 transferred to other QUT courses, 11 transferred to other Engineering courses.

Table 4 summarises the above information which can be compared with the data in Table 2.

Table 4: Summary of Engineering students that used the MAC programs.

Unit	No of MAC users	Completed before the end of 2006	% completed	Discontinued by end of 2006	% discontinued
MAB180	70	30	42.86%	24	34.29%
MAB131	101	55	54.46%	19	18.81%
All Engineering	171	85	49.71%	43	25.15%

2.2 Discussion

In comparing Tables 2 and 4 we see that students that use the QUTMAC either in the support sessions or the exam workshops are nearly twice as likely to complete the course as the whole cohort and half as likely to discontinue engineering. Because the QUTMAC programs are voluntary, the students self-selected. That is they knew that they had gaps in their mathematical knowledge and skills and made an effort to use the QUTMAC to fill those gaps. Such students are more likely to complete the course than those who had gaps and could not see them and/or did not bother to use the resources that were available to them.

3. Conclusion

Students who commence engineering come with a range of prior learning experiences that impact on the type of mathematics program that they can most appropriately engage in at tertiary level. Those who come with only the core mathematics (Maths B) or equivalent from high school are directed to MAB180. These students have lower completion rates and higher discontinuation rates than those students who come with the advanced mathematics (Math B&C) and enter into MAB131.

Students who choose to do the optional programs offered by the QUTMAC have improved completion rates and less discontinuation than the cohort as a whole regardless of mathematics unit studied.

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‘Life is just too complex!’ Let’s entwine mathematics learning with complex theory!

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‘Life is just too complex!’ is uttered many times and funnily enough that is the way life ought to be. The word ‘complex’ originates from Classical Latin word ‘*complexe’* that translates ‘bend, curve, turn, fold, twine, twist, interweave, and weave’. Life proceeds via a pathway that weaves through exchanges with people and environment defining the personality. Some students studying the compulsory mathematics unit feel it is a case of survival, evolving, and adapting to complete the course. This sentiment holds all the attributes of Complexity Theory so why not entwine and interweave it with mathematical learning. In recent years, group work has taken a more prominent role in an attempt to resurrect mathematical interest and creativity. Tutorials questions are designed to maximise understanding through interactions with content and fellow students rather than just complete a set amount of selected exercises. Students expressed a greater interest in mathematics’ group work after the course which differed from their prior expectations and experiences. Students’ remarks showed that an interest in mathematics had been stirred and all achieved by using ‘complex’ styled learning techniques. Interest and enthusiasm will help ensure that so-called ‘penny has dropped’ moments materialise more often leading to a new appreciation of mathematics.

Keywords: Group work, Complexity Theory, First year service mathematics course

1 Introduction

> ‘In the company of friends, writers can discuss their books, economists the state of the economy, lawyers their latest cases, and businessmen their latest acquisitions, but mathematicians cannot discuss their mathematics at all. And the more profound their work, the less understandable it is.’ Adler, Alfred (1870 – 1937)

The essence of this quote is still true in the 21st century in the minds of most first year students taking the compulsory mathematics unit. ‘Don’t bother with the details just give me the answer’ is often heard and accepted as the norm in mathematics service units. There is no enthusiasm for mathematics from students; the service unit is seen as a chore that must be done. Exuberance and passion are sentiments more commonly found on a football field than in a mathematics class. This qualitative paper will show, by exploring students’ recollections and lecturer’s observations, how Complexity Theory may be utilised in the transformation of mathematical ambivalence to a more positive attitude towards mathematics. Group work and an innovative style of tutorial exercises may be the pathway where students can regain the thrill of mathematics. The use of complexity theory in association with mathematics teaching is emerging albeit slowly. The world today is ever evolving and one has to appreciate and live fully within the complex systems [1].

2 Complexity Theory

As mathematics educators are aware their courses are never the most popular from secondary to tertiary education. What a turnaround if being complex could aid students to enjoy and understand mathematics better. A definition of complexity theory is: Critically interacting components self-organise to form potentially evolving structures exhibiting a hierarchy of emergent system properties ‘[2-3]’. Complex systems have a simple set of

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starting conditions that create more elaborate and unpredictable outcomes '[4]'. The elements of such systems can influence and be influenced by others in the system and so produce new states. Complex systems are constantly reacting to influences, adapting via feedback, and reorganising to become more connected. The occurrence of this connection is certain but the outcome of the interaction is unpredictable. Complexity theory can be incorporated in both tutorial activities and organisation so that mathematical learning may behave like a complex system. Tutorial exercises have a basic blueprint of many simple clues that combine to form a solution rather than just one direct question. Each clue can identify mathematical strengths and weaknesses sending students in numerous directions and hopefully to a final solution with understanding. The small groups provide an environment for interaction, feedback and mathematical inspiration to occur.

3 Characteristics of Complex systems

Complexity is never far away from reality; thoughts are often spontaneous and unpredictable connecting to other thoughts producing more reflections. Major characteristics of complex systems include connectedness and unspecified number of parts, self-organisation, order without control, communication and relationships, non-linear networks and unpredictability, continuous feedback with continual adaptation and emergence of new states. These characteristics of complexity theory are integrated into tutorials by the group work environment and evoking curiosity through the presentation of the questions [5]. Tutorials have a game (or challenge) ingredient accompanied with brief interludes of revision when needed. The challenges range from matching cards of similar algebraic forms of a derivative to writing hints on the board about a topic. This atmosphere hopes to enhance mathematical comprehension and appreciation. My tutorials may be described as a series of 'many little unpredictable mathematical events' with the following elements of complex system:

a) Connectedness and unspecified number of parts

Students, though independent learners, are all attending class together to gain more knowledge of mathematics. The universal aspiration of obtaining a pass in this compulsory mathematics unit fulfils the element of connectedness. Student numbers vary from group to group and week to week so interactions in each tutorial differ in both amount and content. If learning is viewed as an interaction of an interaction producing a change of the knowledge base then in mathematics class complexity theory begins to make sense. '[6]'. The tutorial questions are structured in many parts so that students may really feel the progression of understanding at each stage. Cards containing lines from a solution are handed to the students to rearrange into a correct order. This works well with a variety of topics from applied problems to simple procedural applications.

b) Self-organisation and order without control

Students convene groups and arrangements within groups transform without outside control choosing the most suitable environment for learning. Tutorial exercises are designed so they can be adapted to any order to develop skills by group or individual. Information can be discussed in the group then individually put to paper and later, hopefully, into a lasting memory in the mind. The organisation of material proves difficult for a proportion of students. Tutorial problems where students select cards with equivalent algebraic expressions associated with derivatives, integrals or simple fractions for example

c) Communication and relationships

Students need support structures in mathematics to grow in mathematical knowledge. Peer discussion promotes proliferation of ideas, strategies and topic comprehension. Both individual students and the group entity collaborate to form bonds with each other as people, with the tutor and most importantly with mathematics. All of this may not ever eventuate in other circumstance where similar exercises are set directly from the textbook each week '[7]'. Tutorial questions attempt to show and communicate the relationship between concepts by creating pathways within the problem. Tutorials may take a reverse approach where entire solutions are given and students have to offer reasons for the method. This addresses the issue of understanding rather than mechanics of problems. This is an important learning tool.

d) Non-linear networks and unpredictability

Each group interacts with each other producing a forum of ideas with outcomes that release more debate at all levels '[8]'. Independent of subject matter, students' thoughts are often most intriguing to experts and peers alike adding new perspectives and insights to how information has been translated from instruction to students' brains. There is no schemata applied, all activities happen by impulse often through unconnected cognitive pathways mirroring a complex system. The understanding of more intricate problems is not always the result of scaffolding of simpler problems, sometimes leaps and bounds in comprehension happen spontaneously. The so-called 'penny has dropped' moment can materialise at any time with any type of interaction or words.

e) Continuous feedback and continual adaptation and emergence of new states

Innovative discussion provides sources of vital feedback, often by students using examples with language that is attractive and comprehensible to fellow students. Listening, inputting and assimilating contributions by group members incessantly modify the dynamics of groups and individual members '[9]'. In some way, no matter how minuscule, students emerge with new knowledge and techniques with every discourse. The tutorial problems are designed so students may practice each stage in context as much or as little as desired. The constant reviewing of material builds knowledge in all areas not just the topic being tested. Students adapt and modify information stored in memory with each new interaction. A tutorial may consist of several linear programming problems divided into stages such as; finding inequalities, graphing inequalities, shading feasible area and the optimal solution. Students can stay at any stage until they feel confident to proceed.

4 Methodology

In 2007, current students experienced group work only in tutorials whilst group work played a major role in lectures and tutorials for the whole year for former students. 107 emails (see Appendix A) were sent to former students who were taught in years, 2005 and 2006, seeking responses about group work, whilst 30 current students answered during class. The participants were students, mainly international, studying foundation mathematics as a requirement for entrance to first year university courses. The course provides a core set of topics mainly algebraic and statistical content suitable for entry to nursing, architecture, commerce and finance. Additional topics in calculus, vectors and trigonometry are taught to students entering sciences such as pharmacy, occupational therapy, physiotherapy and engineering. The content is comparable to most first year service mathematics unit. Lectures and tutorials integrate group work with problems of an innovative nature and format to promote active participation and discussion and thus create a complex scenario [10]. The following is an illustration of a typical tutorial exercise, this one involving quadratics. Six different graphs of quadratic functions along with relevant information on small strips of different coloured paper are given to the students. Examples of the graphs are shown in Figures 1 – 3 with figure 4 illustrating the relevant information

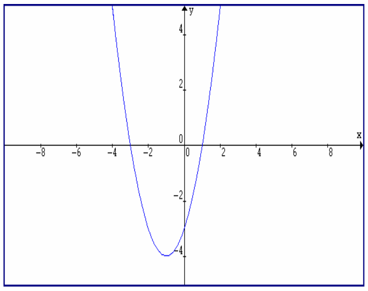

Figure1

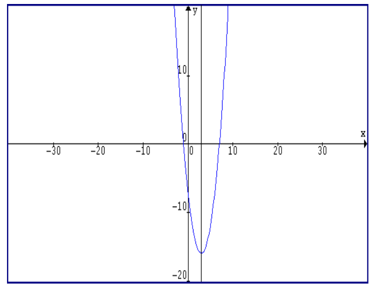

Figure 2

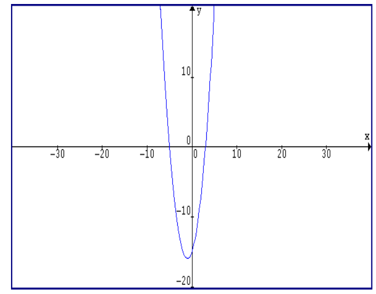

Figure 3

Sheet One	Sheet Two	Sheet Three
$y = x^2 – 6x– 7$	$y = x^2 + 2x – 15$	$f(x) = x^2 + 2x – 3$
Graph opens up because a is positive	*The graph opens up because a is positive*	*The graph opens up because a is positive*
x-intercepts are x = 7, and x = - 1	*The vertex point is (-1, -16)*	*Vertex point is at (-1, -4)*
The *y-intercept (0, -7)*	*Axis of symmetry x = - 1*	*Axis of symmetry is the line x = - 1*
The vertex point is (3, -16)	*Cuts the line x=0 at 15*	*X-intercepts are at x = -3, x = 1*
The axis of symmetry is x = 3	*Cuts the line y =0 at -5 and 3*	*Y-intercept (0, -3)*

Figure 4

The aim of the tutorial is to link correct algebraic and graphical components of quadratic functions. The task has a number of attributes of a complex system: each set of clues is connected to one graph, clues intermingle with each other, clues can be solved in any order, and each clue provides feedback with many unpredictable pathways to the final solution. Complex environments can be created via interactions with tutorial problems and fellow students. This paper will relate the tenets of complexity theory to content, delivery and style of mathematics tutorial using observations during tutorials and students' quotes.

5 Results

Problems encountered using e-mail as a medium are e-mails with multiple recipients being treated as junk mail or valid e-mail address no longer accessed by the student. These cases could not be quantified. Ten e-mails had invalid e-mail addresses. 65 completed questionnaires were received giving an overall response rate of 51.2%. A consideration of response bias should be made since the responses were made directly to me with a self-selection bias due to method of polling. Table 1 shows the progression of students' attitudes to group work before, during and after the course. These responses to the questionnaire will be used to discuss the educational benefit to students' understanding, interest and enjoyment of mathematics and explain the attitude shift to group work. This is expanded in the discussion section.

	Attitudes to Group Work		
Time	Positive	Neutral	Negative
Before Course	28	16	21
During course	40	10	15
After course	54	7	4

Table 1 Survey responses from all students

6 Discussion

Complexity theory is relevant to all aspects of education from the practice of teaching, problems through to curriculum development '[11]' Students are initially wary of group work with the usual concerns such as not all students will put the same amount of effort, communication problems, and working with unfamiliar people [12]. The majority of students had never experienced group work in mathematics tutorials. Such activities often finish at primary or early high school level. Mathematics is perceived by students to be learnt in relative isolation and hence file in to sit in rows at the first tutorial. In students' eyes, group work in mathematics at tertiary level, is somehow not an acceptable learning platform. This was reflected with comments concerning feelings before starting group work in the course:

> No, I did not have any experience in group work in Mathematics before because in my country, we don't have group work in mathematics subject, as we do it all on our own.
>
> Basically I hate group work
>
> Before the maths foundations course, I only had just one or two group works to do which I found out to be a bit boring I should say.
>
> My only concern was that people would be graded unfairly as some students may not put in the same amount of effort as others
>
> I felt that it was not possible to do group work when it came to Mathematics due to the nature of the subject. However, I was not averse to the idea of doing group work itself because I had done some in other subjects before and find it to be enriching.
>
> You will never learn from group work.
>
> When I was in Mathematics Foundations course, I'm very concern on the group work because I'm afraid that it would drop me off from the Foundation and I don't think it's even useful.

It is interesting how quickly they adapt and after only one or two tutorial sessions, groups are eagerly waiting at rearranged tables for mathematics problems. Preconceived ideas and fears of the mechanics of group work can be diminished, once students experience carefully planned mathematical tutorial sessions. The tutorial concerning quadratics will reflect, as a specific example for all tutorials, of the ambiance created by using complex ideas and group work. The many facets of group work are illustrated by my observations of the "quadratics" tutorial. Every interaction, whether it is student to student or students to topics, increases the knowledge base and has future use in a many varied situations '[13 -15]'. My role facilitates the flow of ideas and students' involvement. Here are descriptions of the three main strategies implemented, with no outside influence, by students during this tutorial:

Group One selected a combination of individual and group effort; each student selected one equation to investigate algebraically by factorising, completing the square, and finding other useful features such as intersections. Collaboration on how and when to use different algebraic techniques occurred when required. They pooled all their ideas and workings to start the process of matching their calculated information with the original facts. Workings had to be interpreted algebraically and graphically in the context of quadratics. All students were involved with the formation of clue sets. Finally the reunion of graphs and clue sets happened though not always first time!

Group Two used the whole group's knowledge on one graph checking for co-ordinates, intersections and other features. The next step was to fit pieces using facts that had been found and eliminating with further calculations unsuitable ones for the graph. Each graph is completed before moving to the next one. The debate was fierce with explanations for inclusion and exclusion of clues until finally all clues are in place.

Group Three is reminiscent of playing Rubik's Cube, completion looks far away then at the next twist it is solved. The group chose one graph matching and attaching all obvious clues before proceeding to the next one. After viewing all graphs, the process began again with unattached clues and individual graphs. Attached and unattached clues traded places in a jigsaw-like effect until all clue sets are complete. A mess of clues one second, then an ordered clue and graph set the next.

In the learning environment of the 'quadratics' tutorial, as with other tutorials during semester, aspects of complexity theory are evident through both material content and contact with fellow classmates independent of adopted approaches. Within each group, all students are actively taking part with individuals organised to complete tasks. The overall final solution is the same albeit that material is organised and processed in different fashions and pathways. This shows students the creative side of mathematics often hidden in textbook exercises. The conversation is ubiquitous satisfying communication and self-organisation aspects. Different ideas and concepts float in and out constantly being revised when solutions, calculations, and observations reveal more information.

There is a plethora of suggestions provoking more thoughts, how and why these connect is not always clear.

Observations of all groups clearly showed non-linear networks, and continuous feedback through many movements of knowledge, clues and students. A number of students find learning in mathematics as many 'penny has dropped' moments where these events are as unpredictable as the reasons why they happen. Tutorials are always accompanied with plenty of noise, activity and movement that is uncharacteristic of a 'normal' mathematics tutorial. This divergence was echoed in the sentiments expressed by students:

> Different to any other tutor that I met before, totally different. More interesting
>
> It is just different. Well your class was fun you tried hard to teach students with many different methods, tried many different ways to teach

Groups can start off in chaos, literally, with coloured papers, files and pens all over the table. Slowly during the tutorial recognisable progress emerges as order begins to be seen by groups and tutor. The link between graphs and different forms of quadratic functions is promoted by a game-like investigation '[16-17]'.

Positive comments about the tutorial environment and content experienced during the semester relating the enjoyment and learning factors:

> Your style was more interactive with the students and it brought great interest towards the subject from me as a result. It was motivating and the time spent in class was not in any way boring. Yes it helped in the practical side of life.
>
> I believed that to excel in mathematics does not only limit to good marks and excellent mesmerizing of formulas. It is the way that the student is comfortable in learning, confident, and has a good sense in how the concept works. Therefore, I think it would be benefiting for the students when a different type of approach is used, as long as it helps to embrace the student's ability in learning.
>
> Your style of teaching is good, not that boring, with some jokes....besides, activities in class also interesting nice and comfortable teaching style
>
> I feel you make us all feel involved and inspire us to work. You make the classroom environment very lively which I like about your classes
>
> Yours is much more lively and fun. The group work which provided: really attract people's attention and interest of mathematics. Really interesting. Like fun to enjoy. Excellent experience

A minority of students, even though homework was set, still longed for endless exercises on the same topic to be completed in tutorial or at home. Remarks such as:

> When I was in my country my maths teacher always gave us a lot of homework to do in tutorial class. Doing homework every day
>
> Only teacher talk in class and students study by themselves
>
> It is too relaxed I beg for homework many students are unmotivated to study or work on mathematics without a tug or push from teacher
>
> Give some practice to solve the question together in class and give more homework or little test, so students have more chance to know more question and ready to face final exam.

It can be difficult for students to adapt when the mindset is that learning can only take place by repetition so to alleviate these fears, time is allocated to more traditional exercises in some tutorials. After many years of traditional 'chalk and talk' teaching, students worry with new teaching styles that they will miss 'something' without even knowing exactly what that 'something' is. Many students enjoy and appreciate the carefully planned tutorial sessions and lectures describing the experience with words such as 'joy' and 'fun'

> I enjoyed your style of teaching always a joy coming to your class. The lectures are presented in an easy to comprehend format and I took a lot back at the end of each class.
>
> I passed her module only because of her guided teaching approach which I learnt so much from.

The tutorials are intentionally planned to be unpredictable in style, content and delivery, so interactions with lecturer, mathematical content and peers may produce new mathematical knowledge and understanding.

> Frankly speaking, you make maths more interesting. I was never interested in maths until you taught me last year. You made it fun to study maths. I guess most students last year enjoyed your lectures and tutorials. You can attract students' attention with that, well,

that's happened to me, and I think that was a great job since my grades in maths improved very very significantly.

Your style of teaching is never boring, always expecting something fresh or interesting in every upcoming class.

From my opinion, I guess the best studying style that I had before was in the maths class. I really can see the improvements from that, from a student that hate maths but now it turned me into a student that love maths. I seriously learn a lot from the studying environment that I had in the foundation classes.

Your style was more interactive with the students and it brought great interest towards the subject from me as a result. It was motivating and the time spent in class was not in any way boring.

7 Conclusion

Learning in mathematics has a complex and non-linear portion that makes teaching in mathematics so random and intriguing. Interaction with mathematics is important as it impinges on everyday life and there are times when communication involving mathematics, no matter how trivial, is required. Carefully planned questions and activities entwined with group work enhance a more confident curiosity of mathematics as shown by students' comments and shift in attitudes towards group work. The combination of students, problems and delivery fashions a stage where the understanding of a concept may be as simple as; a different word in a sentence, a simple gesture, a quick sketch, little word from a fellow group member or a different voice. All are possible when group work is the norm rather than an added extra and when the attributes of complex systems are recognised and utilised. The tutorials are welcomed by students for the unpredictability of the format, for the interactions that have consequences far beyond the boundaries of the classroom and for the non-linear progression by leaps and bounds of knowledge. Who would ever have predicted that a male student with a high distinction would find that group work helped with grocery shopping!

Appendix A

Gender Year Foundations Mathematics course was taken Course you are now undertaking

Before taking the Mathematics Foundations course, had you experienced group work in mathematics tutorials? If so, was it similar to that of Mathematics Foundations course?

Before taking the Mathematics Foundations course, what were your feelings about group work?

When you were taking the Mathematics Foundations course, what concerns did you have about group work? At the time did you think the group work would be useful?

After taking the Mathematics Foundations course, do you now feel that the group work helped your understanding of Mathematics? Has the group work helped in other areas of your student life?

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Towards an agenda for research on mathematics teaching at the university

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Abstract

Studies of mathematics teaching and classroom learning, which are at the forefront of research on elementary and high school mathematics, are less prevalent at the tertiary level. Our purpose is to consider possible reasons for this state of affairs and to discuss relevant issues towards a viable research agenda for tertiary mathematics education.
For example, we propose to discuss:

- *Teacher knowledge*. Mathematics teachers at the university level are usually either graduate students or professional mathematicians. Can it be assumed that studies on teacher knowledge of mathematics, which are so widespread at the elementary and high school levels, are not relevant at this level? To what extent and why should pedagogical content knowledge at this level of teaching be addressed by research? How do tertiary teachers make their curriculum/textbook choices?
- *Teaching practices*. Many universities have established teaching improvement units. How can this be related to professional development aimed at improving practice, which is a main concern in the agenda of primary and high school teaching?
- *Teaching and research*. Many tertiary-level teachers do research in mathematics. What is the interaction, if at all, between the research component of their work and their teaching?
- *Classroom studies*. Classroom management, classroom conversations and interactions (such as the production of mathematical arguments and counterarguments) are being intensively studied at the elementary and high school level as a main source of meaningful learning. Why would this be (or not be) a research topic worth pursuing at the university level?
- *Research paradigms*. The two predominant theoretical and methodological paradigms in mathematics educational research are the cognitivist and the socio-cultural. It seems that cognitivism is the leading paradigm in research on learning advanced mathematics. To what extent is this a general trend, and why?
- *Assessment*. Should assessment in university-level mathematics courses be inherently different from its counterpart in elementary and high schools? What would be the role of "alternative assessments"?
- *Multiculturalism*. Mathematics education in elementary and high schools is concerned with issues of equity, inclusiveness and ethnomathematics. Why and how should this concern be addressed (or, alternatively, ignored) at the university level?

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Entry level preparedness for calculus – a reason for concern?

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In 2005 first year engineering students at the University of Pretoria completed a mathematics diagnostic survey at the beginning of the year. The aim with the survey was to determine the students' entry level preparedness for calculus, as there is a growing concern regarding engineering students' mathematical ability. The mathematical content of the survey focuses on aspects in pre-calculus that can be regarded as 'must knows' for calculus. In this paper an overview of the content of the survey is given, the results of the 2005 implementation are analysed and compared with students' final school marks in mathematics. The data of two groups of students are considered. Analysis of results indicates that students in both groups lack the pre-calculus skills and ability that lecturers, presenting the first course in calculus, readily assume students should have. Of the 782 students, 90% performed below the anticipated 80% correct answers.

Keywords: Calculus preparedness; Pre-calculus skills; Mathematics diagnostic survey
2000 Mathematics Subject Classifications: 97D30; 97D40; 97D70; 97C90

1 Background

There is a body of research on first year tertiary students' mathematical skills and the implications thereof for the learning and teaching of calculus. A sample of reported research includes studies by Bottomley, Hollebrands and Parry [1]; Carpenter and Hanna [2]; Ferrini-Mundy and Gaudard [3]; Frith, Frith and Conradie [4]; Hooper [5] and Jourdan and Cretchley [6]. Inevitably, these studies are locally focused on specific students who come from a particular school system. It is envisaged that a forum like Delta ’07 can provide the opportunity to identify commonalities in students' perceived incompetence in mathematics at first year level and, more importantly, to discuss solutions regarding the learning and teaching of mathematics that may be applicable to a wider student body than in one's own institution.

In South Africa, extensive changes have taken place in education since the mid 1990s. These include pedagogical and curriculum changes at school level and increasing differences between secondary and tertiary mathematics [7]. The question arises whether teachers at (all) school levels are equipped to deal with the demands of the new mathematics syllabus. In addition, the total effect of the changes regarding the mathematics curriculum will only be noticed when a generation of students has been educated in this manner. At time of writing, students who were educated according to the new mathematics curriculum have not yet entered tertiary study.

Competence in mathematics is regarded as a key component for engineering study and eventually for the practice of engineering. All engineering disciplines at the University of Pretoria have the same compulsory mathematics modules in the first two years of study. The School of Engineering at the University of Pretoria offers a four year standard engineering

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degree program which is regulated by the Engineering Council of South Africa, as well as an extended five year study programme. The purpose of the Five Year Study Programme (5YSP) is to create opportunities for students who have the potential to do engineering but who do not meet the entrance requirements of the Four Year Study Programme (4YSP). Most of the students on the 5YSP do not meet the entrant criterion set for mathematics: a mark of at least 60% (C-symbol) in mathematics at higher grade in the final examination in Grade 12 [8]. The 5YSP is structured in such a way that the courses of the first two years of the 4YSP are spread over the first three years of the 5YSP. Students on the 5YSP attend the same classes, have the same lecturers and write the same tests and examination papers as students on the 4YSP.

Students on both the 4YSP and the 5YSP do the compulsory mathematics modules in the first two years of study. Continuation to the final year of engineering study is not possible if a student has not passed these mathematics modules. In the School of Engineering our experience has revealed that a large number of students lack understanding in fundamental mathematical concepts. This mathematical inability seems to be increasing [9]. Similar observations are noted in the USA where a survey revealed that an increasing number of incoming students need remedial courses in mathematics [10].

Since the mid 1990s, experienced lecturers in the Department of Mathematics have also expressed a growing concern regarding the entry level mathematical preparedness of first year students. In 2000 a strategy of 'technique mastering' [11] was implemented to address the shortcomings in students' assumed pre-calculus knowledge. The technique mastering exercises were initially paper based and students had to achieve an 80% pass mark in each test. For each of the tests, students were granted three (paper based) attempts to achieve this mark. In 2001, the paper based re-attempts were replaced by computer-based tests. However, this intervention strategy did not work. A possible reason for this is that students did not perceive the effort as necessary as the marks for the technique mastering tests did not contribute to their final mark. Furthermore, the logistics of the additional testing and re-testing became a burden for teaching staff who already had full course loads. The technique mastering strategy was eventually terminated in 2004.

In response to the concern expressed by the School of Engineering [9] as well as by lecturers in the Department of Mathematics who present the second year modules to the engineering students, an inventory, the Mathematics Diagnostic Survey [12], was compiled. The aim in setting up the inventory was to:

- determine the entry level knowledge ('must knows') in mathematics of first year students;
- gain insight into students' information processing ability;
- gain insight into students' problem solving behaviour; and
- establish the students' confidence in their mathematical ability.

'Must knows' are viewed as knowledge components that are *vital for some activity and become so ingrained that it can be recalled effortlessly* ([11], p.10). Information processing in mathematics refers to reading strategies, critical thinking and understanding strategies [13]. Problem solving behaviour in mathematics includes planning, self-monitoring, self-evaluation and decision making during the process of problem solving in mathematics [13].

2 Mathematics Diagnostic Survey (MDS)

The format of the 2005 survey is paper based and questions are answered on forms marked by an optic reader form. In Section A (Questions 1-6) students' details are required. Section B (Questions 7-27) deals with mathematical proficiency. The questions in Section B are grouped in seven content domains, with each content domain focusing on a specific topic (see Table 1). Each content domain comprises two parts. Part one contains the questions and possible answers and in part two the student has to indicate how sure he/she is that the answer is correct. The format of the answers in part one is either multiple choice or match. The given answers apply to all the questions in a specific content domain. The number of answer options per content domain varies between six and ten (see Table 4). The aim with listing answers that apply to various questions, is to discourage guessing but rather encourage reasoning and computing to determine an answer. In setting up the survey, the choice of distractor answers was based on our experience of mistakes that students make. The format of the answers in part two of each content domain is a choice of one option out of four. Each of the four possible options indicates a level of confidence with which the given answer has been determined (selected). The confidence level options are: completely sure; reasonably sure; unsure; I am guessing.

The total score for the MDS is 21 with one mark per correct answer. Of the 21 questions, 19 are regarded as pre-calculus 'must knows' for calculus. Therefore a score ≥80% is regarded as indicative of a satisfactory entry level for calculus. In the instructions for completing the survey it is clearly stated that a student has to achieve at least 80% for part one (mathematics questions) but that the answers in part two (confidence level) is not to be taken into consideration for the final score.

3 Research project

In 2005 the MDS was implemented to ascertain first year engineering students' mathematical preparedness for study in calculus. Students completed the MDS at the beginning of the academic year, within the first week of classes during a formal lecture period. In this paper an overview of the results obtained in the project is given. Only the results pertaining to the mathematics questions (part one of the content domains) are discussed. The mathematics results of the MDS are analysed and the Grade 12 (final school) mathematics marks are compared to the performance in the MDS. Results are given for the 4YSP and 5YSP groups respectively.

In the research reported in this paper, we consider only first entrant engineering students who enrolled at the University of Pretoria in 2005. First entrant engineering students exclude students who repeat their first year of study, as well as students who previously had done any tertiary course. The research involved 782 students including 664 students on the 4YSP and 118 students on the 5YSP.

4 Results and discussion

The data in Table 2 summarises the detail of the research participants showing the mean M-scores, the mean of the Grade 12 mathematics mark and the mean of the MDS scores for the 4YSP and 5YSP students, respectively. Calculation of the M-score is based on final school results. Admission to the School of Engineering at the University of Pretoria requires a minimum M-score of 18 out of a possible 30 points as well as a minimum C-symbol in both mathematics and physical science at Higher Grade (HG). Prospective students with at least a D symbol in Mathematics (HG) and/or Physical Science (HG), or an M score of between 12 and 18 are required to do an additional admissions test.

Table 1: Summary of the content domains of the Mathematics Diagnostic Survey (MDS)

MDS Content domain	Topic	Some examples where topic is applicable in calculus in the first study year	Number of possible answers		'Must knows'	Information Processing	Problem solving behaviour
Content domain 1	Solving equations	Intercepts Domain	10	Q7	✓	✓	
				Q8	✓	✓	
				Q9	✓	✓	
				Q10	✓		
				Q11	✓		
Content domain 2	Solving inequalities	Increasing Decreasing Concavity	10	Q12	✓	✓	
				Q13	✓	✓	
				Q14	✓	✓	✓
Content domain 3	Trigonometry identities	Integration Solving equations Solving inequalities	10	Q15	✓		
				Q16	✓	✓	
				Q17		✓	✓
Content domain 4	Absolute value graphs	Solving equations Solving inequalities Numerical integration	8	Q18	✓		✓
				Q19	✓		
Content domain 5	Radian measure Completing square	Inverse trig functions Extreme values Conic sections	6	Q20		✓	✓
				Q21	✓	✓	✓
Content domain 6	Fractions	Limits Differentiation Partial fractions Integration Asymptotes	10	Q22	✓		
				Q23	✓		
				Q24	✓		
Content domain 7	Translations of sine and cosine graphs	Wave lengths Integration Parametric representations	9	Q25	✓		
				Q26	✓		
				Q27	✓	✓	

Table 2: Means of M-scores, of Grade 12 mark in mathematics and of MDS score

Group	N	M-score	Grade 12 math mark	MDS
4YSP	664	25	77.7%	59.6%
5YSP	118	19	63.2%	52.0%

Although the mean M-score of the 5YSP group in 2005 is above the required minimum of 18 for admittance to engineering study, it is significantly lower than that of the 4YSP group. A low M-score is an indication of deficiencies in schooling and for engineering students with a low M-score, mathematics is a main area for concern. For both the 4YSP and the 5YSP group the means of the MDS are well below the expected minimum of 80%. When comparing the mean of the Grade 12 mathematics mark with the mean of the MDS mark, it is noticeable that for the 4YSP group the MDS mean is 18% lower than the Grade 12 mean and the MDS score is not in par with their high M-score. For the 5YSP group the MDS mean is 11% lower than the Grade 12 mean. Figure 1 illustrates the distribution of the percentage of students in the 4YSP group according to achievement per score interval for the Grade 12 mathematics mark and the MDS mark. Figure 2 illustrates the same data for the 5YSP group.

In **Figure 1** and in **Figure 2** the labels on the horizontal axis indicate the score intervals (where A+>90%; A=80-89%; B=70-79%; C=60-69%; D=50-59%; E=40-49%; F=30-39%; G and H<30%). The values on the vertical axis indicate the percentage of students per score interval. The percentages per score interval are calculated as a percentage of the total number (N=782) of participants in this research project.

Figure 1: Distribution of 4YSP students according to Grade 12 math% and MDS%

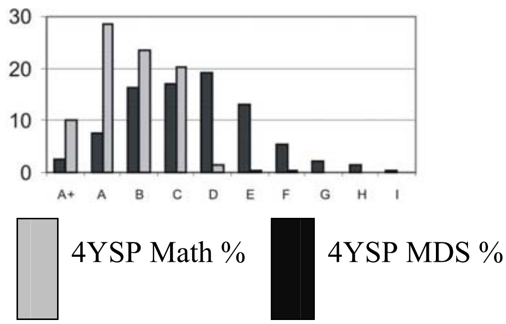
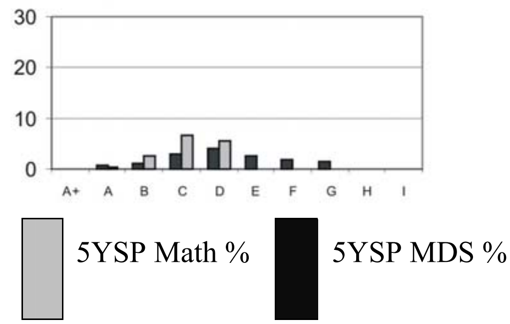


Figure 2: Distribution of 5YSP students according to Grade 12 math% and MDS%



The diagrams in **Figure 1** and **Figure 2** show that for the 4YSP group the distribution of marks for the MDS differs noticeably from the distribution of marks for Grade 12 mathematics and that for the 5YSP group there is less of a difference in the distribution of marks for the MDS compared to the distribution of their marks for Grade 12 mathematics.

Further analysis of the results of the MDS reveals that only 10% of the 782 students who completed the MDS in 2005 achieved the expected minimum score of 80%. This 10% (N=78) of students comprises:

88% students from the 4YSP group with Grade12 mark in A+ and A score intervals (≥80%);

7% students from the 4YSP group with Grade12 mark in the B score interval (70-79%);

3% students from the 5YSP group also with Grade12 mark in the B score interval; and

2% students from the 5YSP group with Grade12 mark in the D score interval (50-59%).

Of the 664 students in the 4YSP group, 51% achieved less than 60% in the MDS and another 26% scored less than 50%. This is in contrast to their Grade 12 mathematics achievement of a C or higher (≥60%) and not in par with the average M-score for the 4YSP. It should be pointed out that a C or higher for Grade 12 mathematics and an average M-score of 25 is an indication of mathematical capability for engineering study.

Analysis of results for the 5YSP group is more alarming as 74% of the group (N=118) achieved less than 50% in the MDS.

For both the 4YSP and the 5YSP groups their performances in the MDS in 2005 is not encouraging. If a student's entry level knowledge of mathematics is poor it has consequences for calculus study. Schattschneider [14] confirms that an inadequate grounding in pre-calculus can be a barrier in the study of calculus.

The data in **Table 3** gives the mean scores for the content domains of the MDS for the 4YSP group, the 5YSP group and the groups combined.

Table 3: Mean scores for the content domains of the MDS

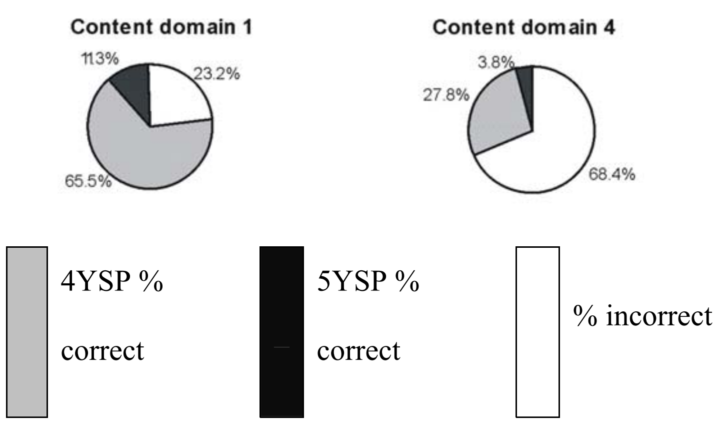
MDS Content domain	Topic	Mean score		
		4YSP N=664	5YSP N=118	All N=782
Content domain 1	Solving equations	65.5%	11.3%	76.8%
Content domain 2	Solving inequalities	56.6%	8.9%	65.5%
Content domain 3	Trigonometry identities	39.6%	5.1%	44.7%
Content domain 4	Absolute value graphs	27.8%	3.8%	31.6%
Content domain 5	Radian measure. Completing the square	50.5%	7.9%	58.4%
Content domain 6	Fractions	43.8%	6.1%	49.8%
Content domain 7	Translations of sine and cosine graphs	53.0%	8.3%	61.3%

Although a detailed analysis of the results of the content domains is beyond the scope of this paper, the content domain with the highest mean score and that with the lowest mean score are briefly analysed. This analysis is done for the groups combined. A short analysis is also given of the question with the highest percentage correct answers and the question with the lowest percentage correct answer.

The expected minimum score of 80% was not reached in any of the content domains. The highest mean score of 76.8% occurred in Content domain 1. **Figure 3**A shows the distribution of correct answers for the 4YSP and the 5YSP groups as well as the incorrect answers of the mean score in Content domain 1. The lowest mean score of 31.6% occurred in Content domain 4. **Figure 3**B shows the same distribution for Content domain 4.

Figure 3: Distribution of correct answers in two content domains

Figure 3A Figure 3B



We did a further analysis of performance in specific questions of content domains one and four.

In these content domains, all of the questions are regarded as 'must knows'; three of the questions require additional skills in information processing and for one of the questions, additional skills in problem solving behaviour is necessary (see **Table 1**). The questions and possible answers of Content domain 1 are listed in **Table 4** and that of Content domain 4 in **Table 5**.

Students were exposed to the topic in Content domain 1 (solving equations) for at least five years of secondary schooling and it was expected that they should perform well. In each of questions seven, nine, ten and eleven a mean score of 80+% was achieved but the mean score for Question 8 is only 48%.

Table 4: Questions and answers in Content domain 1 (Solving equations)

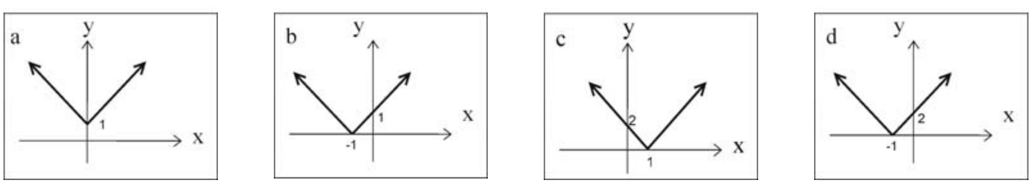
		Possible answers to questions 7 to 11	
Question 7	If $(x+1)^2 = 4$ then $x =$	(a) 1	(f) -1 or 1
Question 8	If $\sqrt{(x+1)^2} = 2$ then $x =$	(b) 2	(g) -3 or 1
Question 9	If $\lvert x+1 \rvert = 4$ then $x =$	(c) 3	(h) -3 or 2
Question 10	If $\frac{4}{x+1} = 2$ then $x =$	(d) ±2	(i) -5 or 3
Question 11	If $-x^2 + x - 1 = 0$ then $x =$	(e) ±3	(j) none of these

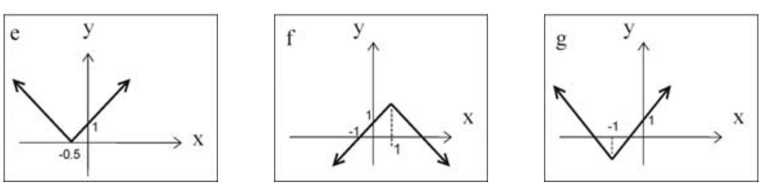
In Content domain 4, Questions 18 and 19 are regarded as must knows and require skills in information processing and problem solving behaviour. Both questions also require skills in transforming graphs. A possible explanation for students' inability to determine the correct option is their lack of transfer of knowledge ([5], p.52). In this case the visual images of possible solutions are given (see **Figure 4**) and the correct answer can easily be found by checking the intercepts with the axes (a fundamental 'must know').

Table 5: Questions in Content domain 4 (Absolute value graphs)

Question 18	The graph of $y = \lvert 2x+1 \rvert$ is
Question 19	The graph of $y = 2\lvert x \rvert + 1$ is

Figure 4: Possible answers to questions 18 and 19



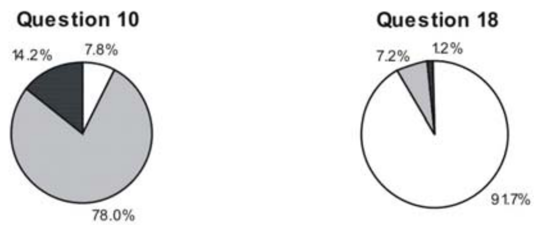


g None of these

The diagrams in **Figure 5** show the distribution of the correct answers to the selected questions for the 4YSP and the 5YSP groups respectively. Question 10 in Content domain 1 has the highest number of correct answers and Question 18 in Content domain 4 has the lowest number of correct answers. The percentages in **Figure 5** indicate the distribution of correct answers by the 4YSP and the 5YSP group respectively as well as the incorrect answers for the research group as a whole.

Figure 5: Distribution of correct answers for selected questions

Figure5A Figure5B



4YSP % correct

5YSP % correct

% incorrect

The outcomes of the 2005 implementation of the MDS confirm that there is reason for concern regarding the entry level preparedness of first year engineering students at the University of Pretoria. The inevitable question now arises what should be done to address the problem. Continuing speculation about possible reasons for the students' low level of preparedness such as: inadequate schooling; poor qualified school teachers; coaching for the Grade 12 examination and inadequate content of the school syllabus will not solve the problem. We cannot start a calculus course at a knowledge level where we assume the students are, we need to meet the students where they are [15] and get them actively involved [16]. This will require rethinking our learning facilitation strategies, the process of learning and the learning content. In all our endeavours we should also keep in mind that: *No style of teaching mathematics can substitute for insisting that students pick up their share of the work, unless one is willing to compromise standards* ([17], p. 277).

5 Conclusion

Analysis of data concerning the mathematical ability of first entrant engineering students who enrolled at the University of Pretoria in 2005 indicates that 90% of the students did not demonstrate the level of preparedness expected from students who enrol for a first course in calculus. As competence in mathematics is a key aspect in engineering study, the 2005 results of the Mathematics Diagnostic Survey are not encouraging and reason for concern. The challenge is that action, which will involve not only commitment from teaching staff, but also from students, should to be taken to address the shortcomings in the entry level preparedness of first year engineering students.

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Checking the accuracy of ODE solvers in the classroom, laboratory and workplace

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Abstract

Most computer algebra systems (CAS) have built-in ordinary differential equation (ODE) solvers, but the accuracy of the solutions produced is not always obvious. Various ways of estimating the accuracy of ODE solvers are discussed here, extending work presented at the "Remarkable Delta 2003" conference in New Zealand. Our methods are easy enough for undergraduates to implement because the needed mathematics is accessible to them. Many students (and their teachers) have an in-depth knowledge of how to check the accuracy of numerical routines, but many trust them blindly. On the other hand, testing the accuracy of a routine takes more time than just running the routine to produce a solution and this is another reason for taking a solution at face value. Such blind trust could have negative connotations if carried through to industry and elsewhere after the student graduates. We cite an example of how experienced mathematical scientists (academics) have fallen into the trap of assuming numerical solutions to be correct. There already exist a number of routines to test the accuracy of ODE solvers, some of them time intensive, and some not. The routines introduced here add to this collection of routines and one of them substantially reduces the calculation time of an existing routine.

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The role of graphical representation involving analysis of one variable functions

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Abstract

At the beginning of an introductory mathematical short course, especially for new university students, serious problems are observed with basic algebraic calculations. For instance, students do not know how to multiply and divide, solve easy problems, apply different properties, represent functions graphically or model geometrical situations that may help them in the comprehension of a particular situation.

According to Baroody, informal learning is the fundamental basis to understanding and learning mathematics at school, when children undertake formal mathematics it replaces the informal mathematics. The early graphic representations help students understand the process of learning from simple operations through to modelling more complicated functions. There are several phases associated with the process of learning these basic operations. According to Rico, the learning of functions represents a process with different phases well delimited. These phases in the process, must be based on simple basic calculations, geometric graphics, 3-D intuition, and recognition of the problem variables. It can be observed that most of the students do not have these abilities when finishing their secondary studies

For this reason students fail the introductory examinations for the university. Therefore when attempting calculations, they just try to memorize algorithms without achieving a mental representation that may allow them get the basics of calculations

Fucson (1986) considers that if we use concrete materials in the teaching of the math as a strategy of learning, this will help the students to organise their knowledge. If each concept has a concrete representation for students, it will be simpler for them to identify variables and therefore, model the proposed problem.

If each operation, learned in previous courses, was acquired with the confidence from a representation, students may be able to develop strategies of problem solving, without mathematics being the major hurdle for them when entering university.

If students have developed and matured their mathematical expertise at the three levels of acquisition of the abstraction, i.e., conceptual level, connection level and abstraction level (Rico et al.) and they successfully solve the problem, identifying variables, therefore they will be not afraid of failure, since their mathematical structure is built on solid foundations.

If teachers of primary and secondary levels consider mathematics as a modelling activity, this would help to provide analytic tools to allow students understanding of the real world by interpreting physically and modelling its environment whilst knowing that there is not a

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unique model for those representations. Students will understand the physical qualities of magnitudes, opening gates for understanding of variables and functions modelling.

Comments about experiences in the Intelligence Harvard Project

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Abstract

At the end of the seventies, Intelligence Harvard Project (I.P.H) was developed by researchers from both University of Harvard and several Venezuelan institutions: Hernstein, Nickerson, Perking, Jaeger Adams, Margarita, Amestoy, Catalina Laserna, etc… Twenty years later, the outcomes of this project are being implemented in our courses, with a combination of students; secondary students, first year university students and students receiving individualised attention at private institutions. The basic purpose was to improve the quality of the mathematics learning, developing abilities of intelligence, such as:

- Enlarging the intellectual competence (intellectual abilities) in tasks as systematic observation, etc.
- Learning different approaches to specific tasks (strategies or heuristics), using generalisation methods.
- Utilise the expertise of conventional matters for the improvement of thinking.
- Promote specific attitudes that favour the progress and implementation of the intellect

Achievement of the development of several abilities: ability to classify patterns; to reasoning inductively and deductively; to develop and use conceptual models; to understand and modify the adaptive behaviour.

We worked with the six steps mentioned by the project, such as:

Bases of the reasoning: It tries to develop attitudes, awareness and basic processes that form the foundations to build upon: Observation and classification, Codification, Hierarchical classification, Analogies, 3-D Reasoning.

Comprehension of the language: It tries to teach how to deal with difficulties in the comprehension of texts such as: relations among words, language Structure, Read for understanding.

Verbal reasoning: The deductive reasoning can be stated as propositional reasoning, based on the elaboration and analysis of proposals that are related, forming arguments that can be logical: Assertions, Arguments.

Problem solving: Strategies of problem solving are on different basic types: Linear representations, Table Representations, Representations by Simulation, a systematic approach by estimation, discuss and explain common misconceptions

Taking decisions: It tries to instruct students in the intricacies of the decision problems, where there is a choice of alternatives in order to arrive at the final goal, keeping in mind the following issues: Introduction to decisions taking, explore and evaluate information in order to reduce uncertainty, Analysis of situations where there is difficulty making decisions.

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Innovative thinking: This item impacts in the routine habits, trying to teach how to see objects and ordinary procedures as designs; product of creativity, for instance: design, procedures of design.

The application of the project has been very positive for the teaching of mathematics mainly at the pre-university level and in short courses at the initial stages of university studies.

A drawback was the number of students in the courses that often created difficulties with the individualised attention required by this system.

A professional development course for mathematics teaching assistants

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We assess the impact of a three credit-hour graduate level mathematics course which focuses on issues related the teaching of mathematics at the college level. We do this by analyses of student course evaluation data from classes taught by our mathematics graduate student teaching assistants and from interviews with teaching assistants who had taken the course. Our primary conclusion is that our course had a significant positive effect on the teaching assistants' confidence and comfort levels, with related impact on their teaching practise.

Keywords: Mathematics, Teaching Assistant, Professional Development

1. Introduction

A significant percentage of all lower division courses, at doctoral-degree-granting universities in the United States are taught by graduate student teaching assistants (GTAs). (Marincovich, Prostko, & Stout, 1998) Moreover there are indications that such reliance on GTAs will only increase in the near future. (Mcgivney-Burelle, et al, 2001) The need for graduate-degree-granting departments to provide teacher training to GTAs is clear, especially for departments in such common core areas as mathematics.

1.1. Local Context. Our context is that of a large (~30 000 students) state-supported university. The department of mathematics and statistics consists of 45 tenured or tenure-track faculty, eight to10 lecturers, and 80 or so mathematics graduate student teaching assistants (MTAs). The department offers bachelor through Ph. D. degrees, but the bulk of the department's teaching load is service courses at the undergraduate level (maths for business students, for engineers, etc.).

During the fall semester of 2000 just over 8000 students were enrolled in our undergraduate mathematics classes, while in fall 2005 our enrollment was over 9000. Always crucial, the role of MTAs in meeting the department's undergraduate teaching responsibilities has become all the more critical since the number of full-time faculty positions has remained static during this period of enrollment increases.

Prior to fall 2000 the only preparation for MTAs with sufficient English communication skills, as defined by the university, consisted of a two-hour orientation session just before the start of each term. A typical assignment for MTAs with at least 18 graduate mathematics credit-hours is to teach, as instructor of record, two sections of one of the multi-section courses such as college algebra, taught in 25-30 sections of 40-50 students each. A full-time

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faculty member is assigned to teach one section and is designated as the course coordinator. This person meets for one to two hours, once every two weeks with the MTAs, for the purpose of coordinating the conduct of the course and dealing with any issues that might arise. MTAs with fewer than 18 hours of graduate mathemathics credit can not be assigned as instructor of record for a class, so there is pressure for all new MTAs to obtain 18 credit-hours as quickly as possible.

1.2. Project Impetus. In fall 1999 our chair came to the conclusion that our MTA professional development programme was inadequate. Moreover, as indicated above, it is in the department's best interest for new MTAs to obtain as many credit hours as possible during their first semester. So he asked the Director of Undergraduate Programs (DUP), second author of this paper, to develop a three-hour, graduate-level course devoted to the professional development of our MTAs. The DUP was given explicit instructions that the course not place such a burden on the MTAs as to hinder progress in their 'real math classes'. The resulting course, referred to as The Course, is the topic of this paper. Our basic question is the following. What effect does The Course have on the MTAs who take it?

2. The Course

Using extensive records maintained by each instructor of The Course, we compiled a detailed description of its evolution from its inception in fall 2000, through fall 2004. (Froman, 2005). The Course has always consisted of three primary components.

2.1. MTA Presentations. Each MTA gives a 20 minute, video-taped presentation on a topic he/she expects to be teaching in a college maths class. After the presentation the MTA is evaluated by the other MTAs using the Mini Lecture Critique form. (Appendix A) The critiques and the recording are given to the presenter. The purpose is to introduce the MTAs to self-reflection and constructive criticism as mechanisms for improving their classroom practise.

2.2. Case Studies. Working in groups, the MTAs analyse case studies developed by the Boston College mathematics case studies project. (Friedberg, 2001) These case studies involve a wide range of situations and issues that commonly occur in the teaching of mathematics at the college level. The intent is to help the MTAs gain experience for use in their own classrooms and professional interactions.

2.3. Reading Assignments. The MTAs read selected materials which are then covered in group and/or whole class discussions. The readings focus primarily on ethical, philosophical, and theoretical issues related to teaching mathematics at the college level. (Appendix B) The objective is to give the MTAs a broader understanding of the profession and help them begin the life-long process of developing their own teaching philosophies.

The role of the instructor of The Course is to choose the assignments and case studies and facilitate the in-class discussions. The expectation is that the MTAs, working together with their peers, develop their own teaching practises and philosophy. While the instructor provides examples and participates in the discussions, it is not his/her role to teach either methodology or philosophy. Grades are not an issue since the MTAs are guaranteed an 'A' if they attend and participate.

All new MTAs entering our program in the fall with fewer than 18 hours of graduate mathematics credit are required to take The Course. MTAs who enter in the spring and have earned 18 hours prior to the subsequent fall, or who enter our program with 18 hours, are not required to take The Course. On average, 17 MTAs are enrolled in The Course each fall and none have any prior college level teaching experience.

3. Assessment

In spring 2005 we decided to see what effect, if any, The Course was having on the MTAs taking it. The primary sources of data available to us were the university's teaching evaluation and grade data from all courses taught by MTAs, and the MTAs themselves. Since there is no uniform system for assigning course grades, we decided the grade data was not useful.

3.1. Student Evaluations. The end-of-term student course evaluations, Student Evaluation of Course and Instructor (SECI), used by the university has 16 statements, the first 10 with the subheading 'Instructor's Performance' and the last six with subheading 'Course Evaluation'. The students are asked to respond to each statement by choosing one of the following: strongly agree (5), agree (4), neutral (3), disagree (2), or strongly disagree (1). We considered only those responses to the 10 statements in the 'Instructor's Performance' category:

1. Overall the instructor was effective.
2. The instructor was available for consultation during office hours or by appointment.
3. The instructor stimulated student leaning.
4. The instructor treated all students fairly.
5. The instructor treated all students with respect.
6. The instructor welcomed and encouraged questions and comments.
7. The instructor presented the information clearly.
8. The instructor emphasized the major points and concepts
9. The instructor went beyond presenting the information in the text.
10. The instructor demonstrated knowledge of the subject.

We analysed the SECI results from all fall 2004 classes taught by the 48 MTAs on staff in spring 2005. This included responses from 1377 students taught by 28 MTAs who had taken The Course (MTA-C) and 906 students taught by 20 MTAs who had not taken it (MTA-nC). For each statement we compared the scores for MTA-C to the scores for MTA-nC. Here we choose to present the data for statements 5 and 6, tables 1 and 2, because statement 5 is an example of a statement that yielded no significant difference and statement 6 is the one that yielded the most significant difference between the MTA-C and the MTA-nC. Similar tables for all ten statements are available on line. (Froman, 2005, pp 21 - 30) The results of our final analyses on all ten statements are tabulated in tables 5 and 6 below.

In all tables a p-value marked with "***" is interpreted as highly significant (p-value < 0.01), with "**" as significant (0.01 ≤ p-value < 0.05), and with "*" as marginally significant (0.05 ≤ p-value < 0.10). To maintain a 95% overall confidence level, we applied a Bonferroni adjustment using $\alpha = .05/10$ with $|z| > 2.807$ for each comparison. (Keppel, 1999) There was significant difference in favor of MTA-C for all statements except numbers 4, 5, and 10.

Table 1: Statement 5 analyses: Treated students with respect.

Course	Avg.	n	St. Dev.	z-value	p-value
MTA-C	4.510654	1361	0.696681		
MTA-nC	4.487859	906	0.707002		
Sig.				0.7563	1.0000

Table 2: Statement 6 analyses: Encouraged questions.

Course	Avg.	n	St. Dev.	z-value	p-value	
MTA-C	4.532151	1353	0.703358			
MTA-nC	4.326923	884	0.840993			
Sig.				6.0109	0.0000	***

However, of the 28 MTA-C, five were international, while 14 of the 20 MTA-nC were international. This led to the question of whether the scores were the result of an international, IMTA, versus national, NMTA, issue. So we compared all groups with the same significance test. Results for statements 5 and 6 are given in the tables 3 and 4.

Table 3: Statement 5 analyses by IMTA or NMTA

		Avg.	Total	St. Dev	z-value	p-value
IMTA-C		4.533040	227	0.705548		
IMTA-nC		4.511111	585	0.670026		
NMTA-nC		4.475083	301	0.763900		
NMTA-C		4.497832	1153	0.704798		
IMTA-C	IMTA-nC				0.4030	1.0000
IMTA-C	NMTA-nC				0.9017	1.0000
IMTA-C	NMTA-C				0.5477	1.0000
IMTA-nC	NMTA-nC				0.6926	1.0000
IMTA-nC	NMTA-C				0.3836	1.0000
NMTA-nC	NMTA-C				-0.4673	1.0000

Table 4: Statement 6 analyses by IMTA or NMTA

		Avg.	Total	St. Dev.	z-value	p-value	
IMTA-C		4.576763	241	0.721197			
IMTA-nC		4.304274	585	0.769539			
NMTA-nC		4.371237	299	0.965500			
NMTA-C		4.511091	1127	0.716224			
IMTA-C	IMTA-nC				4.8394	0.0008	***
IMTA-C	NMTA-nC				2.8296	0.2796	
IMTA-C	NMTA-C				0.9041	1.0000	
IMTA-nC	NMTA-nC				-1.0420	1.0000	
IMTA-nC	NMTA-C				-5.3989	0.0000	***
NMTA-nC	NMTA-C				-2.3397	1.0000	

Here we have made 60 comparisons, so to maintain an overall confidence level of 95% we apply the Bonferroni adjustment $\alpha = .05/60$ with $|z| > 3.34$. Again there appeared to be significance in all statements except 4, 5, and 10. (Froman, 2005)

This model, however, cannot account for all the dependence occurring due to possible variation between individual MTAs within each group. Thus we implemented a two-way factorial analysis of variance (ANOVA). (Anderson/Mclean, 1974) Tables 5 and 6 show the results of the two-way ANOVA model on the course effect and the MTA type effect for each of the 10 statements.

Table 5: 2-Way ANOVA analyses: course effect

Statement	F (df1, df2)	p-value	
1: Overall Instructor Effective	2.58 (1, 46)	0.1153	
2: Instructor Available	5.09 (1, 49)	0.0287	**
3: Stimulated Learning	3.70 (1,46)	0.0608	*
4: Treated Students Fairly	1.86 (1, 47)	0.1781	
5: Treated Students With Respect	1.78 (1, 47)	0.1881	
6: Encouraged Questions	10.29 (1, 46)	0.0024	***
7: Presented Material Clearly	3.55 (1, 46)	0.0658	*
8: Emphasized Major Points	3.22 (1, 47)	0.0790	*
9: Went Beyond Information in Text	5.34 (1, 46)	0.0254	**
10: Demonstrated Knowledge	1.78 (1, 47)	0.1893	

Table 6: 2-Way ANOVA analyses: MTA type effect

Statement	F (df1, df2)	p-value
1: Overall Instructor Effective	0.11 (1, 46)	0.7418
2: Instructor Available	0.40 (1, 49)	0.5322
3: Stimulated Learning	0.11 (1, 46)	0.7437
4: Treated Students Fairly	0.74 (1, 47)	0.3930
5: Treated Students With Respect	1.97 (1, 47)	0.1674
6: Encouraged Questions	1.33 (1, 46)	0.2546
7: Presented Material Clearly	1.05 (1, 46)	0.3107
8: Emphasized Major Points	0.24 (1, 47)	0.6269
9: Went Beyond Information in Text	0.04 (1, 46)	0.8348
10: Demonstrated Knowledge	0.18 (1, 47)	0.6765

3.2. Interview Data. We interviewed 22 MTAs (13 males, nine female) on staff in spring 2005 who had taken The Course. We used a uniform interview instrument (Appendix C) in which all MTAs were given the same instructions and asked the same questions, in the same order. The DUP was not present when the first author conducted the interviews, which were audio-recorded. We anticipated the MTAs would assume the DUP would listen to the interviews. To mitigate this possibly intimidating factor we chose interview questions that focused attention on the content of The Course and solicited criticism. During each interview the first author noted his observations on the interview instrument. After each interview, he reviewed the recording to insure that his noted observations were consistent with the interviewee's actual statements. Then the DUP listened to each interview and used the interview instrument to record his observations. Independently we made a subjective assessment of each MTA's overall reaction using the following descriptors: highly positive, positive, neutral, negative, highly negative. After our independent observations and assessments we met to compare conclusions. When there appeared to be inconsistencies we listened to the appropriate portions of the recorded interviews together and arrived at a consensus. Extensive accounts of the interviews are available on line. (Froman, 2005)

Fall 2000 (Taught by DUP). Three MTAs (all male) were interviewed from the fall 2000 course. One was rated as very negative and the other two as positive. The very negative MTA claimed to have gotten 'nothing out of the course', although later in the interview admitted to realizing that he learned that he 'needed to relax a little more'. He thought the video-recorded lecture was least beneficial, while the grading exercise from the case studies (Friedberg, 2001) was the most beneficial thing from the course. Of the other two MTAs, one felt he gained more confidence while the other gained more humility and 'became more patient'. Both rated the video-recorded lecture as the most beneficial and one rated the grading exercise as the least beneficial: 'Everyone had their own way of grading, it didn't change them'. All three thought the course needed to have greater emphasis on practical issues: 'things we do every day'.

Fall 2001 (Taught by our Chair). Three MTAs (all male) were interviewed from the fall 2001 course. One was rated as negative, the other two as neutral. Two claimed no effect on either attitudes or teaching practise. One observed 'I'm old and set in my ways anyway'. The other confirmed a much held belief: 'How to teach a math course was based on my experience as an undergrad'. However the third pointed to the grading exercise as affecting his practise, making him 'more lenient in my grading' and 'more patient with students'. One MTA thought that watching others teach was the most beneficial and the readings least beneficial. Two MTAs felt The Course was a good introduction to graduate school and the mathematics department. When asked what was lacking in the course two MTAs had very similar responses:

> *What they're doing now, having lessons on how students learn, dealing with students, video lessons, how to deal with complaints and false accusations, how to deal with parents and administration.*

One MTA added 'Since it is required it should count toward your degree'.

Fall 2002 (Taught by DUP). Three MTAs (all female) were interviewed from the fall 2002 course. One was rated as neutral and the other two as positive. Two noted a change in their attitudes towards teaching, while the third did not. However the one claiming no change in attitude, claimed to have made significant changes in her practise, observing the need to 'try to incorporate different styles' and 'make students do examples in class. If they just watch me do it then they're not going to learn'. Video-taped presentations and the grading exercise case study were listed as most beneficial. An interesting response to the least beneficial aspect of the class was 'no closure to the discussions, all gray, not necessarily a right answer, maybe that's the point, there are gray areas'. When asked why she took the course one MTA responded: 'Required, well not required, but they say everyone takes it'. Two thought The Course was worthwhile, with one commenting 'Yes, recommend everyone to take it, even professors'. After hesitation, the third thought the course needed to include 'more mechanical aspects of teaching'.

Fall 2003 (Taught by another colleague). Six MTAs (three male and three female) were interviewed from the fall 2003 course. Three (two males and one female) were rated as highly positive, two (one male and one female) as positive, and the other female as neutral. One male MTA claimed no attitude change: 'I went in wanting to be a teacher, that goal was not changed'. The other MTAs said the course had contributed to changes in their attitudes, with one male observing 'It made me a little more calm. I was worried about teaching students of the same age or older'. The same MTA thought a main benefit of The Course was ‘talking to fellow students (MTAs), knowing we’re in the same boat’. One female MTA viewed the course as ‘a therapy session’. Four MTAs felt The Course had affected their teaching practise with respect to improved classroom technique. Four MTAs viewed the case studies as being most beneficial and the other two thought the video-recordings and peer critiques were most beneficial. Two female MTAs mentioned the articles as being the least beneficial, with one admitting to not reading a lot of them. One female MTA summarized the goal of The Course as follows:

> *Prepare you to teach, to make you more comfortable and let you know that you’re not alone…course is a good example of how the department really cares about its’ TAs. I heard horror stories about how they hand you a book and say go teach.*

All six MTAs believed the goals had been achieved with ‘gaining confidence’ and ‘feeling more comfortable teaching college’ being the two most often mentioned outcomes. The hours not counting towards degree plans was a concern.

Fall 2004 (Taught by DUP). Seven MTAs (four male and three female) were interviewed from the fall 2004 course. One male was rated as highly positive, two females and one male as positive, two males as neutral, and one female as negative. The negative MTA claimed no change in her attitude, while all the other MTAs reported a change in attitude, with increase in comfort level being mentioned by several. One female MTA summarized her attitude change as follows:

> *I feel more privileged to teach students the same age.....at first I hated it, still intimidated. I realize now it’s not just giving out info, but communicating with people.*

Regarding teaching practise, one male and one female MTA referred to the video-taped lecture as pointing out things they will try to improve upon. Another female MTA observed ‘I learned that I have the authority to gain control of class…became more comfortable using teaching skills’. Five MTAs mentioned the case studies as being the most beneficial, with the case studies involving grading, cheating, harassment, and meeting students outside of class all mentioned. One female and two male MTAs said everything was beneficial, while one female thought the articles were ‘not helpful in general’, and one male MTA observed ‘A lot of days, just argued over opinions. Some would dominate and basically shut down discussions’. A summary of the course goals as perceived by these MTAs follows:

> *To prepare us to represent the department and university as educators and to set standards.*
>
> *To expose us to different styles of teaching, how to handle students, other TAs and faculty.*
>
> *Teach you how to teach.*
>
> *To get an overview of what to expect as a TA and to gain confidence in teaching. To get us ready to teach at the college level.*
>
> *To produce better teachers, not just teach, but implicate moral values and present material beyond the requirements.*

All MTAs thought The Course achieved its goals. When asked why they took The Course, four simply responded that it was required; one adding ‘but I’m glad I took it, learned some stuff I didn’t know I needed’. Six could not think of anything lacking in The Course, while one female MTA said ‘Maybe some more role playing in groups’. Two male MTAs thought the course was not worthwhile because it didn't count towards their degree and one female responded 'No, I don't feel I learned that much as far as how to teach. I think you learn to teach by teaching'.

4. Discussion

From the analyses of the SECI data we conclude that their students' perceptions, with respect to the specified attributes, were not measurably affected by the MTA type, international or national. The MTAs who took The Course were viewed by their students as much more likely to welcome and encourage questions and comments, and as more likely to be available for out-of-class consultation and to present information beyond the text, than were the MTAs who had not taken The Course.

Analyses of the interview data suggests the common attitude changes involved gaining confidence and becoming more comfortable with their role as teachers. The case studies appear to be the primary impetus for change of attitudes and the video tapes for the changes in teaching practise. It appears The Course has become an accepted part of our MTA culture; however, there is lingering concern about hours not counting and the need for more practical information.

5. Conclusion

We believe the SECI and interview data are consistent in that an instructor's comfort and confidence levels should be directly related to his/her willingness to encourage questions and comments during class, being available and encouraging out-of-class consultation, and presenting material beyond the text. Thus, in answer to our original question, we believe The Course has increased the confidence and comfort levels of the MTAs taking it (121 as of fall 2006), with related impact on their teaching practise.

Appendix A

Mini Lecture Critique (Comment space on original form has been omitted here.)

Name of Lecturer: ________________________

(From the Texas Tech University Student Evaluation of Course and Instructor)

Please respond to the questions below by marking the appropriate oval. The ovals form a rating scale of 5 (strongly agree) to 1 (strongly disagree).

The instructor stimulated student learning.	5	4	3	2	1
. . . welcomed questions and comment	5	4	3	2	1
. . . presented the information clearly	5	4	3	2	1
. . . emphasized the major points and concepts	5	4	3	2	1
. . . demonstrated knowledge of the subject	5	4	3	2	1

(Specific for these mini lectures)

Comment on strengths of the presentation.

Comment on weaknesses of the presentation.

Appendix B

Course Resources

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Appendix C

Interviews of students who have taken the pedagogy course. (Space for observer notes on original form has been omitted here.)

Name of TA:________________ Date and time of interview: ________________

Term in which TA took the course:______________

(To be read at beginning of each interview) We are trying to determine how the Pedagogy course (MATH 5360) has effected our TA's beliefs and practise with regard to the teaching of math to college students. The purpose of this interview is to seek your opinion about some issues related to this endeavor. The results of our study will appear in a Masters Thesis and possibly in a journal article. No individual will be identified in either the Thesis or the possible journal article. By agreeing to this interview you are granting us permission to use data obtained from it in the thesis and possible article. In order to accurately interpret your responses we wish to audio tape this interview. Do you agree to allow this interview to be audio taped?

(If "yes" start audio taping, if "no" make note that audio recorder remained off.)

In what ways, if at all, did taking the pedagogy course affect your attitudes about teaching math to college students?

In what ways, if at all, has the pedagogy course affected your teaching practice?

What topics or content in the course do you believe to have been the most beneficial?

What topics or content in the course do you believe to have been the least beneficial?

What do you think was the goal of the course and do you think it achieved that goal?

Did you watch your video tape lesson?

If "yes" what was your assessment? If "no" why not?

Why did you take the course?

Is there anything you think should be in the course that was not there?

Do you believe you got your money's worth out of the course?

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Understanding students' misconceptions of statistics at Botswana College of Agriculture

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Abstract

Botswana College of agriculture (BCA) admits about 200 students per year to pursue programs at undergraduate level in various agricultural disciplines such as General Agriculture, Agricultural Education, Horticulture, Forestry, Animal Health and Production and Agricultural Engineering. These students complete normal courses for their respective disciplines, but they are also required to complete and pass courses in Statistics as a requirement for graduation. Students' grades for the past ten years have so far indicated that Statistics courses are not well appreciated at BCA with the majority of students obtaining below average marks and a low overall pass rates. Statistics is also one of those courses which delay students to graduate since they have to do it repeatedly until they succeed. This paper looks at the way in which Statistics courses are viewed by students pursuing agricultural programs at BCA. A survey of students doing a course in Introduction to Statistics is used to look at their attitudes, expectations and their general perceptions of Statistics in relation to other courses taken and their career expectations. The outcome of this research will be used as a yardstick to provide strategies that will help both students and instructors address attitudes that hinder students' effective learning.

Keywords: *attitudes, statistics, agricultural students*

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The mathematical education in the current scenario

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Abstract

The development of computer technology has affected university education along with other changes in the past few years. The difficulty of accommodating and retaining new students (tutorship system has helped this in both public and private universities) combined with lack of mathematical skills greatly hinder the teaching and learning of mathematics at this level.

Some notable differences between the average level of mathematics education and above, with reference to the purposes, objectives, methods and approaches to teaching lead to many problems. The mathematics teachers of the Universidad Tecnológica Nacional Facultad Regional Buenos Aires, in particular those teaching discrete mathematics, face a double challenge with students whose average level of preparation, knowledge and attitudes still are questionable as university students and other topics must have the level and quality of study in each subject deserves.

The dropout rate is a constant disappointment that impacts not only on the educational mission of the department but on the university institution. To attempt to rectify this situation, it is necessary to provide innovative teaching where the educators assist the growth of cognitive strategies to motivate students to become main players rather than spectators in their mathematical development.

The curricular activities of the course will encourage the potential of each student with the use of new technologies (computers) and traditional methods (tutor system) taking into account the individuality of each student. This research will show that respect of the individuality of each student and the uses of alternative methodologies, such as computer technologies, small integrated pieces of work, will assist in the development of how to surmise, build and design.

Keywords: university, permanency, tutor, teaching, mathematics, learning

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Recruitment, entrance and retention of students to university mathematics studies in different countries or where have all the mathematicians gone?

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Workshop abstract

I have been asked to head a group of five people who are to prepare a survey on the "Recruitment, entrance and retention of students to university mathematics studies in different countries" for ICME 11 in Mexico next year. This topic I suspect was predicated on the assumption that numbers in mathematics are declining worldwide. The group has already sent a questionnaire to the Delta mailing list and a summary of the results of that are now available on the web.

In my session I plan to summarise these results and to look at the form of the final ICME presentation. But more importantly I would like to use much of my time to discuss the issues raised by the survey. These are

(i) are the numbers of students in mathematics declining?;

(ii) if so, why?; and

(iii) what is being done to combat any decline and how successful have these attempts been?

I will be very keen to hear your comments.

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Transfer of learning in linear algebra

GULDEN KARAKOK and BARBARA EDWARDS

Abstract

One of the main problems in college education is stated to be students lacking ability to transfer of learning. Researchers designed a study to investigate if the same problem apparent at higher level undergraduate courses such as linear algebra. The concepts from linear algebra courses are used in many different courses in mathematics, science and engineering departments. Researchers first wanted to investigate how students' understand some typical linear algebra concepts such as vectors, matrices, linear transformations, eigenvalues and eigenvectors, etc. Then, students' ways of making connections within these linear algebra concepts was explored further. Students' conceptual connections between and among different concepts in linear algebra may or may not be determined by the type of linear algebra course they take. Thus we invited students from different majors who have taken some sort of linear algebra courses. Our preliminary finding shows that certain activities help students make stronger connections among concepts. We wish to share these preliminary results. We hope theses findings will help educators improve the linear algebra curriculum.

Teaching university mathematics and coaching youth soccer

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This note reflects on similarities between best practice in teaching mathematics and in coaching children's soccer. The focus is on two examples: The key role, importance, and design of problem solving activities, and the necessary restraint on what kind of feedback and corrections to provide in a student centered classroom or practice. A central observation is that, in the end, learning and teaching in no matter what subject, involves the same human organ. Hence one may find helpful ideas and solutions for teaching one's subject even in domains that appear very distant. Indeed, as laid out in the author's personal experiences, sometimes it may even be easier to become aware of, understand, and improve on learning issues when experiencing them in a very different discipline.

Keywords: Problem solving, student-centered, best practices.

1 Introduction

This article presents a different twist of the popular saying [1] *"from sage on the stage to guide* (or *coach*) *on the side"*. It reflects on similarities between best practices in teaching mathematics in a student centred environment, and in coaching youth sports, with focus on children's soccer. It is strongly motivated by the very similar misconceptions about either side: The colleagues and even the instructors in the coaching licensing courses, much like the general public, thought of mathematics teaching as being mostly about supervising the memorization of recipes and drill of algorithmic procedures. On the other hand, fellow mathematics instructors and e.g. colleagues at international conferences related to mathematics education thought of coaching youth soccer as being all about repetitive drill exercises designed to shape *muscle memory*, and at older ages, about perfecting the execution of a library of predesigned combination plays. Moreover, many more experienced, and more reflective members of either side claimed that, unlike the other side, effective teaching/coaching in their own area is mainly about designing problem solving activities.

As a research mathematician, with a secondary interest in undergraduate mathematics education research, the author found that this is more than just an amusing coincidence. As a student in coaching license courses, and as a coach routinely designing activities for each practice, he observed that this is a two-way street. The two groups are very dissimilar: On one side there are university students and primarily mental exercises in mathematics. On the other side, there are consider children of ages 8 through 12, and primarily physical activities in soccer. But it is precisely this contrast that makes it even more compelling to reflect on the common principles of effective learning strategies. Almost certainly there are deep reasons why in either setting the professional organizations seem to converge on very similar recommendations. While it is fun to speculate about such deeper reasons in terms of the make-up of our brains and general learning theories, this is beyond the scope of this brief note. Instead, we shall be very descriptive and practically oriented.

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This article does not present hard statistical data, but the author has considerable anecdotal evidence that either practice, together with abstraction and reflection, has considerably improved the effectiveness of both his mathematics teaching and his soccer coaching. This includes superior success (retention, being selected by top graduate schools, or by elite teams at the teen level) as well as numerical student teacher evaluations, and verbal feedback by players, parents, and other coaches. From the point of view of mathematics education researchers, or physical education researchers, these observations may not come as a surprise. Yet the teacher working in the ditches is often torn between heeding traditions and implementing lofty recommendations that come from education researchers and which often are considered controversial among students, parents and colleagues.

Observing the positive results of implementing the recommended principles in such different arenas has made the author feel considerably more at ease with following the recommendations, even in the presence of considerable local antagonism. He hopes that by sharing this experience, others, too will find strength to continue the path of implementing the recommendations based on research.

Since this note is primarily addressed at mathematics educators, and mathematics education researchers, the next section first gives an overview of guidelines for coaching youth soccer by the top professional organizations. This is followed by comparisons to teaching strategies in mathematics, and selected more detailed examples of teaching situations. The final section summarizes our observations, and makes a few suggestions for deeper reflection, and for looking beyond the traditional narrow disciplines for proven methods for effective teaching and coaching.

2 Coaching youth soccer in the United States

2.1 Youth soccer in the United States

Soccer is widely considered the world's most popular sport, as measured by the number of active participants and spectators. However, it has arrived in the United States only comparatively recently. At the professional level it cannot compete in the U.S. with the highly developed traditional sports with their intricate commercial networks. However, at the youth level, already in the late 1990s soccer became the number one sport (measured by number of active players) in the United States, too.

While in most parts of the world, soccer is played everywhere in the streets, parks, plazas, on the beaches with often only minimal equipment and formal organization, and still predominantly by male players, the situation in the United States is quite different.

Unlike, say, the continental European countries, in which schools and universities concentrate on academics, the United States schools and colleges have a long tradition of fielding highly competitive athletic teams. These not only usually are major commercial enterprises by themselves, but they also come with a long tradition of athletic scholarships. Given ever more sky-rocketing costs for private schools and colleges, such athletic scholarships, which may be worth thousands, even tens of thousands of dollars, are by many seen as a unique enabler for accessing a postsecondary degree. Historically, the male dominated team-sports with their huge rosters (up to a hundred players) such as American Football have been the biggest providers of such scholarships. However, in 1972, the famous *Title IX* of the Education Amendments of 1972 [2] completely changed the playing field. Among many other changes, it requires that colleges and universities give females equal access to athletic scholarships. This has resulted in the termination of numerous male sports, and the creation of many female teams. The natural counterpart to the male American Football teams are female soccer teams at colleges across the continent.

In view of this background, it may not come as a big surprise that parents are willing to invest large amounts for competitive soccer programs for their children. Typical costs for even twelve to fifteen year olds in competitive teams are anywhere from USD 1500 per year, and up, plus additional costs for interstate tournament travel.

2.2 Coaching education and best practices

Given the ensuing high demand for competitive coaching, it is natural that the professional organizations, foremost the United States Soccer Federation (USSF), have instituted carefully monitored professional training and licensing programs for coaches. The desire to compete and win at the highest levels, together with a large community of researchers in university physical education and related departments, provide ample resources for science based guidelines. A detailed discussion of the literature in this area is beyond the scope of this note. We refer to the *"Player development guidelines"* as in the USSF's "*Best Practices For Coaching Soccer In The United States"* [3], which have been written in consultation with a broad spectrum of academic researchers in kinesiology, psychology physical education, sports sociology etc. Notable primary references are [4,5], which advocate e.g. *"athlete centred coaching"* with the objective of *Developing Decision Makers*. Through the affiliated state organizations, the USSF offers a multi-tiered system of coaching education and certification courses. These start with weekend long seminars, and extend to multi-week, all day courses. They combine both theoretical and practical (on the field) instruction and examinations.

As a candidate in such courses, the author was particularly intrigued by the careful balance between technical content and pedagogy, which reminded him most strongly of the heated controversies about the corresponding proper balance between content knowledge and pedagogy in mathematics teacher education.

The lower level USSF coaching courses very much focus on pedagogical issues, what is age-appropriate, and what characterizes effective practices (a.k.a. lesson plans). The analogue of *content knowledge* plays a much smaller role: Most coaching candidates have substantial technical knowledge from an earlier career as a professional player, and continuing technical education relies strongly on constant collaboration with ones' peers. Possible mathematical analogues would be future teachers have prior work experience applying mathematics in research, industry, business, government jobs, before training to become mathematics teachers, and teachers continually learning from their senior colleagues as e.g. described by Liping Ma [4].

Both as a candidate in coaching courses and as a practicing coach, the author found many analogies with his idealized teaching of university mathematics. Many of these involve rather *common sense* issues about a dignified treatment of the students/players, dealing with physical limitations (available space and time), the importance of a deep understanding of how different topics are interrelated and how this affects the order in which they can be learnt, etc. However, this note shall focus on two specific examples: The primacy of *problem solving activities* and clear *focus* on *one topic at a time* with strictly tailored immediate *corrections*.

2.3 Problem solving

Many outsiders (and unfortunately, some students, coaches and teachers) think of endless repetitive drills as the core of practices in sports and in mathematics classes, alike. While we cannot speak for other team sports (but doubt that it is correct for these). Indeed, a key design principle for soccer practices is to build these around activities that require continuous development of problem solving skills.

The obvious motivation is that there are no two game situations that are exactly alike. Moreover, without timeouts or electronic communication devices to obtain directions from the coaching staff (as prevalent in some other team sports) the game of soccer is distinguished by the requirement for continuous decision making by the eleven individuals on each team.

Due to evolutionary reasons humans may have a natural inclination to categorize and systematize all possible game situations and develop catalogues of automatic responses. However, the beauty of the game and the reason why it continues to attract hordes of spectators may well be attributed to its defiance and resistance to be confined in such ways. (Think ahead: what makes mathematics so beautiful and attractive?) Indeed, up to the very highest levels of international competition, the most common complaint is a widely deplored lack of *creativity* of, in particular, the US men's National team, compare e. g. the USSF guidelines [3], or the widely popular recent article [7]. This stands in stark contrast with the broadly acknowledged technical perfection and amazing athletic capabilities of the players.

Guided by this final objective, the recommendations call for developing creative problem solving skills right from the beginning. For obvious developmental reasons it is clear that at the younger ages (e.g. age eleven and younger) hardly any instruction is about tactics (and none about strategy), instead the primary focus is on developing basic ball skills. But this is to be done in an environment encouraging creativity and problem solving. The word *drill* is an anathema: *drills kill creativity*. Following the same principle for the aspiring coaches as for the players, coaches are not given a manual of sessions. Instead, the focus is on design principles, and coaches are asked to be creative themselves!

At its easiest, we want practices to not repeat activities. Instead, every week we have exercises with different rules. At the younger ages these almost always take the form of some game, with rules rigged in whatever way to make the players learn a new skill or technique. A fundamental design principle (heeding the rule of the *three Ls:" No laps, no lines, no lectures"*) is to start with incomplete instructions. We want the players to start exercising with the ball as quickly as possible, further instructions are added with time. While initially considered frustrating for the players, this simple principle does an amazing job at training players to come up with creative interpretations of the rules, make new rules, and create unscripted solutions. We invite the reader to pause for a second, and reflect whether this applies to mathematics instruction? Can we do it? Should we do it? Do we do it?

A more specific example in the same spirit was recently nicely reiterated by M. Beale who works at the Chelsea FC Academy (Chelsea FC in London is one of the premiere clubs in professional soccer in the world):

A crucial idea in coaching young players is to introduce an element of choice into drills and practices. For example, instead of telling a player simply to dribble up to a cone and shoot, put another player several meters to his right or left and offer the attacking player the choice of shot or lay-off. This approach encourages individuality and self-expression and helps fight against the cookie-cutter mentality. Ultimately, this will help develop a player's flare and confidence.

If we want to foster creativity, make innovative choices, we must accept that many of these choices are not the best ones, many may even quite bad choices. One cannot expect anyone to freely explore new avenues if there is any fear of punishment. On the soccer field, this means constant encouragement for bold choices and questioning about whether the player thinks this was a good choice. During competitive games, this primarily involves discounting numerical losses, making players and parents feel comfortable with loosing a game today, but having learnt a lot by making some bold choices. Especially at the youngest age, there is widespread agreement that one should deemphasize counting goals and wins as much as possible—as a focus on winning all too often only impedes creative player development.

In a nutshell, we never say "*never do this*", we do not prescribe scripted actions, but instead encourage making choices and trying unconventional solutions starting at the earliest stages. This approach is strongly motivated by designing the curriculum from the final objectives backwards.

2.4 Focus and corrections

As a 20 year veteran of teaching mathematics at the university level, one of the hardest items to learn for the author as a soccer coach was not to try to correct every mistake all the time. Indeed, learning from his soccer coaching courses, he has changed his approach to teaching mathematics!

A fundamental design principle of any class/practice is to clearly identify the topic that is to be learnt in any single session. In youth soccer these may be simple items as good balance (with slightly bent knees), *locking the ankle* when striking the ball, alignment of the shoulders when making a pass, protecting the ball from an encroaching opponent, letting the ball *come across the body* when receiving a pass. The typical session starts with some unrestricted activities that explore the concept to be learnt/practiced, then restricts the space and, in stages, adds pressure (opponents), continues with practicing the concept/skill in game like situations, and ends with small-sided games with rules rigged in a way to foster work on the concept.

As explained above, given limited time resources, the activities start as quickly as possible, and usually with incomplete instructions. The experienced coach looks for *coachable moments*, incidents where a player made a questionable choice, and will immediately interrupt play to question the decisions. Most typically the player is aware of a bad choice, and now is asked to come up with better choices. These are explored and reworked until a desirable performance is achieved. What really impressed this author, and what he found so hard to do, is the emphatic guideline to *only correct* missteps that involve this session's topic: If today's topic is *receiving passes*, then the coach is instructed to only correct mistakes involving receiving passes, and to (from the player's side) completely ignore whether e.g. a player strikes the ball inappropriately, say, with the tip of their toes.

Does it work? The guidelines by the professional organization say so. But just like the mathematics teacher who already *knows what works bes*t, many a coaching candidate ignores these guidelines. This author has seen many coaches discount the guidelines, and in their own clubs try to correct everything at the same time. Anecdotal evidence, players developing much better in some environments than in others, has convinced him that it pays to heed the advice from experts in developmental psychology. We will revisit this item from a mathematics perspective below—but the reader is encouraged to reflect on this now.

3. Teaching university mathematics

3.1. Learning on the job

The author has been teaching mathematics at the post-secondary level for over 20 years—yet he never received any formal training on how to teach until long after he started. Even then, it was purely voluntary participation in workshops which almost counted against him as they took time and effort away from the primary research mission of his institution. Basically, he was thrown into the water, to *swim or sink*.

Today's graduate students are usually a little better off as they are typically required to participate in teaching assistant training workshops. Yet the majority of college and university teachers have precious little formal training on pedagogy, and psychology of mathematics education. Moreover, in many countries, heated debates continue about the proper balance of teaching content knowledge and pedagogy to mathematics (and science) teachers at the secondary (and even the elementary) level.

This author acquired his fragmented understanding of how to teach mathematics in a long and painful trial-and-error process, augmented by voluntary participation in workshops, courses, and conferences which are in tight competition with efforts related to his primary research area. Very limited time is available to go beyond this, to e.g. familiarize himself with findings in the mathematics education literature—there are too few executive summaries! Curiously, some of the strongest impacts to consciously redesign the classroom teaching come from outside the discipline, e.g. from encountering *active learning* as in *Workshop Physics*, compare e.g. [8], and *cooperative learning* as e.g. [9].

This experience does not seem to be unusual, and we note that our discipline is lacking systematic mechanisms that ensure that findings from education research are implemented into teaching practice in a timely manner. While at the younger levels, recommendations such as [10] do have measurable effects, at the higher college levels, the time scale seems to be measured only in generations. Let us consider mathematics analogues of the items highlighted in the previous section.

3.2 Problem solving

It is hard to find someone who does not agree that mathematics is about problem solving. However, once we try to more precisely nail down what we mean by *problem solving*, the agreements quickly end. This author shakes his head at the designations given to the last part of each section in typical mathematics textbooks, usually *"Problems"*, and only rarely *"Exercises"*. It takes little reflection to note that the large majority of items in these parts of popular algebra or calculus books do not at all satisfy the following characterization [11]:

A problem is only a problem (as mathematicians use the word) if you don't know how to go about solving it. A problem that has no 'surprises' in store, and can be solved comfortably by routine or familiar procedures (no matter how difficult!) is an exercise.

Let us accept that the starting point for curriculum design should be its end: what we want the successful graduates to be able to do and to know. One may question the importance of problem solving skills in mathematics. This author takes the stance, that, in particular, with the virtually ubiquitous access to networked universal information systems, problem solving skills as in Schoenfeld's characterization above [11] are becoming ever more important in mathematics. On the other hand, drill of manually executing routine algorithms becomes ever less important—as the very straightforward nature of traditionally taught algorithms makes them predestined to be programmed on and executed by machines.

If we accept this view, the question arises how to most effectively design mathematics classes at all levels. There has been much controversy and discussion over the balance of drill of basic *manual manipulations* versus other contents such as conceptual understanding and problem solving skills. One of the most fervent opinions about abolishing traditional items was elaborated in [12], and it appears that this opinion is standing the test of time! Certainly, the large majority of the secondary and college entry-level mathematics courses taught around the world emphasize drill over problem solving, even while there are calls from all sides to devote more attention to problem solving. But how to do this?

This author took lessons learnt while coaching soccer into his classrooms (and into activities with his children when at the primary school level). The first guideline is so simple: start right away, typically with incomplete instructions! Initially the reaction is one of frustration, sometimes even anger—but it does not take long and students at all ages turn out to be a lot more resourceful than many a teacher may believe. No long lectures, long demonstrations, detailed worksheets—instead, start student centred activities with only rudimentary instructions. In the last few years, the author has been mostly teaching courses at the upper undergraduate level such as *Advanced Calculus / Intro to Analysis*, *Abstract Algebra*, *Intro to Topology*. At this level, the key is to reverse the traditional order of axiom - definition - lemma - theorem -proof - example. Instead, we start with a vaguely circumscribed setting, some intriguing observations and invite exploration and conjecture. Theorems are proposed, but they do not have precisely stated hypotheses—instead the hypotheses, and often even good definitions are outcomes of the process of trying to prove a theorem that captures the main idea. At this first sight this approach may appear to be opposite to the *Moore method*[13] but our teaching experiments, even in Moore's *home turf* of point set topology indicate otherwise: The common goal is to foster problem solving skills by providing suitably challenging activities. Over the last years we have experimented with various details of this approach—but the highly successful classes attest to the focus on *problem solving* being the key. A critical ingredient for making these successful is to establish an environment of *no fear*, of mutual respect for even the wildest ideas. In practical terms, this involves the whole class, or small groups engaging in prolonged brain-storming sessions during which no criticism is allowed. These are followed by periods in which all ideas are explored—and quite often we develop ideas and proofs on the board which turn out to be dead-ends. These are not at all futile efforts, in some sense they are so much more important than the polished proofs presented in the textbooks. Credit is always given for the creative ideas!

As a simple, easily accessible example from (advanced) calculus consider the inverse function theorem, which captures the intuitive picture of a function being locally invertible near a point where its graph has a nonzero slope. The mathematics problem (in the advanced calculus class) is to make this picture into a theorem, and prove it. The picture is quick, and provides incomplete information. Indeed, one of the main challenges is to decide what should be part of the hypothesis, what be part of the conclusion—the main issues are regularity assumptions on the function and the existence of the inverse. A sampling of a handful popular textbooks reveals almost any combination of such choices.

Let us end with one practical example from arithmetic: This father all too often was annoyed with the rigidity of his children's math worksheets. One extreme example involved long division. Rather than arguing its intrinsic merits and importance [12], our point here is its similarity to the above described repetitive soccer drills. Just like in a game situation, there should always be choices that have to be made. The easiest, and likely most natural one for any mathematician is to rewrite the division problem as a fraction, simplify it by cancelling common factors, or by multiplying numerator and denominator by powers of two or five, and then work a simpler division problem, if still necessary. Yet, the author's daughter insisted that this was not allowed—she had to exactly follow the prescription, no creative shortcuts were allowed.

In summary, starting from the ultimate objectives for the graduates, the author considers the development of creative problem solving skills one of the most important tasks in mathematics teaching. In order to achieve the goals, he consciously chooses activities that demand creative work, making choices, exploring without fear of punishment, and at the same time absolutely minimizing rote repetitive drill exercises.

3.3. Clear focus: one item at a time

It may sound so simple—focus on one item at a time. After all, don't our textbooks and syllabi typically specify one single topic for each class? This is comparatively easy for lectures. For student centred classes the first major challenge is to break the tasks into bite-size chunks—which can be attacked one at a time. See [14,15] for a masterful implementation in a modified Moore style of such break-up, and a detailed discussion and reflection on these challenges.

Yet, in his experiments with student centred classrooms, this author *did not get the message* until he consciously reflected on his correcting players as a soccer coach: Indeed, he found it exceedingly difficult to only correct mistakes (or rather interrupt for questionable choices) that were directly related to the particular session's topic. It is very hard to quell the urge to admonish any unrelated "*why did you not take the shot?*", *"did you not make that open pass?"*, *"do not kick with your toes"*, *"use the other foot," "let the ball come across your body?*", ... Indeed, the routine way focuses on one topic only, and uses questioning as opposed to *telling* what to do: "*if you could do it over again, would you make the same choice?" what else could you have done?*"

Yet personal experience convinced this coach that the professional guidelines were right after all, and he eventually learnt to follow them.

The next step: Reflect how this relates to teaching (in this case, university level) mathematics. Clearly we focus on one topic at a time (do we?). In lecture-format classes one is to avoid distracting digressions. But what about teaching in the modern student centred classroom, where students work in teams, present work on the board, routinely interject ideas, some of which are really far off the wall? Once he realized the analogies, this author caught himself looking for all possible kinds of mistakes and less than perfect choices at the same time, wanting to correct all of these all the time. In USSF soccer coaching education, this is an absolute no-no, and a guarantor for failing the examination, not earning a coaching license.

It is a long way from recognizing a likely deficiency (trying to correct everything at all times), to mediating it. The author has consciously tried hard to not only clearly identify the technical mathematics topics that are the subject of each class' meeting, but also the items that shall be subject to discussion and correction upon student discussion and presentation. Typical items, again at the levels of *Advanced Calculus*, *Intro to Topology*, *Abstract Algebra*, and, most recently, a *Second Course in Differential Equations* include writing, arithmetic and basic algebra, logic (implications and their converses, alternating quantifiers), meticulous attention to all hypotheses (especially regularity assumptions), utilizing (building on) previously established results. While purely anecdotal, the author has become convinced that in the student centred mathematics classroom—just like in the youth soccer practice—learning success substantially increases if the instructor does not try to fix all problems at the same time. In his most recent topology class, he informally laid out a parallel time-table for correctible items as above. These include e.g. grammar and punctuation, slang versus formal language, proper use of prepositions for mathematical relationships, not using any symbols that have not been quantified before unless they have been assigned a value). This accompanied the official syllabus which focused on technical topological structures. Rather than being a linearly ordered sequence, it naturally had a semester-long, more circular structure with various items being revisited repeatedly. A typical example is the simple logical structure of most compactness proofs that use the open cover definition: While being addressed relatively late in the semester, this is a typical point where to again focus on the logical structure of the arguments, especially the innocent looking part *"every open cover of ..."* which all too often is incorrectly employed in various proofs. In practical terms, this simply means to develop a set of activities involving items that addresses both related logical issues and some compactness property. In presentations and discussion, we downplay, or even ignore, all sorts of other mistakes, misconceptions, misstatements, and focus on every student getting the key logical issue right 100 percent of the time.

At the author's institution, the second course in Differential Equations (the first course on their qualitative theory) is distinguished by having several different prerequisites: A first (calculation oriented) course in Differential Equations (DE), a first course in Linear Algebra (LA), and Advanced Calculus (AC). The large majority of students in this course have major deficiencies in more than one of these, often due to mismatches to courses taken at the institutions from which they transferred. While barely changing the core syllabus for this class, the author consciously aligned the learning objectives in such a way that students had a chance to make up any gaps in the prerequisites—one issue at a time. While having three such different prerequisites for one course first appeared to cause extra problems, it turned out that this forced the author to consciously reflect on and plan how to bring in each one of these. In daily practice this meant that mistakes in any areas other than the ones focused on were most casually downplayed, or sometimes even ignored. This required choosing the examples addressed in class activities in such a way that *desired* mistakes would likely appear repeatedly, whereas off-topic mistakes be less frequent.

In the end, it remains a major challenge when grading papers (homework, exams, ...) to focus on one designated item to be corrected, and not get side-tracked into marking every mistake at the same time. The ultimate goal is that all students master all topics, satisfy all criteria. It is possible to focus on clearly identified subsets on individual exams. A careful design of problems assigned or test items selected helps a long way, but it will take many rounds of experimentation to perfect this.

In summary, for traditional lecture-style classes it is comparatively easy to focus on one item, and in the wrap-up at the end summarize the one (or three) items to take home. In student centred classes, a critical component is learning from one's mistakes—and it is a real challenge for the teacher (coach) to focus on correcting only one kind of mistake at a time.

4. Summary

This article reflected on similarities between effective strategies for teaching mathematics at the postsecondary level and coaching youth sports. Rather than attempting deep explanations in terms of general learning theories, the focus was on practical, day-to-day activities. We focused on two specific topics: the desired focus on activities that enhance problem solving skills, and on the desirable habit of the teacher/coach restrict and focus her/his feedback on one item at a time.

Whereas neither of these two individual parallels may prove truly important, we hope that this note encourages all teachers to become ever more open-minded, and actively search for insights into what constitute best practices, and for innovative teaching strategies in even the most far-fetched and most remote places. In the end, teaching and learning of no matter which subject all involve the same human organ!

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Dynamic visualization in advanced undergraduate courses

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This article presents and discusses selected uses of interactive dynamic visualization tools, such as the *JAVA Vector Field Analyzer II* in advanced undergraduate and beginning graduate courses, foremost advanced courses in differential equation and dynamical systems.

Keywords: Dynamic visualization, Lyapunov, Poincare-Bendixson.

1 Introduction

The advent of inexpensive computing technology has dramatically changed the way many mathematicians do mathematics, and the way many students and teachers learn and teach mathematics. This is an ongoing revolution, and no end is in sight. Among the many different ways that computing technology may be employed in mathematics, one of the most intriguing ones is for interactive and dynamic visualization. It started in the 1980s when then first personal computers became available, with such visionary work as e.g. [1]. The technical revolution continues at an ever faster pace. Nowadays one needs little more than a JAVA-enabled WWW-browser and an internet connection to do truly exciting mathematics. And we still are looking forward to many possibilities that currently we only dream about—such as instant computations and rendering in 3D. The author's dream is an instant computation of curvature and geodesic spheres for control and subRiemannian geometry, manipulating theses instantaneously by dragging the mouse.

The technology has invaded classrooms in different areas and levels at very different rates, for reasons that we may only speculate about—but this is not the place to do so. Likely the late secondary and introductory post-secondary level are among the leaders with topics such as parameterized families of functions in precalculus, calculus, and the first course differential equations. Both at the earlier and the later stages we seem to see comparatively less use of computing technology for interactive visualization. Similarly, there have been many formal studies about the benefits of such technology and the utility of visual images for developing concepts primarily at these same levels centred about calculus, but also some others, including at the postcalculus levels—we only refer to the recent study [2].

One common issue that severely restricted the use of such technology was the, real or perceived, start-up costs. Computer algebra systems and similar packages are widely perceived as demanding very high start-up costs. These include both the learning of syntax and the actual launching of the program. A notable opposite is the well-known *Famous Curves Applet* [3] which requires virtually no preparation, yet invites almost limitless experimentation. In our context of an advanced course in Differential Equations, we mainly rely on the similar JAVA applet *Vector Field Analyzer II* (VFA2) [4]. It is based on an earlier program [5] that simply tried to allow experimentation in response to the question [6]:

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"If zooming is so much better (than disappearing secant lines) for understanding derivatives in calculus I, why not zoom in on vector fields to see and understand the curl and the divergence?"

This applet also includes additional functionality to visually connect vector calculus and differential equations. The key is to experiment with the flow of a vector field (or differential equation) acting on a solid region (as opposed to just plotting integral curves for initial conditions specified as discrete set of points.)

There are many similar dynamic visualization packages available for differential equations and dynamical systems, compare e.g. the extensive suite of the ODE Architect [7] or the many resources at the Dynamical Systems and Technology Project at Boston University [8]. We like the VFA2 for its instantaneous start-up and trivial syntax, while inviting new kinds of experiments far beyond the original design.

There does not exist a comprehensive write-up of suggested experiments and uses of this tool at the level of vector calculus and the first course in differential equations, but many users around the world have found their ways of employing it. This article and presentation is focused on unanticipated uses in higher level courses, foremost the advanced undergraduate or graduate level theoretical courses on qualitative studies of differential equations. We also have used the tool very effectively in powerful ways in graduate level courses in complex analysis (no surprise, given the ideas in [9]), differential geometry and control theory.

Such higher level courses usually have very densely packed syllabi of theoretical material, and high expectations on the students' prior knowledge. Thus it is no surprise that many an instructor does not believe that it is possible to squeeze in some computer work. However, we found that with the extremely quick start-up we routinely use the VFA2 now also in many graduate courses, as each demo or experiment only takes a minute or two, or even less, yet imparts such powerful images that many students indeed did not bring with them, but which are the foundation for many modern theories, compare e.g. [10].

Typically, we employ the tools as a *backdrop* for a class discussion, often to perform experiments suggested by students. This usually takes much less time than a minute. Where available, we also have used the tool in small group settings where each table of students has its own computer. Moreover, students use the tool routinely at home for quick explorations and cross-checking. Distinctively different from common uses of e.g. computer algebra systems or graphing calculators in calculus classrooms, our experiments are never scripted. Instead, the tools are used instantaneously when a mathematical question comes up which is such that the students *want to take a closer look*.

2 Existence of periodic orbits in one dimension

Our first example is from the first chapter of the textbook [11], which is a modernized version of the classic [12]. It takes into account the influences of three decades of technology on the modern study of differential equations. In its modern style it immediately sets the stage for the topics to be studied later in the course, and for the character of the modern approach. Here we consider the logistic differential equation subject to periodic harvesting

$$y' = ay(1-y) - h \cdot (1+\sin^2(2\pi x))$$

This example follows a first foray into the world of bifurcation theory. More specifically, if e.g. $a = 1$, then the logistic model with constant rate h of harvesting $y' = ay(1-y) - h$ will have two distinct positive equilibrium solutions if $0 \leq h \leq \frac{1}{4}$ corresponding to sustainable harvesting. These coalesce into a single one at $h = \frac{1}{4}$. If $h > \frac{1}{4}$ then every solution that starts with a positive initial population faces extinction in a finite time.

Most students have seen the logistic model without harvesting in their first course in differential equations. Already for the case of constant harvesting $y' = ay(1-y) - h$ it is quite challenging to obtain closed form solution formulas. Even computer algebra systems such as MAPLE require quite sophisticated tweaking before they yield all solutions (on both side of the bifurcation). Indeed, this example beautifully coaxes the students towards more graphic arguments and qualitative studies.

The next question, which really is a lead-in into Poincare-Bendixson theory studied at the end of the course, asks whether the equilibria persist as periodic orbits if the harvesting are and oscillations are small. The teacher's objective is to encourage topological ways of thinking. However, the author's students had very little experience thinking graphically.

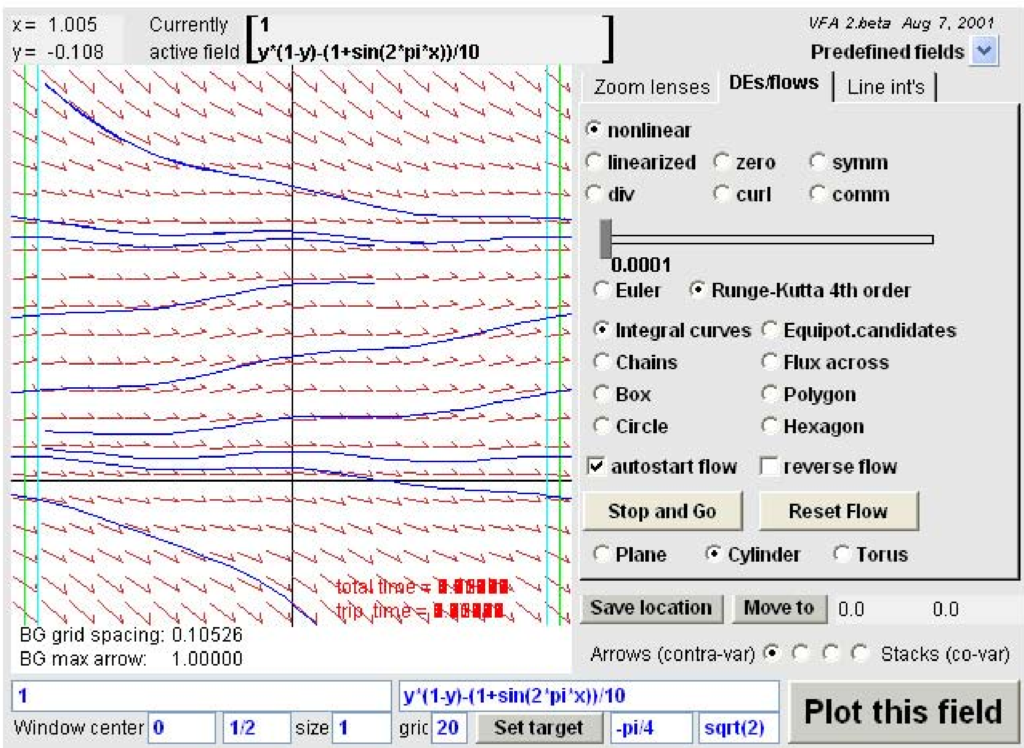

Figure 1. Existence of periodic orbits for logistic growth with periodic harvesting

Starting up the VFA2 is instantaneous, enter the formula for the differential equations, press plot, and scatter some initial conditions with the mouse. The only advanced feature is to turn on the cylindrical model for a state-space which is the natural home for periodic vector fields. This means that as solutions leave on the right edge they come back on the left. One or two minutes of experimenting with different initial conditions led most everyone in the class to have a strategy in mind on how to prove the existence of at least two distinct periodic solutions, compare the static screenshot shown in figure 1. Given just this little help with the VFA2, a few minutes only, these students basically discovered the concept of a Poincare map on their own. Still having almost the whole class period available, it was now very easy and fast to develop the theory in detail.

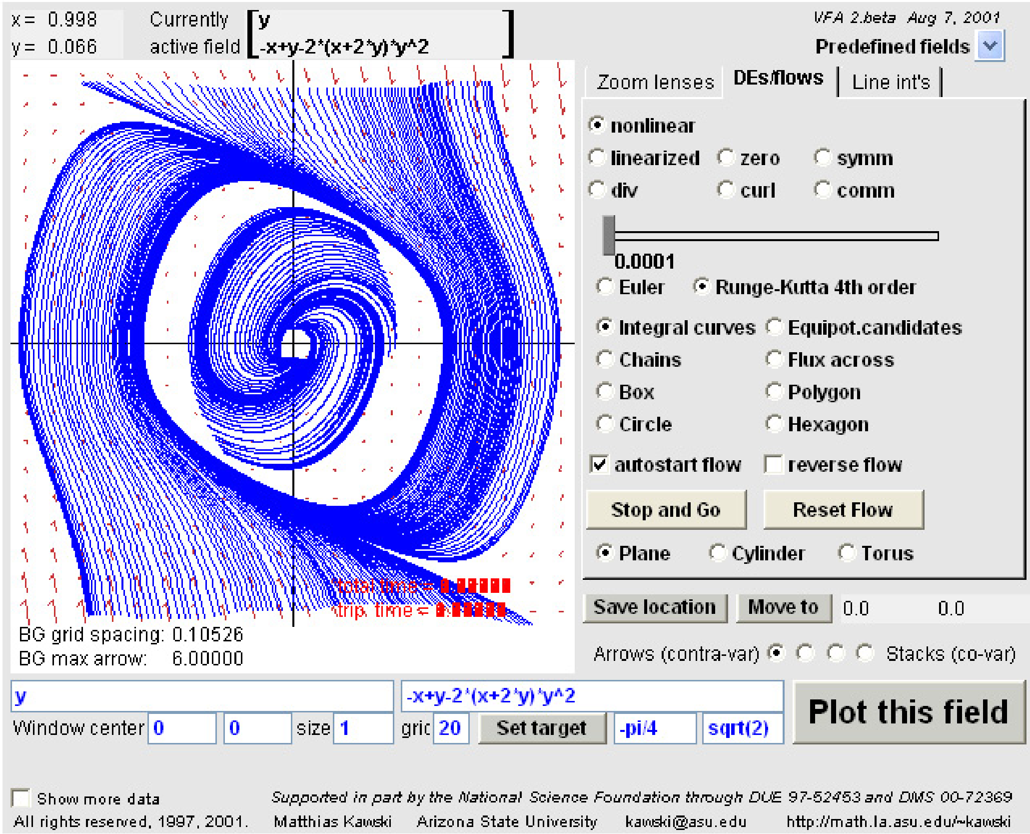

Figure 2. Poincare Bendixson theory

3 Poincare Bendixson theory

At the end of the semester, our senior and first graduate level course in differential course studies the existence or lack of periodic orbits for planar systems. The main tool is the Poincare Bendixson theorem, compare e.g. [13].

Theorem 3.1. Suppose that $D \subseteq R^2$ is open and connected, $f : D \mapsto R^2$ is locally Lipschitz continuous. Suppose C is a bounded positive semi-orbit of $y' = f(y)$ with positive limit set L^+. If L^+ contains no equilibrium point, then it is a periodic orbit.

While the technical statement is quite intimidating, its message is extremely intuitive from a graphical point of view. Unfortunately, the majority of the students in the author's class did not have well developed graphical reasoning skills and were hesitant to use even the little they had. Again we found that it took just one or two minutes to start up the VFA2 and provide some stunning and compelling dynamic images that clearly demonstrate what this theorem says, and how it is typically employed. Figure 2 shows a screen shot for the following example taken from [13, p.309]

$$x' = y\,, \quad y' = -x + y - 2(x + 2y)\, y^2$$

We found that after students could play, modify, and experiment so effortlessly with various systems on the VFA2, they found it easier to systematically work the usual exercises, to proceed strategically, establishing the invariance of some bounded region (usually topologically an annulus) that does not contain any equilibria. Moreover, the visual experiences did not only have helped with the acceptance of the theorem, its application, but also with the understanding of the outline of the proof which is an indispensable item on the syllabus.

4 Absence of periodic orbits: Bendixson Dulac criterion

The Poincare Bendixson theorem is routinely used to establish the existence of periodic orbits. Complementary to it is the Bendixson Dulac criterion which is used to establish the absence of periodic orbits, compare e.g. [14].

Theorem 4.1. Suppose that $\Omega \subseteq R^2$ is simply connected bounded open domain, $f : \Omega \mapsto R^2$ and $\beta : \Omega \mapsto R$ are continuously differentiable. If $\operatorname{div}(\beta f)$ does not change sign in Ω and is not identically zero on any open set, then the system $y' = f(y)$ has no periodic solution lying entirely in Ω.

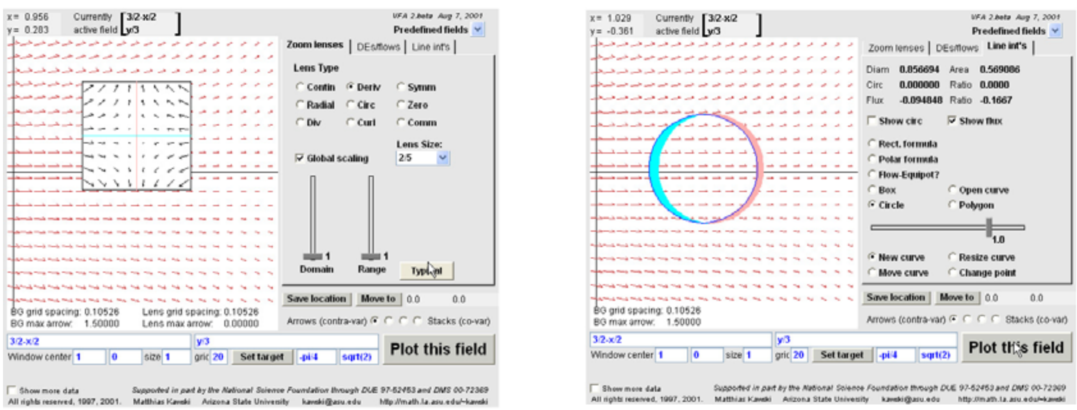

Figure 3. Two graphical views of divergence

Again, for a graphically thinking person this is completely intuitive, and the picture actually provides a feasible strategy for a simple proof, using Green's theorem in the plane. In a nutshell, if C is a periodic orbit, then the vector field f must be tangent to C at all points on C. Consequently the flux of f across C is zero. Hence, by the divergence form of Green's theorem, the integral of the divergence of f over the bounded region inside C is also zero. This establishes the contrapositive of the theorem. The function β changes the lengths of the arrows, leaving the argument intact.

For students who learned their vector calculus in a graphic way similar to the development in [15] which in turn follows the classic physics text [16], both of which motivated the VFA2, this is a very translucent argument. Figure 3 shows two different screen-shots providing the zooming and the flux integral point of view. However, the large majority of our students have much weaker mental graphical images of divergence. Our test case was the vector field $f(x,y) = \left(\frac{3}{2} - \frac{x}{2}, \frac{y}{3}\right)$ whose divergence $\operatorname{div} f(x,y) = -\frac{1}{6} < 0$ is constant negative.

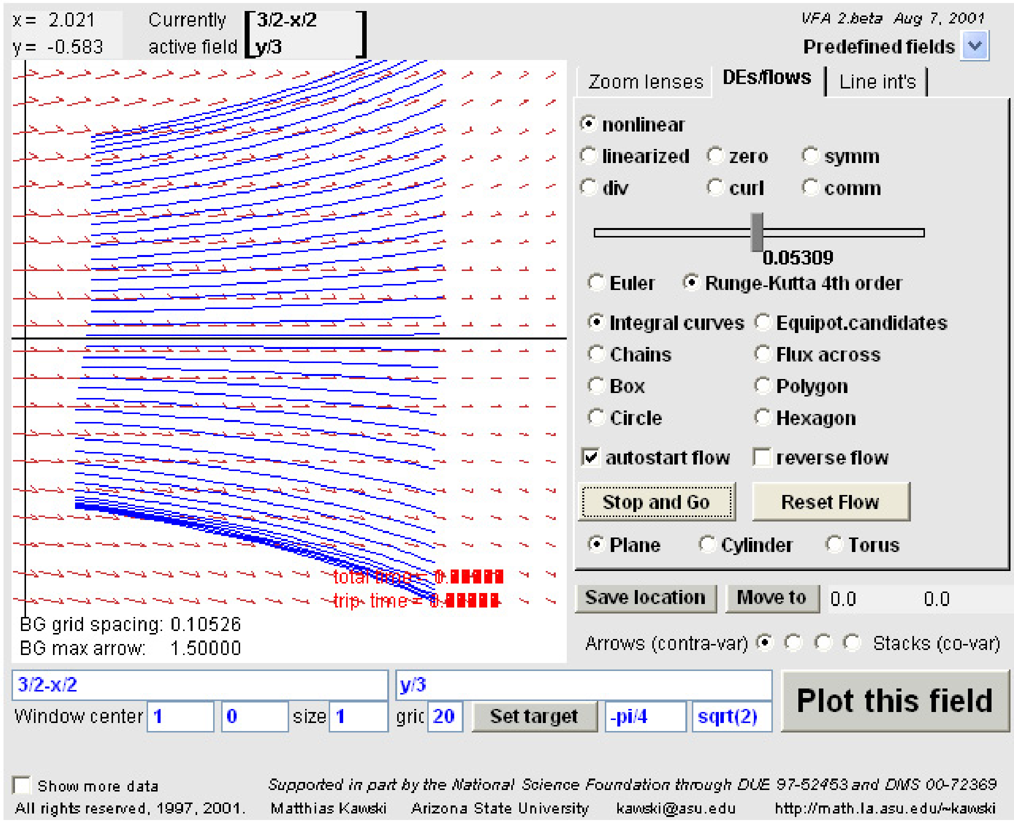

Figure 4. Misconceptions about (positive) divergence

The graphical image of the vector field, with overlaid solution curves of the system $(\dot{x}, \dot{y}) = f(x, y)$, compare figure 4, nicely shows that (in this region) all solution curves go away from each other. Indeed, in the author's class last year, more than 20 advanced undergraduate and beginning graduate students, many with very good credentials unanimously agreed that this vector field must have positive divergence (in the region shown on the screen). This is not at all unexpected misunderstanding.

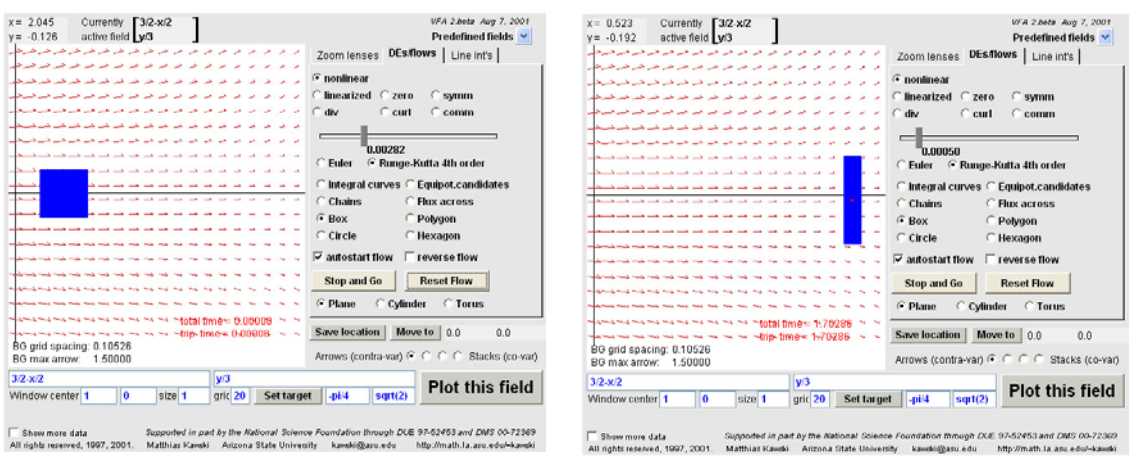

Figure 5. Misconceptions about (positive) divergence

What we did differently was to use the VFA2 for a minute to dynamically visually set the record straight. Indeed divergence does NOT measure whether solution curves go away from each other (a concept more related to stability, and of primary importance in numerical computations), but instead divergence measures the infinitesimal rate of expansion (rate of change of area)—one of the primary objectives for the VFA2. Unfortunately in this paper version, figure 5 can show only two suggestive screen shots and again have to appeal to the imagination of the reader. All students immediately grasped what was happening when confronted with the dynamic version: The vector field is such that solution curves *go away from each other*, but they *slow down* even more. Consequently, a rectangle (aligned with the coordinate axes) of initial conditions stretches vertically but becomes squished even faster horizontally. This results in a net decrease in volume/area.

5 Lyapunov stability, invariant regions, and Lasalle's theorem

Elementary stability theory relies on two basic pillars: The more algebraic approach analyzes the spectrum (set of eigenvalues) of the linearized system about an equilibrium point. The much more graphic direct method of Lyapunov abstracts the concept of passivity and an energy function that is nonincreasing along trajectories. A simple version of the main theorem is:

Theorem 5.1. Suppose $f : R^n \mapsto R^n$ and $V : R^n \mapsto R$ are continuously differentiable. $f(0) = 0$ and $V(0) = 0$, V is proper and positive definite and the directional derivative $L_g V = \langle \text{grad}\, V, f \rangle$ is negative definite. Then the origin is a locally asymptotically stable equilibrium of $y' = f(y)$.

Figure 6 shows pictorially the main ideas, and, in particular, demonstrates the importance that V be proper (i.e. preimages of compact sets are compact), as otherwise V might be decreasing along solution curves, yet these still *run away* (as depicted in the slowly increasing trench). This artfully crafted image has its own story: the key to a *nice* and useful picture is to have asymptotically linear growth, not the expected quadratic growth of typical convex functions. But this is a static image, and it did not help much with the large majority of our students who were weak both in their fragmented understanding of *positive definiteness* and of invariant subsets.

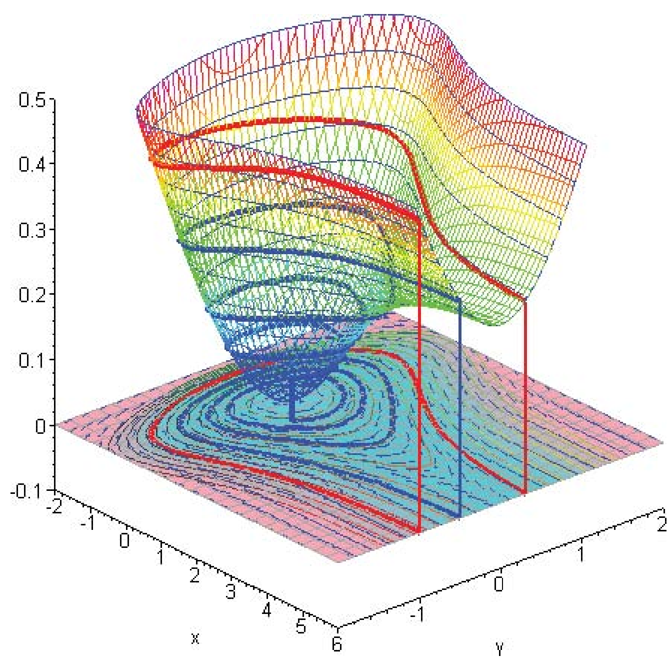

Figure 6. Pictorially: The importance of being proper

Again some *playful* experiments with the VFA2 made all the difference. In particular, we used the visual representation of the flux across a curve to study the invariance of the bounded region inside that curve. In turn, positive definiteness and properness went along with connected, simply connected and bounded sublevel sets, and hence with level sets that are simple closed curves. It turned out to be quite fortunate that the VFA2 only allows to study one curve at a time (on the line integral panel) as this focuses the students' attention. Studying an image that simultaneously shows many level curves is simply too demanding. Of unexpected utility was the feature of the VFA2 that allows one to translate, resize or reshape the curves for line integrals—and thus explore other candidates for level curves of Lyapunov functions. (Mathematically, homogeneity with respect to some general group of dilations often leads to the general argument.)

The VFA2 was even helpful when trying to go one step further, to Lasalle's invariance principle. This relaxes the requirement that the orbital derivative $L_f V$ of V along solution curves be strictly negative definite to negative semidefinite. In turn it requires that the largest *f*-invariant subset contained in the preimage $(L_f V)^{-1}(0)$ be the origin itself. In this case the same conclusion holds. Figure 7 shows one poor (when compared to dynamic experiments) static screenshot of investigation of $V(x,y) = x^2 + y^2$ as a candidate Lyapunov function for a viscously damped pendulum $x'' + \sin x = \beta\, x'$. This illustrates that $L_f V(x,0) = 0 \not< 0$, i.e. $(L_f V)$ is only negative semidefinite, hence calling for the use of Lasalle's theorem. In turn, pictorially it is clear that the largest invariant subset of $(L_f V)^{-1}(0)$ is $\{0, 0\}$.

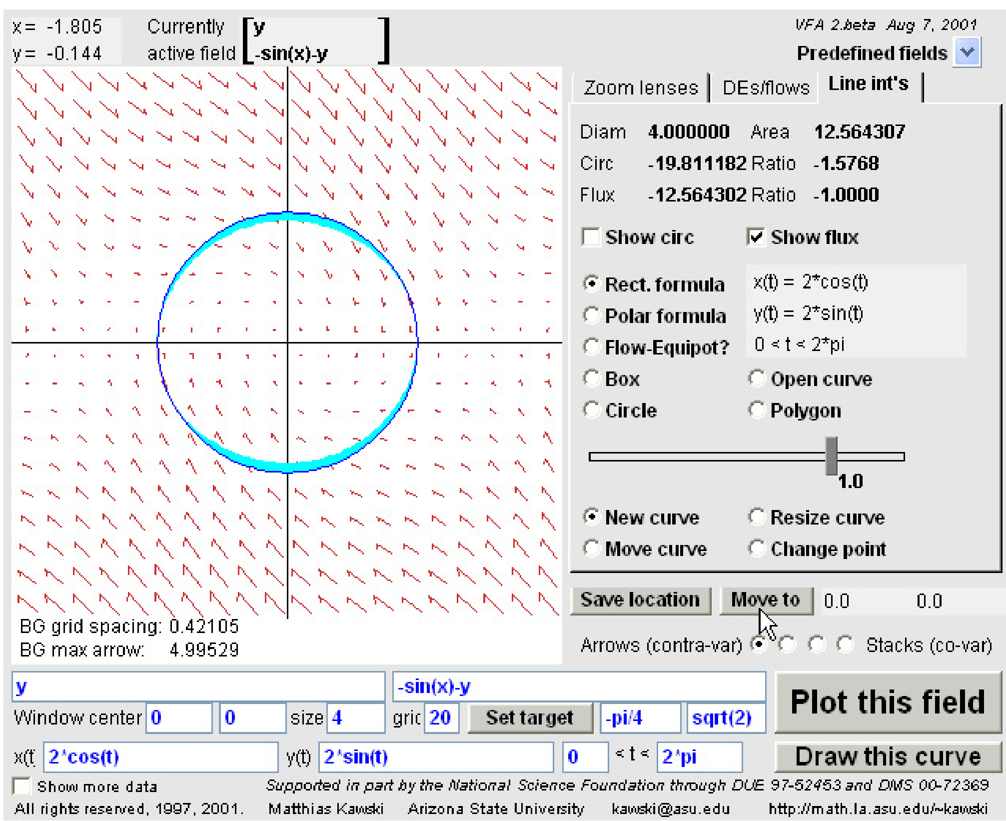

Figure 7. Lasalle's theorem and a viscously damped pendulum

4. Summary

In summary, this article discussed the need for, and usefulness of live, interactive, and dynamic visualization tools in advanced undergraduate and graduate classes. Anecdotal evidence suggests that the students' visualization skills and capabilities are all too easily overestimated. In turn, modern tools such as the VFA2 which was designed for open-ended experimentation may be used successfully beyond lower level undergraduate courses. Critical to make the use of such tools feasible in courses with packed curricula are minimal start-up time (both syntax and launching time). Traditional computer algebra systems have shown to be less practical in advanced undergraduate classes where students did not have much prior experience. Custom-made JAVA tools such as the VFA2 designed for open-ended experimentation requiring minimal investment (i.e. learning how to use them and launching time) appear more suitable for theoretical classes.

Our experience strongly suggests that even in advanced courses with a traditional emphasis on theoretical issues there are many places to employ modern dynamic visualization and experimentation tools.

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Two facets of the use of dynamic geometry in preservice teacher education: learning mathematics and learning how to teach, by making use of Cabri-geometry

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Abstract

It is now generally accepted that pre-service teacher training is a critical issue in the integration of ICT into the teaching of mathematics. This talk will address two facets of the use of the dynamic geometry environment Cabri-geometry in pre-service teacher education for secondary school: as an environment for learning mathematics on the one hand, preparing future teachers how to integrate dynamic geometry into their teaching on the other hand. In both kinds of use of technology, the change in the nature of the representations of mathematical objects and in the ways of operating on them may deeply affect the mathematical activity.

One of the main novelties of dynamic geometry lies in the mediation of the notions of variable object and of variation through the dynamic manipulable representations of objects. In the first part, it will be shown how dynamic geometry may reveal the conceptual difficulties of university students about the notions of differential equation and of solution of such an equation and how the tangible representation of a variable solution may foster more understanding of these notions. In the second part, by means of the example of a year-long teacher preparation to integrate dynamic geometry into their teaching practice, the instrumentation process of the dragging by teachers in order to design tasks promoting mathematical learning will be analyzed.

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Exploring 3D Visualization and Measurement Using Cabri 3D

JEAN-MARIE LABORDE*

Abstract

Cabri 3D is a dynamic math environment with dedicated 3D tools. Presentation will focus on ways to support the development of students' intuitions about formulas and relationships between objects. Explorations and visualizations are enhanced with Cabri3D bringing this time, direct manipulation and dynamic philosophy to the world of 3D-objects.

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Didactic sequence for the construction of the concept of a function limit

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Abstract

Students may appear to understand the delta-epsilon definition of the limit of a function though they can complete the algorithm process without any conceptual awareness. In an attempt to improve this process, a didactic strategy for the teaching of this topic for first year students has been introduced at the Facultad de Ingeniería. This paper will discuss the design, activities and the results of the implementation of this strategy

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The reality of the transition from high school to university

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Abstract

One of the main purposes of mathematics teaching in the primary and secondary level is to develop logical thought processes and a capacity to build upon previous knowledge. Unless this is moderately successful, the student may experience difficulties in further studies at university.

The mathematical content at secondary level connects the student with simplified situations of real life. The development of logical tinking is usually unattended, and the difficulty of mathematical content taking precedence in the learning plan, which often leads to students' tendency to generalise from particular cases. This generalisation includes concepts and procedures, creating epistemological learning difficulties.

The causes of student disengagement with the university are related firstly to the mismatch between the students' expectations and the big demand of a complex university institution, and on the other hand, with the lack of elements and guidance related to an impoverished and insufficient preparedness for tertiary studies.

A frequent complaint is that students 'don't study theory' or 'they don't read the text books'. If learning mathematics is compared to learning a new language, then the domain of that language should be verified, with regard to its rules, symbols and involved logic that can be notoriously different to that of common usage.

General guidelines can be formulated for teaching and learning strategies in common core of mathematics that allow the development of abilities and may solve these difficulties.

Keywords: transition from high school to university

2000 Mathematicas Subject Classification: 97D70; 97C60; 97B20

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Scene analysis as a source of statistical and numerical problems: experiences in informatics' engineering courses

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One of the essential issues in Mathematics Education at the undergraduate level is Problem Solving, due to the close relation this activity has with students motivation. It has been stated that for making mathematics courses relevant to engineering students "the subject has to be made to seem valuable for their own specialisation and future cases" (Wood *et al.*). According to this, many books focus in "real life problems", but in the majority of cases these are physical problems (examples from areas like Mechanics, Electromagnetism, Thermodinamics; etc. had been traditionally used). Despite this kind of examples could be appropiate for some engineering courses, they are not closely related with some other courses, as for example, Informatics. In this paper, we examine Scene Analysis problems for mathematics courses. This is a novel area of Artificial Intelligence in Informatics, but particularly interesting for mathematics education due to its Statistical, Numerical Analysis and Discrete Maths contents. We describe some experiences performed in University Engineering Courses, where problem examples are presented in a simplified version to a group of students and they are asked to solve different aspects of it using statistical tools and numerical methods. The mathematical richness of this new problem, allowed teachers of Mathematics and Artificial Intelligence, to propose interesting projects to the students of Statistics and Numerical Analysis courses.

Keywords: Scene Analysis; Project-work; Numerical problems.

2000 Mathematics Subject Classifications: 15A18; 62F15; 68T45

1 Introduction

Problem Solving is one of the main topics in Mathematics Education at the undergraduate level. Actually, it is closely related with motivation when Mathematics is taught as a service course. For example, Wood *et al.* [1] stated that "To make a mathematics course seems relevant to engineering students –and hence worth an investment of time –the subject has to be made to seem valuable for their own specialisation and future cases".

In most text books "real life problems" means "Physical problems". Moreover, the word "applications" usually refers to different situations of Mechanics, Electromagnetism, Thermodynamics, etc. (see for example Zill [2]). In other cases, this word is used in connection with Geometrical problems [3]. As a result, in Informatics field, it is not easy to choose through real-life problems closely related with their main areas of interest.

Scene Analysis provides interesting problems that can be used to illustrate how to use different mathematical tools in several courses like Statistics, Numerical Calculus, etc.

When Mathematics is taught to University students, problems must be chosen carefully in order to motivate without confusing them. This can be the case when tools and/or concepts which are very difficult to be understood at a certain level are introduced.

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In many cases, an interesting choice is presenting a sophisticated mathematical tool or concept in a preliminary version, immersed in a motivating context. To achieve this goal, these concepts and tools must be in the Zone of Proximal Development of the students [4].

In this experience, real-life problems are used as a source for project-work, at the ORT. University, in the Republic of Uruguay. In this paper, a real-life problem is presented in a simplified version to a group of students and they are asked to solve different aspects of it using statistical tools, numerical methods, etc.

A statistical, analytical or numerical approach is required, depending on the course considered. In ORT University, students of Engineering careers have different mathematical courses such as Numerical Calculus, Statistics and also Artificial Intelligence. In these courses mathematical methods are implemented. Depending on course's methodologhy, sometimes students can choose between the "traditional" assessment (consisting in typical written evaluations) or a project-work. In some courses, they can join in a group of no more than three students where they are supposed to solve a simplified real-life problem, with some help and/or orientation from teachers.

In this experience, the proposed problem must be a non-trivial one, but, on the other hand, it must be an interesting problem with a high degree of difficulty, so it usually needs a Didactic Transposition [5], in order to convert the original problem in a suitable version for university students.

Several examples of this kind of problems for these courses were presented in previous papers ([3] and [6]). In the next section, an even more complicated problem than those already mentioned, will be described. The mathematical richness of this new problem, allowed teachers of Mathematics and Artificial Intelligence, to propose interesting projects to the students of Statistics and Numerical Analysis courses.

2. Numerical Techniques for Scene Analysis

Artificial Vision is a field that had great development in recent years. Different techniques have been applied, such as Neural Networks, Mathematic Morfology, Machine Learning, Clustering and also Numeric Techniques. Despite of this, several problems are still unsolved, specially those related to what is called *scene* analysis [7][8], that means recognizing different objects and their background from an image. Several researchers are working in this unresolved task [7][9][10]. In this research line, lets see how numerical techniques, specially Principal Component Analysis (PCA) can be combined with Neural Networks for this task.

As our goal is proposing general techniques –not only applicable for "toy problems" – we analyzed general case images (real colour of 32 bits) usually called RGB, because each pixel has three components: Red, Green and Blue, and their proportion (the range is 0-255) determines the colour. First, we need to normalize values with a conversion function from three-component vectors to unique values. This function has to fulfil the following properties:

1. The function has to be biyective, as its necessary both normalize and un-normalize.
2. It has to be continuous and monotone, for allowing interpolation.
3. It needs to have a good dispersión level, to avoid bad conditioning problems. [11].

To achieve this points, we used an aggregation model, that can be expressed as a polynomial:

$$y = r + 256 * g + 256^2 b$$

By the way, as every polynom, this conversion fulfils continuity and it is also monotone for RGB inputs. Dispersion level is acceptable, altough it could probably could be optimised using non-linear functions.

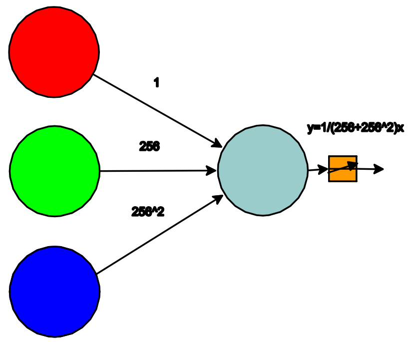

Figure 1. RGB standarizing

In the following phase we proceeded to reduce image's matrix dimension using Principal Component Analysis. This is a technique of feature detection based in statistic measures. The goal is to detect data correlation level, in order to extract data with the higher level of information.

Here we describe the procedure used:

We have a vector of n elements with p dimensions.

1. First we calculate Covariance matrix
2. We apply the following calculation

$$\mathbf{C_x (x_j, x_k) = \sum_{1}^{p} (x_{ij} - \mu_j) (x_{ik} - \mu_k)} \qquad \text{where} \quad \mathbf{\mu_j = (\sum x_{ij}) / n.}$$

This represents the dispersion level of each variable with their means.

3. Now, we need to calculate matrix *eigenvalues*

This can be analytically calculated (power method) or numerically (Givens or Jacobi methods). In our case we applied Jacobi method for performance reasons.

4. With the eigenvalues, now we calculate the *eigenvectors*

$$\mathbf{C_x e_i = \lambda_i e_i}$$

Once obtained, we order them in descendent order, with the result that first elements represent the "maximum energy level" [12]. With this order vector, a fixed (or even variable) number of less significant values are removed.

5. The following step is calculating the transformed matrix

This can be done by using a linear transformation to the eigenvectors.

Calling **E** the matrix of obtained eigenvectors, we calculate the transformed values **y**, by:

$$\mathbf{y = E(x - \mu)}$$

and the inverse process –the original matrix reconstruction, also vectorially expressed:

$$\mathbf{x} = \mathbf{E}^T \mathbf{y} + \boldsymbol{\mu}$$

taking into account that real symmetrical matrixes fulfill $\mathbf{E}^{-1} = \mathbf{E}^T$.

This matrix represents the original image with reduced dimensions. Finally, we use the un-standarizing function described before to obtain the RGB corresponding image.

These images were later used as an Artificial Neural Network input, to recognize simplified images, using the Conjugate Gradient algorithm. but the description of this complex techniques is beyond the scope of this paper. Some of this techniques we have already described in a previous work [13]. In Figure 2 it can be seen several tools implementation of this numerical techniques for scene analysis:

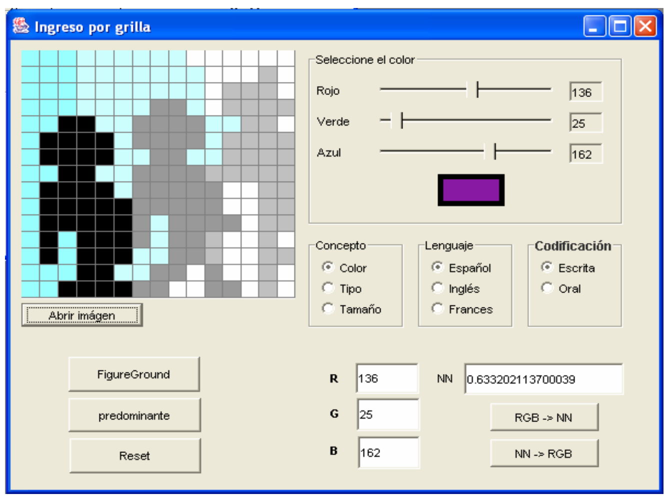

Figure 2. Tool for Scene Analysis

Here we can see the RGB normalize and un-normalize functions used and pattern identification.

3. The educative view-point

The problem proposed in the last section, was effectively used as a source of different tasks, examples and project-work for students. Moreover, this problem can be useful at least for the "islands approach" and/or the "mixing approach", mentioned by Blum and Niss in a well known clasic paper [14].

For instance, the original version has, at least, two different possibilities: the first one consists in using this problem for an Artificial Intelligence course and the second one is to use this problem as a project-work in a Statistics and/or Numerical Analysis course. The following examples illustrate this possibility:

Project-work Nº 1: *Numerical methods in PCA. Eigenvalues and eigenvectors.*

Project-work Nº 2: *Statistical measures for reduce data dimensions. Covariance matrix, functions to use.*

Project-work Nº 3: *Descendent Gradient and Conjugate Gradient methods for convergence in Neural Networks.*

Obviously, there are other possibilities and combinations that can be proposed to the students, depending on the topics to be assessed and the total time to be used for these activities.

As it was mentioned before, this problem and others (see for example [3] and [6]) were widely used in the classes (as a source of examples) in the course notes and in different project-work proposed to the students.

This project-work can be chosen instead of the traditional evaluation (written exercises and problems) in order to approve the courses. It is important to mention that only the students who were approved in the courses (by traditional assessment or by project-work), can be evaluated in the final examinations.

4 Results and Conclusions

In a previous paper, an expert group was consulted about Mathematics teaching and learning at the undergraduate level, focusing in the specific case of Engineering careers [15]. Almost all the experts remarked the importance of teaching significant concepts and procedures in service courses. On the other hand, Engineering students showed an important preference for teachers who make the effort of presenting real-life problems, related with their own careers [14].

In the same way, when students of Statistic and Numerical Calculus were consulted about these courses, they react positively to this style of teaching, were mathematical problem solving and applications are taught through motivating examples. Moreover, they enjoyed working together in project-work, applying the different concepts, tools and techniques to solve them.

Other aspect, very important to be considered is assessment. The evaluative process must not be dissociated from the style of teaching. So, if courses are oriented through problem-solving, modelling (no estoy seguro si es asi, el corrector no la reconoce, pero en un paper anterior figura de esa manera), etc., then, assessment must be carried out in the same way. This purpose can be put into practice through project-work, where students – with orientation of an interdisciplinary team of teachers and lecturers – try to solve real problems of their careers, in order to approve their mathematical courses.

It is important to remark that scene analysis is an excellent source for this kind of problems. Moreover, there exists an important set of real-life problems from these areas, which remain almost unexplored from the point of view of their mathematical education richness.

Searching new real-life problems to be used for project-work in Engineering courses, represents an interesting challenge for engineers, mathematicians and Mathematical Education researchers and, at the same time, it provides a good opportunity for interdisciplinary work in research and teaching.

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Inverse-modelling problems in chemical engineering courses

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This manuscript discusses *mixing problems* in a first course on differential equations for Chemical Engineering students. Mixing problems are easily accessible as they rely on only mass-balance, and they are suitable for a first course as they naturally lead to linear systems. This much facilitates technical issues such as existence of solutions. On the other hand they form a sufficiently rich class that provides for ample opportunities for exploration. In one direction, a typical modelling task involves setting up the systems of equations for different geometries in which tanks are interconnected. This paper works an explicit example in detail, and shows how one may be led to discovering conservation laws. In the other direction, the paper asks whether any given linear system arises from a mixing problem for some geometry of tanks, whether this is uniquely determined, and whether this relationship is stable under perturbations. The answers are shown to involve whether certain conservation laws are respected, and that due to linearity the scales of the tanks cannot be recovered from the equations. The paper continues with the description of related teaching experiments in Chemical Engineering and other related careers. The manuscript ends with some general reflections about including this kind of examples into service courses.

Keywords: Inverse-modelling; differential equations; Chemical Engineering courses

2000 Mathematics Subject Classifications: 97D50; 34A30; 34A55

1. Introduction

In engineering careers courses, differential equations are widely used to solve problems concerned with modelling and so, they may be used to motivate students [1]. Problems involving tanks and chemical reactors (*mixing problems*) provide interesting examples [2] and at same time, these problems are easily accessible for second year university students, as they rely on only mass-balance.

In a previous paper [3], modelling and inverse-modelling issues were discussed simultaneously for chemical kinetics problems. In this article, all the real-life problems will refer to chemical solutions, mixtures, tanks and reactors (i.e., mixing problems).

From the educative view point, it is important to note that mixing problems are more suitable than chemical kinetics ones for a first ordinary differential equations (O.D.E.) course, as they lead to linear systems. On the other hand they form a sufficiently rich class that provides for ample opportunities for exploration in Chemical Engineering and other related careers as Food Technology Engineering and Environmental Engineering.

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The discussion in this paper will deal with the important issues such as existence, uniqueness and stability for mixtures problems in relation to teaching experiences. Conclusions based on the results of the teaching methods used, will be drawn for differential equations courses and other mathematical service courses.

2. A concrete example

The following mixing problem is considered (figure 1), where a three tank system containing water solutions of salt:

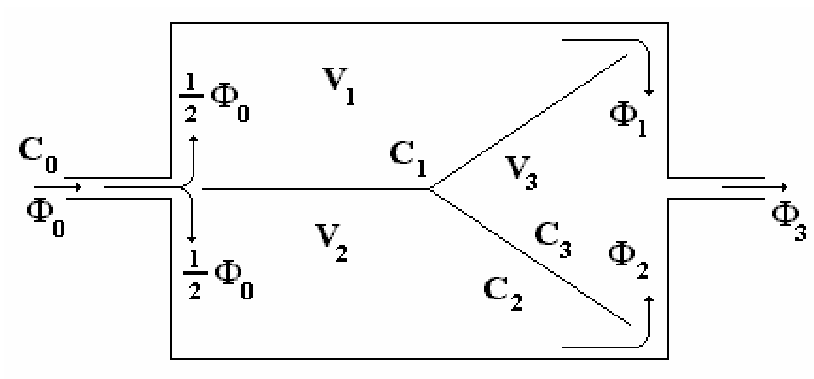

Figure 1. Three tanks system.

In figure 1, Φ_i represents flux of salt/water mixture (in litres per second), V_i is volume (in litres) and C_i is salt concentration in water (in grams per litre or any other mass unit per litre).

As previously shown [2], this problem can be modelled by the following O.D.E. system:

$$\begin{cases} V_1 \dfrac{dC_1}{dt} = \dfrac{1}{2}\Phi_0 C_0 - \Phi_1 C_1 \\ V_2 \dfrac{dC_2}{dt} = \dfrac{1}{2}\Phi_0 C_0 - \Phi_2 C_2 \\ V_3 \dfrac{dC_3}{dt} = \Phi_1 C_1 + \Phi_2 C_2 - \Phi_3 C_3 \end{cases} \qquad (1)$$

Taking into account the following flux's equations (if all volumes remain constant):

$$\Phi_1 = \frac{1}{2}\Phi_0$$

$$\Phi_2 = \frac{1}{2}\Phi_0 \qquad (2)$$

$$\Phi_1 + \Phi_2 = \Phi_3 \quad \text{(and then, } \Phi_0 = \Phi_3 \text{)}$$

The O.D.E. system can be written as:

$$\begin{cases} V_1 \dfrac{dC_1}{dt} = \dfrac{1}{2}\Phi_0 (C_0 - C_1) \\ V_2 \dfrac{dC_2}{dt} = \dfrac{1}{2}\Phi_0 (C_0 - C_2) \\ V_3 \dfrac{dC_3}{dt} = \dfrac{1}{2}\Phi_0 (C_1 + C_2 - 2C_3) \end{cases} \tag{3}$$

The sum of all these equations gives the following one:

$$V_1 \frac{dC_1}{dt} + V_2 \frac{dC_2}{dt} + V_3 \frac{dC_3}{dt} = \Phi_0 (C_0 - C_3) \tag{4}$$

This result is independent of the number of compartments and the geometry of the system. Lets suppose a tank system with n compartments with volumes V_i and concentrations C_i (the internal geometry is unknown). If Φ_0 and C_0 represent flux and concentration at the input, Φ_f and C_f are the final ones at the output, then, equation (4) can be generalised to get this new one:

$$\sum_{i=1}^{n} V_i \frac{dC_i}{dt} = \Phi_0 (C_0 - C_f) \tag{5}$$

The intention is to prove this equation and other features of mixing problems in a subsequent paper.

Engineering students know that each mixture problem has a corresponding mathematical model. This model can be written as a linear O.D.E. system and it can be obtained by just performing mass balances in all the compartments [2]. In class, several questions are frequently posed:

1. Is the converse true?, i.e., if every linear O.D.E. system corresponds to a particular certain mixtures problem, where the geometry and volumes of the tanks (or compartments) can be chosen to fit with the given equations. In other words, they are asking about existence of the inverse-modelling problem.
2. Are these problems about unique? (i.e., if two different problems can led to the same O.D.E linear system, for example, as it happens in mechanic and electric oscillations).
3. What about stability? This question is not asked so often, and depends on previous studies of the student. It should be noted that stability is a new concept for several students.

All these issues (existence, uniqueness and stability) will be analysed here, but first let go back to the three tanks problem considered above [2]. If the following matrixes are introduced:

$$\mathbf{C}=\begin{pmatrix}C_1\\C_2\\C_3\end{pmatrix}\quad,\quad \mathbf{A}=\begin{pmatrix}-\Phi_1/V_1 & 0 & 0\\ 0 & -\Phi_2/V_2 & 0\\ \Phi_1/V_3 & \Phi_2/V_3 & -\Phi_3/V_3\end{pmatrix}\quad \text{and}\quad \mathbf{B}=\begin{pmatrix}\Phi_1/V_1\\ \Phi_2/V_2\\ 0\end{pmatrix} \qquad (6)$$

Then, the mathematical model can be written as

$$\mathbf{C}' = \mathbf{AC} + C_0\mathbf{B} \qquad (7)$$

It is interesting to note that if $\mathbf{V}=(V_1,V_2,V_3)$ is the volume's vector, then:

$$\mathbf{V}^{\mathbf{T}}\mathbf{C}' = \sum_{i=1}^{3} V_i \frac{dC_i}{dt} = \Phi_0(C_0 - C_3) \qquad (8)$$

this is a different form of equation (4), or a particular case of equation (5).

Multiplying by $\mathbf{V}^{\mathbf{T}}$ the other side of equation (7), other observations can be made. Concretely, the following formulas can be obtained straightforward:

$$\mathbf{V}^{\mathbf{T}}\mathbf{AC} = -\Phi_3 C_3 \qquad (9)$$

$$\text{and} \qquad C_0\mathbf{V}^{\mathbf{T}}\mathbf{B} = \Phi_0 C_0 \qquad (10)$$

Both formulae obviously correspond to the right hand side of equation (8).

Moreover, it is very easy to show that:

$$\mathbf{V}^{\mathbf{T}}\mathbf{A} = \begin{pmatrix}0 & 0 & -\Phi_3\end{pmatrix} \qquad (11)$$

only depends of the "final" flux and, in the same way,

$$\mathbf{V}^{\mathbf{T}}\mathbf{B} = \Phi_0 \qquad (12)$$

only depends of the initial flux.

Similar conclusions may be found using equations (9) and (10).

Taking into account all these observations, what happens if a slight change in $\mathbf{A}$ or $\mathbf{B}$ is made?

If $\mathbf{A}$ is changed (for example, in position (1,1)) to this new one:

$$\mathbf{A}_\varepsilon=\begin{pmatrix}-\Phi_1/V_1+\varepsilon & 0 & 0\\ 0 & -\Phi_2/V_2 & 0\\ \Phi_1/V_3 & \Phi_2/V_3 & -\Phi_3/V_3\end{pmatrix}$$

Then, $\mathbf{V}^{\mathbf{T}}\mathbf{A}_\varepsilon$ will give $\begin{pmatrix}\varepsilon V_1 & 0 & -\Phi_3\end{pmatrix}$ which depends of the "final" flux, but also of the compartment volume V_1 (note the difference with equation (11)).

Similarly, if $\mathbf{B}$ is changed to $\mathbf{B}_\varepsilon = \begin{pmatrix} \Phi_1/V_1 + \varepsilon \\ \Phi_2/V_2 \\ 0 \end{pmatrix}$, then $\mathbf{V}^\mathbf{T}\mathbf{B}_\varepsilon$ will give $\Phi_0 + \varepsilon V_1$, which also depends on the volume V_1, which is an important difference with equation (12).

It is now easily recognised that in both cases, the modified O.D.E. system does not verify the general equation (5), so, there is no mixture problem associated to this mathematical model. As a consequence of these facts, existence and stability questions for the inverse-modelling problem have a negative answer.

Let's now consider the effect of a scale factor, i.e., lets multiply volumes V_i and fluxes Φ_i by the same number without changing concentrations. For example, in the three tanks problem, volumes can be

$$V_1 = V_2 = V_3 = 10L$$

fluxes can be: $\Phi_0 = \Phi_3 = 2\,{}^{L}\!/_{s}$ and $\Phi_1 = \Phi_2 = 1\,{}^{L}\!/_{s}$

and finally, $C_0 = 10\,{}^{gr}\!/_{L}$

For this problem, the mathematical model is:

$$\begin{cases} 10\dfrac{dC_1}{dt} = 10 - C_1 \\ 10\dfrac{dC_2}{dt} = 10 - C_2 \\ 10\dfrac{dC_3}{dt} = C_1 + C_2 - 2C_3 \end{cases} \qquad (13)$$

Now, if volumes and fluxes are duplicated, i.e.,

$$V_1 = V_2 = V_3 = 20L$$

$$\Phi_0 = \Phi_3 = 4\,{}^{L}\!/_{s} \quad \text{and} \quad \Phi_1 = \Phi_2 = 2\,{}^{L}\!/_{s}$$

while C_0 remain unchanged (that means $C_0 = 10\,{}^{gr}\!/_{L}$ in this example), then, the corresponding mathematical model will be:

$$\begin{cases} 20\dfrac{dC_1}{dt} = 20 - 2C_1 \\ 20\dfrac{dC_2}{dt} = 20 - 2C_2 \\ 20\dfrac{dC_3}{dt} = 2C_1 + 2C_2 - 4C_3 \end{cases} \qquad (14)$$

after a straightforward simplification, this now becomes, the same as (13).

So, a scale factor in geometry, not in concentrations, produces exactly the same mathematical model, giving a new negative answer, adding, to the question of uniqueness.

These examples show that existence, uniqueness and stability does not occur, at least for these typical mixing problems. Other situations like multiple inputs and/or outputs, recirculation, etc., would be excessively difficult for a second year university course, and are studied in courses like Reactor's Design [4] for Chemical Engineering and other related careers. These situations will not be included in this paper.

3. The educative view point

Mixing problems are relevant for students of several careers such as Chemical Engineering, Food Technology Engineering and Environmental Engineering. The students react more positively to these problems than to others which are not so specific (as circuits or mechanics problems). In fact, there are several reasons because they become interested in those problems:

a) specificity (already mentioned)

b) low pre-requisites (as it was mentioned, they only need to know how to perform mass balances)

c) relevance to other subjects (Physical Chemistry, Reactors Design, etc.)

Modelling was introduced in UDELAR Differential Equations courses for Chemical Engineering and related careers courses in 1996, and since then, tanks and mixtures problems have appeared in the final examinations [2]. Inverse problems appeared in the assessment of this course two years later, in 1998. The questions had two different settings: firstly, tank dimensions and geometry were given, and students were asked to obtain an input for a desired output; and secondly, both input and output were given and the question was about what to put in the middle (i.e., how many tanks, which volumes and fluxes, what connections occurred between them, etc.). Finally, inverse-modelling issues were considered specifically since year 2005 [3], although inverse-modelling students' questions, appeared since the beginning of all this experience, in 1996.

In assessment, inverse-modelling was included in two different forms:

a) Asking the question whether or not a given O.D.E. linear system corresponds to a certain mixing problem. In the first experience (in year 2005), these problems were very simple, because mass balances or fluxes equations did not fit, and then, the answer was trivially negative. In those problems, students only need to explain this fact.

b) A more sophisticated inverse-modelling problem, consisting in a group of tanks, where the geometry was not given. In this case, students were asked to give an accurate geometry to fit with a certain O.D.E. linear system. This kind of problem appeared at the end of year 2005 and the beginning of 2006.

As it was mentioned before, all this teaching experience incorporated modelling, problem-solving and inverse-modelling. All of them were not just discussed in the classes, but played an important part of the assessment. This is a very important issue, for example Smith and Wood said that "…appropriate assessment methods are of major importance in encouraging students to adopt successful approaches to their learning. Changing teaching without due attention to assessment is not sufficient" [5].

5. Results and Conclusions

Previously, an expert group was consulted about mathematics teaching and learning at the undergraduate level, focusing in the specific case of engineering careers [6]. Most remarked the importance of teaching significant concepts and procedures in service courses. On the other hand, chemistry and engineering students showed an important preference for teachers who make the effort of presenting real-life problems, related with their own careers [6].

When students of Differential Equations were consulted about these courses, there were positive reactions, when motivating examples are used to promote mathematical modelling and applications. Moreover, they enjoyed working together in project-work, trying to propose mathematical models and/or applying the different concepts, tools and techniques to solve them analytically or numerically.

The need for relevance was highlighted by many writers as being important in assisting students with learning mathematics. For example, Bajpai *et al.* [7] suggested a range of improvements including a modelling approach and providing more relevant examples. According to Wood *et al.* [8] 'To make a mathematics course seem relevant to engineering students – and hence worth an investment of time – the subject has to be made to seem valuable for their own specialisation and future cases'. Finally, Mc Alevey and Sullivan [9], asserted that there is a need for using real-life problems since, 'Students are best motivated by exposure to real applications, problems, cases and projects'.

One aspect that must not be ignored, is the type of assessment must reflect the teaching method of the topic. The evaluation process must not be dissociated from the style of teaching. So, if courses have been instructed through problem-solving, modelling, etc., then assessment must be carried out to reflect this. This purpose can be put into practice through project-work, where students – with orientation of an interdisciplinary team of teachers and lecturers – try to solve real problems of their careers, in order to approve their mathematical courses.

As it was mentioned by Blum and Niss in their clasic paper about applications, modelling and applied problem solving [10], there are six different types of basic approaches to including relations to applicational areas in mathematics programmes. In our course, at the beginning (1966 to 2000), the "islands approach" was the selected one. In this approach, the mathematics programme is divided into several segments, each organized according to a two-compartment approach: a first part of a usual course in "pure" mathematics whereas the second one deals with one or more "applied" items, utilizing mathematics established in the firt part or earlier. Gradually, the course changed to a "mixing approach", where elements of applications and modelling are invoked to assist the introduction of mathematical concepts and conversely, newly developed mathematical concepts, methods and results are activated towards applications and modelling situations whenever possible.

Finally, it is important to remark that mixing problems are excellent sources for this purpose. Moreover, there exists an important set of real-life problems from these areas, which remain almost unexplored from the point of view of their mathematical education richness.

Searching for new real-life problems to be used for project-work in chemical and food technology engineering courses, represents an interesting challenge for engineers, mathematicians and mathematical education researchers and, at the same time, it provides a good opportunity for interdisciplinary work in both research and teaching.

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An experimental design course for university specialties

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In this study we discuss student assessment and feedback in an Experimental Design course that has been taught, over a ten-year period, in several faculties of the University of the Republic and other institutions in Uruguay. At the end of each course, students were assessed by means of individual work projects on real life problems. Students produced high quality work, some of which was the basis of future papers. Some of the student projects were used in later courses as examples to illustrate the use of different techniques and to provide teaching material appropriate to the context of the different university specialties. This has allowed a process of continuous improvement to occur, which was reflected in students' positive opinions of the course. An analysis of these opinions as well as students' suggestions, gathered over a decade, is also presented.

Keywords: Experimental design; applied biostatistics; continuous improvement

2000 Mathematics Subject Classifications: 62-01; 62D-5; 62K15

1. Introduction

Since the early 1990s, we have been consulted in an advisory capacity by teachers and researchers at various faculties of the University of the Republic in Uruguay, and by industrial companies and laboratories.

The subject of these consultations gradually changed from Statistics applied to Experimental Design, to Process Control and other more specific areas. The increasing need for training in Experimental Design led to courses of different contents and durations being taught in the following institutions: Chemistry Faculty, Engineering Faculty, Science Faculty, Uruguayan Chemical and Pharmaceutical Association (AQFU) and Technological Laboratory of Uruguay (LATU), among others.

The following is a list of the dates and places of the Experimental Design courses to which we refer, and the number of students involved:

Year	No. of students	Place / Department
1997	7	Faculty of Chemistry
1999	6	Faculty of Chemistry
1999-2000	15	Faculty of Engineering
2001	10	Faculty of Science / Biology

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2003	10	AQFU
2006	22	LATU
2006	8	LATU
	Total 78	

Experimental Design is a regular part of the curriculum at the undergraduate level in university specialties such as Chemical Engineering, Biology and Agronomy, while in others it is a topic dealt with at the postgraduate or continuing education levels. Our courses used a blend of the four different approaches discussed by Roiter and Petocz [1] in their article building on previous work by Blum and Niss [2].

From the start, at the Chemistry Faculty in 1997, the final assessment of students taking the course has been carried out in a non-traditional way. Students were required to carry out a short project on a real-world problem they had a personal interest in. As other authors have observed, this is very important for getting students to recognise that mathematical knowledge is relevant to their specialty [3, 4]. In addition, if real problems and using computers are an integral part of the course, it makes sense to make them part of student assessment as well [5]. In fact, efforts to change the contents or teaching methods in a course, without adapting student assessment methods accordingly, are incomplete [6].

This type of assessment has motivated students to produce high quality work, some of which has been published in international journals. For instance, two pieces of research work on biotechnology were published in the Journal of Data Science [7] and the Journal of Applied Quantitative Methods [8], and another on environmental engineering was presented at a Congress of the Asociación Interamericana de Ingeniería Sanitaria y Ambiental (AIDIS) [9]. The problems selected for the student projects were often developed subsequently in other directions. For example, the study presented at the AIDIS Congress [9] was the basis for a problem using partial differential equations applied to mass transfer, which gave rise to a publication in the New Zealand Journal of Mathematics [10] a few years later.

Over the years, real-life examples have not only been used in student assessments, but have also been introduced into subsequent Experimental Design courses, in a continuous feedback process. Many of these examples have proved to have excellent didactic potential when explaining and exemplifying techniques customarily used in Experimental Design. In some cases it has been possible to use the work projects directly as illustrations, while in others a didactical transposition [11] was necessary.

In this study we shall describe one of these examples, and report the results of course evaluations by students, carried out by means of anonymous questionnaires. Finally, we shall arrive at some conclusions and recommendations.

2. A concrete example of a student work project for course assessment

For the final assessment after taking a one-semester (42 hours) Experimental Design course as part of the coursework for a Master's degree in Biotechnology, one student carried out this piece of experimental design work applied to a real life problem posed by his practical laboratory project.

2.1 Study of different final formulations of streptolysin-O

The aim of the project was to discover a simple, stable and low-cost formulation for a freeze-dried recombinant protein, streptolysin-O (rSLO) [12], a protein that is useful in diagnostic tests for *Streptococcus* infections. Its antigenicity and haemolytic activity depend on its three-dimensional conformation, one of the most labile features of a protein, and finding a formulation in which these are preserved during storage is an important task.

2.2 Experimental design

Aliquots of rSLO were treated with different preparations, including bovine serum albumin (BSA), cysteine, sucrose and glycine at different concentrations, and freeze-dried. A full 2^3 experimental design was used to compare formulations containing BSA, sucrose and cysteine [13]. There were 8 possible combinations, together with their duplicates, making a total of 16 experimental trials (Table 1).

A fixed-effects bifactorial design, with replicates, was used to compare formulations containing sucrose and glycine at different concentrations [13], with 4 levels for factor A and 3 levels for factor B, giving 12 combinations. With duplicates, there were therefore a total of 24 experimental trials (Table 2).

Table 1. Factors and levels used in the 2^k factorial design.

Factor	**Levels**	
	-	+
A*	0 mg/mL	1 mg/mL
B§	0 mg/mL	1 mg/mL
C‡	0%	0.48%

*Factor A: BSA

§Factor B: Sucrose

‡Factor C: Cysteine

Table 2. Factors and levels used in the 2- factorial design.

	Levels			
Factor	1	2	3	4
A*	0 mg/mL	1 mg/mL	5 mg/mL	10 mg/mL
B§	0 mg/mL	0.1 mg/mL	1 mg/mL	-

*Factor A: Sucrose

§Factor B: Glycine

2.3 Product quality control

Titre (the reciprocal of the highest effective dilution) of haemolytic activity was used to measure of the quality of the rSLO as formulated, because haemolytic activity is an excellent indicator of the proper three-dimensional conformation of the protein. The different freeze-dried formulations were reconstituted with phosphate buffered saline and stored for 7 days at 4 ºC, and then haemolytic activity was measured. Statistical analysis of the haemolytic activity results was carried out by analysis of variance (ANOVA).

2.4 *Results of comparisons of different formulations*

Table 3 shows the results of the haemolytic assays for each experimental trial in the comparison of BSA, cysteine and sucrose. Table 4 shows the ANOVA table, together with statistical and tabulated values of F.

The preservative effect of the sucrose formulation on rSLO was significant at the 1 % level ($p = 0.01$), while that of cysteine was significant only at the 5% level ($p = 0.05$). BSA alone did not have a significant effect on rSLO preservation; however, BSA in interaction with sucrose had a highly significant effect, even more than the interaction of cysteine with sucrose (Table 4).

Table 5 shows the experimental haemolytic activity results for rSLO in the trials of sucrose and glycine at different concentrations. Table 6 shows the ANOVA, and Table 7 shows the statistics and critical tabulated values of F.

This analysis showed that the effect of sucrose in the formulation was significant at the 1% level, while glycine did not have a significant effect. However, the interaction between the two factors had a preservative effect on the protein which was significant at the 1% level (Tables 6 and 7).

Table 3. Results of haemolytic activity assays.

Experiment	**A (BSA)**	**B (sucrose)**	**C (cysteine)**	**Titre**
1	-	-	-	200
2	+	-	-	<200
3	-	+	-	400
4	+	+	-	1600
5	-	-	+	200
6	+	-	+	<200
7	-	+	+	<200
8	+	+	+	200
9	-	-	-	200
10	+	-	-	<200
11	-	+	-	400
12	+	+	-	800
13	-	-	+	200
14	+	-	+	<200
15	-	+	+	<200
16	+	+	+	400

Table 4. ANOVA table and values of F.

	Sum of Squares	Degrees of fredom	Mean Square	F*
A	122 500	1	122 500	2.882
B	562 500	1	562 500	13.235 §, ‡
C	422 500	1	422 500	9.941 ‡
AB	562 500	1	562 500	13.235 §, ‡
AC	62 500	1	62 500	1.470
BC	422 500	1	422 500	9.941 ‡
ABC	62 500	1	62 500	1.470
Error	340 000	8	42 500	
Total	2 557 500	15		

*Tabulated values of F: $F_{0,95}$ (1,8) = 5.320

$F_{0,99}$ (1,8) = 11.260

§significant at 1% level

‡ significant at 5% level

Table 5. Results of haemolytic activity assays of formulations coded as in Table 2.

Treatment	Titre (Replicates)	Total
A1B1	200 - 200	400
A1B2	400 - 200	600
A1B3	800 - 800	1600
A2B1	400 - 400	800
A2B2	1600 - 1600	3200
A2B3	1600 - 800	2400
A3B1	800 - 800	1600
A3B2	200 - 200	400
A3B3	400 - 400	800
A4B1	1600 - 3200	4800
A4B2	400 - 400	800
A4B3	1600 - 800	2400

Table 6. ANOVA table.

Source of Variation		Sum of Squares	Degrees of freedom	Mean Square
Treatment	A	3 591 666.670	3	1 197 222.220
	B	490 000.000	2	245 000.000
	AB	5 843 333.330	6	973 888.889
Error		1 940 000.000	12	161 666.667
Total		11 865 000.000	23	

Table 7. Statistics and tabulated values of F.

Null hypothesis	Model I factors A and B fixed	Critical values of $F_{0.95}$	Critical values of $F_{0.99}$
A homogeneous	7.405*	3.490	5.950
B homogeneous	1.515	3.890	6.930
No interaction between A and B	6.024*	3.000	4.820

*significant at 1% level

To sum up, most of the formulations studied demonstrated measurable haemolytic activity after reconstituting the lyophilised formulations and storing them at 4°C for 7 days, even those with nothing added to the rSLO. Formulations of rSLO with BSA and cysteine did not have an additional stabilising effect on the product. The best formulation in terms of simplicity and stability was the preparation of rSLO containing sucrose at a concentration of 10 mg/mL, the highest concentration studied. Other formulations containing sucrose in combination with BSA and glycine also produced similar titres of haemolytic activity, but the formulation containing only sucrose at 10 mg/mL was preferred as the simplest and most economical.

2.5 Discussion of the value of the work project

As a learning process, this student's theoretical knowledge from the experimental design course enabled him to structure his experiments to compare different formulations of rSLO in a way that was logical and economic in terms of effort and materials; and to analyse the experimental results to reach clear and statistically significant conclusions. We believe this kind of process not only demonstrates students' ability to apply experimental design knowledge, which is useful for course assessment, but also reinforces it, gives students real-world experience of using it, shows them its relevance as a research tool, and increases their confidence to apply it in future.

3. Results of course evaluation by students' opinion surveys

Students were asked to evaluate the course, in order to continually improve it. At the end of each course a written questionnaire was issued to students to be filled in anonymously. These questionnaires covered a range of specific items on how the course had been conducted, and perceptions of teacher effectiveness, using structured questions that could be analysed quantitatively. There were also a number of semi-open questions, where students could freely express their opinions and contribute suggestions, criticisms, etc. The course evaluation presented here was for a course taken by Biology students, and is representative of the evaluations carried out for the different courses over a decade.

3.1 Results of course evaluation: structured questions

Table 8 shows the items covered by structured questions about how the course was conducted, and the pooled results of students' responses, as percentages of total responses. Table 9 shows the aspects of teacher effectiveness enquired into, and the pooled results of students' responses, as percentages of total responses.

Table 8. Results of course evaluation: How the course was conducted.

Aspect of the course to be evaluated	Bad (%)	Neither Good not Bad (%)	Good (%)	Very Good (%)
Selection of topics covered	_	_	83.3	16.7
Quality of the methodology	_	_	66.6	33.4
Time assigned to each topic	_	33.2	66.8	_
Clarity of the initial objectives	_	_	50	50
Course dynamics (group work)	_	16.6	66.8	16.6
Course material supplied	_	_	83.4	16.6
General appreciation of the course	_	_	33.3	66.7

Table 9. Results for perceptions of teacher effectiveness.

Perceptions of teacher effectiveness	Bad (%)	Neither Good not Bad (%)	Good (%)	Very Good (%)
Competence in theoretical knowledge	_	_	22.2	77.8
Skill and clarity of explanations	_	12.5	25	62.5
Ability to motivate and interest students	_	_	50	50
Interest in clarifying students' problems	_	_	50	50
Ability to communicate theoretical information	_	12.5	62.5	25
Management of the class	_	12.5	37.5	50

3.2 Results for overall evaluation of the course and open-ended questions

When students were asked to evaluate the course overall, in terms of whether the topics covered had satisfied their expectations of the course, 66.6 percent replied "Yes", and 33.3 percent replied "Partly". No students answered "No" to this question.

In the open-ended questions the results showed that, in the case of these Biology students, the topics they would have liked to see covered in more depth were: experimental design more closely applied to biological examples; more examples of the application of non-parametric statistical tests; optimisation of experiments; applications of variable selection; and goodness-of-fit and analysis of variance for simple linear models. The students also said that they would prefer to spend more time on practical work and in the computer laboratory, and these were the greatest overall needs identified by the students on the courses given. They proposed expanding the courses, with more theoretical and practical hours per course; and they suggested having practical classes in parallel with theory classes throughout the course, so that they could apply their new theoretical knowledge immediately to practical problems.

3.3 Discussion of students' evaluation of the course

The results of the student evaluation showed an acceptable level of general satisfaction with the course, its contents, teacher effectiveness, and most other items asked about in the questionnaire. However, the need for a greater connexion between Experimental Design and biological topics was stressed. This was reflected in two parts of the questionnaire: the open-ended part, where more examples applied to biological fields were explicitly requested, and the part asking about time allocated to different activities, where several students mentioned the need for more hours of practical work and computer laboratory access to deal with these examples.

4. Conclusions and Recommendations

It should be noted that the example worked through in Section 2 was a work project carried out after the first course for biology students, and in later courses it served as a source of example problems that aroused the interest of this group of students. In fact, in evaluations of more recent courses, taught in 2006 and 2007, students' perceptions improved with respect to applicability to their main subject, time allocated to practical work, use of appropriate software, and other points.

We found the following to be good practices for increasing both students' and teachers' learning and satisfaction: a dynamic feedback cycle of course improvement and methodological changes, using materials based on students' own work projects; the linking of student assessment to solving real-world problems; careful attention to students' opinions through regular evaluations; and ensuring closer application of examples to students' specific specialties.

These have been part of a process of continuous improvement of Experimental Design courses, which began 10 years ago and is still continuing.

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Students' learning difficulties with volumes of solids of revolution

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This paper offers a diagnostic report on learning difficulties of 37 engineering mathematics students from two Further Education and Training (FET) Colleges in South Africa as they solve problems involving volumes of solids of revolutions. The instrument used in this research consisted of 23 questions classified in five categories. The data collected revealed that the students were not competent in drawing graphs and diagrams (in 2D and 3D). Students were seen to do well in problems that were procedural in nature or that had been seen before in their textbook or in examinations, even if those problems were regarded as more difficult than problems that were conceptual in nature. In general, students were better equipped to translate from visual to algebraic representations than from algebraic to visual. The study revealed that students lack the visual skills to interpret graphs and diagrams appropriately and were also unable to interpret some problems that were formulated in words.

Keywords: Solids of revolution, translations, algebraic and visual representations, 2D and 3D diagrams, procedural and conceptual learning, algebraic skills, cognitive skills.

AMS Subject Classification: 97D40

1. Introduction and background

The aim of this study is to investigate what written and verbal interpretations are produced by engineering mathematics students at two Further Education and Training (FET) Colleges in South Africa after learning the topic of Volumes of Solids of Revolutions (VSOR) in integration, which constitutes between 20-40% of their syllabus.

Traditionally, students have difficulties with this section. When rotating a given area bounded by graphs about the *X*- or the *Y*-axis, students seem to find it difficult to distinguish between the *disc* method; the *ring/washer* method or the *cylindrical shell* method. Students are expected to translate from a graphical representation (drawing) to an algebraic representation (formula) in order to come up with a numerical answer (calculation). This study is diagnostic and identifies aspects that influence students' learning of VSOR with reference to graphing skills; how they translate between visual and algebraic representations (in 2D and 3D); how they translate between two and three dimensions using diagrams and also verbally; how they translate between discrete and continuous situations (in 2D and 3D) and how they solve problems requiring algebraic skills and those that require more general cognitive abilities.

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2. Literature review, research questions and the theoretical framework.

In learning mathematics and symbols students deal with graphs and equations and interpret mathematical language. According to Dreyfus [3: 32], “to be successful in mathematics, it is desirable to have rich mental representations of a concept”, which enables students to interpret the external representation (diagrams) appropriately from their internal representations (mental images). The mental images that students construct cause them to succeed or fail. In solving mathematical problems, Dreyfus [4] believes that students avoid using diagrams and diagrammatic reasoning because of the cognitive obstacles related to diagrams.

Thornton [13] argues that visual thinking should be an integral part of students’ learning process as it plays an important role in the development of *algebraic* learning (a very important aspect to be explored in this study). According to Thornton, “powerful algebraic thinking arises when students attach meaning to variables and visualize the relationship in a number of different ways” [13:252]. The purpose of this study is to investigate these relationships, with the main focus on the development of algebraic thinking as students translate the visual (rotation of graphs after selection of an appropriate strip) to the algebraic representations (of equations) in order to compute the volume using integration and vice versa.

Other studies on translation from visual to algebraic representations and vice versa were conducted by Knuth [8] and Santos [11]. The studies revealed that many students were able to connect between the algebraic and the graphical representations of functions when dealing with familiar routine tasks, where a table of values is used to satisfy a given equation. The students in Santos’s study were more successful working in different levels of representation more than those in Knuth’s study, possibly because they used a dynamic software package, Cabre Geometre to help them visualise the graphical relationships.

Mofolo [9] conducted a study at one FET College which led to a report on *The interplay between the visual and the algebraic abilities of students*. Students were assessed in writing after visual and verbal instruction using a Mathematica demonstration by Kim and Ryan [8], which was in the form of animations and graphics covering the topic of VSOR in detail. The study revealed that the majority of students were unable to translate from the visual (graphical representation) to the algebraic representation so as to compute the volume generated [9].

In learning of VSOR one wants to investigate whether failure to translate from visual to algebraic is as a result of inability of students to relate to the graphs they have drawn (if any), or is it because VSOR is just too difficult a topic for them. Is it maybe too high in terms of their cognitive abilities and preparedness? If a new concept that is to be learned is cognitively high for the students’ internal representation, it is argued that students normally fail to make sense of it or even understand it conceptually [12]. In their report on students’ conceptions, Petocz et al [10] revealed that internationally, students regard maths as being ‘abstract’. Learning concepts that are above the students’ cognitive level is often regarded as abstract but sometimes possible if the students are given enough time to deal with such concepts. Unfortunately, that is not always possible with the FET College students due to the fast pace and volume of work. Eisenberg [5] argues that the abstraction of the new mathematical knowledge and the pace with which it is presented often becomes the downfall of many students.

In teaching and learning of VSOR, students are expected to use procedural knowledge as well as conceptual knowledge, which normally complement one another. Engelbrecht, Harding and Potgieter [6] pointed out that along the process of learning, the conceptual knowledge that is repeatedly presented might end up being procedural knowledge, in that students might not be thinking about what they are doing when presented with repeated problems, since the problems might have been done many times in class. In learning of VSOR the visualisation of the graphical representation and the translation to an algebraic representation can be regarded as conceptual learning since it involves analysis and critical thinking to enable the student to use a particular method. For the given different graphs in most cases, the areas bounded may be different; hence one cannot procedurally proceed, conceptual thinking is necessary. Students must engage with the graph, analyse what area needs to be rotated, what boundaries are given and how it must be rotated. The substitution to the algebraic (formula) and calculating the volume generated can be regarded as being procedural since is involves applications of rules and algorithms.

From the studies discussed above and our experience in learning and teaching of VSOR, the following research question was established: *Why do students have difficulties to understand when learning about Volumes of Solids of Revolutions?*

The way in which the students construct knowledge, interpret and make sense of what they have learned will be located within Bernstein's [1] rules of knowledge acquisition as our theoretical framework. Since one is dealing with students' difficulties, it is necessary to study students' thinking processes and how that impacts on their ways of learning. Using students' written and verbal interpretations, one can investigate their ways of thinking. In the process of learning, knowledge acquisition occurs when students are able to recognise or realise, that is, being able to interpret the question and to give the correct answer. Bernstein [1] refers to that process as involving the recognition and the realisation rules. He refers to the *recognition rules* as the means by 'which individuals are able to recognise the speciality of the context that they are in' during a learning process, while the *realisation rules* allow the production of the 'legitimate text' as in giving the correct answer. During the learning process the recognition rules enable the necessary realisations, while the realisation rule determine how meaning is being put together and made public [1]. In terms of this study the recognition and the realisation rules are related to the students' ability to link their internal representations (mental image) properly with the external representation (visualising and interpreting the graphs correctly) in VSOR. The ability to recognise and realise, using procedural knowledge flexibly may be influenced by the way in which instruction occurred (knowledge transmission) or what the students believe mathematical knowledge to be.

3. The participants and instrument used

The participants in this study were students from two FET colleges (aged 19 to 25 years), one from a township and one in the city in the Gauteng province enrolled for the National Certificate in Engineering (N6), in their second year after secondary school. The sampling was purposive [2] whereby students selected were from the local colleges accessible to the researchers. All the students participated voluntarily. The results presented in this study are only for 37 students (of 52 students) who participated fully in this study by completing all sections of the measuring instrument.

Data was collected over the period of one week during the first trimester of 2007 using an instrument consisting of 23 questions grouped into five different categories. To avoid repetition, examples of questions will only be included in the next section

Category 1: Graphing skills and translation between visual and algebraic. Questions require students to translate a given equation into a graph or to translate a given diagram into an algebraic equation, both in two and three dimensions.

Category 2: Translation between 2D and 3D. Questions require students to translate from two dimensional to a three dimensional diagrams or to translate from three to two dimensions, verbally and visually.

Category 3: Translation between continuous and discrete. Questions require students to draw rectangles to approximate the area under a curve or to draw the rotated strip in the cross-section of the generated solid representing a *disc, shell* or *washer.*

Category 4: General algebraic skills. In this category, students are expected to evaluate a given definite integral.

Category 5: General cognitive skills. In this category, students are expected to finish a full problem by drawing the graph, indicating the representative strip that they would use and to calculate the volume.

All questions were verified by an expert to ensure that proper standards are maintained throughout. The questions were randomised before given to the students to ensure that the categories used were not grouped. The students responded to them either individually or in groups, so as to gather individual and group responses. The students' successes or failures were validated by their written responses. All the responses were marked, analysed and summarised, and discussed with an expert to validate the analysis and the interpretations. The marked responses were then reorganised in the proper categories for further analysis. Since the questions given to the students focused on the problematic section of their syllabus, we assumed that the students would be serious in doing the questions. For ethical considerations, students completed consent forms for willingness to participate and also for ensuring that the results will be treated with confidentiality without their names being revealed [2].

4. Findings

In establishing the results, students' written work was marked and analysed. Each question was rated as either *fully correct, almost correct*, *traces of understanding, no understanding* or *not done.* The results were statistically analysed but for the investigative purpose of this current report, only a descriptive evaluation of student performance in each of the five categories is given rather than a complete statistical analysis of the results.

Category 1 (Graphical skills, translation between visual and algebraic)

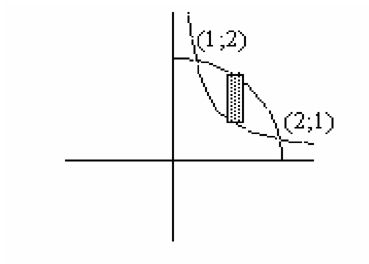
In this category there were 10 questions on graphing skills and translating between visual and algebraic representations, in both directions. We give three examples:

Example 1 (graphical skills): Draw a line with positive gradient passing through the origin for $x \in [0;3]$

Example 2 (algebraic to visual): Draw the 3D-solid of which the volume is given by

$\pi\int_0^1 (1-x)^2\,dx$ and show the representative strip.

Example 3 (visual to algebraic): The figure below represents the first quadrant area bounded by the graphs of $x^2 + y^2 = 5$ and $xy = 2$. Using the selected strip, substitute the equations of the given graphs in a suitable formula to represent the volume generated if the selected area is rotated about the *x*-axis. Do not calculate the volume.



The questions on graphical skills produced some interesting and surprising results. The question in Example 1 seemed simpler than another question that required the sketch of a hyperbola $x^2 - y^2 = 9$, yet most of the students could draw the hyperbola while few could draw the straight line. Most students drew vertical or horizontal lines ending at 3 or lines $y = 3$ or $x = 3$ and other totally different graphs. This reveals that they have difficulty in interpreting a verbal description such as "a line with a positive gradient" and they do not know what $x \in [0,3]$ means.

Student responses on questions (such as in Example 2) in which they should move from an algebraic to a visual representation varied. Most of those who gave meaningful responses drew graphs but not in three dimensions. The students were unable to relate the given equations for volumes to graphs that would represent a disc and a cylindrical shell. In two dimensions the results were somewhat better.

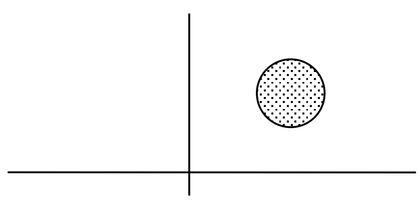
When required to move from a visual to an algebraic representation (such as in Example 3) the majority of responses were correct. Students clearly have fewer problems translating from visual to algebraic than vice versa, both in two and in three dimensions.

Category 2 (Translation between 2D and 3D)

In this category there were 4 questions in both directions. We give one example:

Example 4: Draw a 3-dimensional solid that will be generated if you rotate the circle

Given below about the *y*-axis.



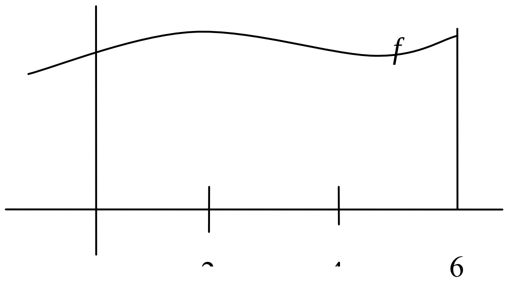
Very few students managed to draw a torus. Students were seen to draw cylinders and other nonsensical diagrams. In this category translation from 2D to 3D and from 3D to 2D was considered as only partially successful and only for simple diagrams such as a straight line that gave rise to a cone and a semi-circle that gave rise to half a sphere. If diagrams involved more imaginative skills at a higher level of conceptualising (such as in Example 4) most students failed. The performance in this category was surprisingly and disappointingly low.

Category 3 (Translation between continuous and discrete)

In this category there were 4 questions in both directions. We give one example:

Example 5: In the sketch below, show in terms of rectangles what the following represents

$$2f(0)+2f(2)+2f(4)$$



The performance in this category was disappointing. Students could not interpret the expression $2f(0)+2f(2)+2f(4)$ visually, and could not relate it to the given function. Students would approximate the given area (as in Example 5), or volume for other questions, by disjoint rectangles or slices, not showing any continuity of points on a continuous function. From the responses given, it was clear that students were not familiar with the concept of a Riemann sum. In short, students were certainly not proficient in moving from continuous to discrete representations and from discrete to continuous representations in 2D and in 3D.

Category 4 (General algebraic skills)

In this category there were 3 questions on algebraic skills. We give two examples:

Example 6: Find the point of intersection of $4x^2+9y^2=36$ and $2x+3y=6$.

Example 7: Evaluate $\int_0^1 2\pi\, x(1-\sin x)\,dx$,

Students showed some competencies in algebraic skills even though there were substantial mathematical errors. Such errors for Example 6 are that some students responded by finding the *x*- and the *y*-intercepts for each equation, some equated graphs in an incorrect way while others took square roots incorrectly: $\sqrt{36-4x^2}=6-2x$. For questions such as in Example 7, although some students made mathematical errors by multiplying $x\sin x$ to be $\sin x^2$ or even $\sin^2 x$, most students were able to solve the integral even if they were making errors with the signs. It can be argued that students were reasonably successful in this category, despite the mathematical errors made.

Category 5 (General cognitive skills)

In this category there were 2 questions on general cognitive skills. We wanted to test whether students can do an entire problem correctly, in this way combining the individual skills tested in categories 1-4. We give one example.

Example 8: Given: $y=\sin x$ and $y=1$, where $x\in\left[0\,;\frac{\pi}{2}\right]$.

(i) Sketch the graphs and shade the area bounded by the graphs and $x=0$.

(ii) Show the rotated area about the y-axis and the representative strip to be used to calculate the volume generated.

(iii) Calculate the volume generated when this area is rotated about the y-axis.

The performance for this question was below expectation, with very few students producing correct responses. Summarising:

Some students did everything correctly, arrived at the correct formula but could not do the integration.

Others drew correct graphs, shaded correctly, drew a disc, but used formulas for the disc- and shell-method simultaneously: $2\pi \int_0^1 y(\sin^{-1} y)^2 \, dy$

Some drew the correct graphs, shaded correctly, drew a Δy-strip and used an incorrect formula such as $\pi \int_0^1 x \sin x \, dx$ and other students drew a sine graph with a Δy-strip then used the formula $\pi \int_0^{\frac{\pi}{2}} \left(\sin^2 x - 1\right) dx$.

The different interpretations given for this question reveal how confused the students were, and how failure in one facet of the problem can make them fail in the rest of the problem. The majority of the students lacked the broader cognitive skills for questions in this category.

In general the responses given by individual students and students who were working in groups were not significantly different. When discussing with students as they were solving problems, it appeared that most students had problems understanding what a 3D-diagram means. Many students also believe that when asked to rotate about the y–axis one must use a Δy-strip and when rotating about the x-axis one must use a Δx-strip. This conception justifies why most of the students use the disc method in many cases. If one rotates a Δx-strip about the x-axis one will always get a disc or a washer and the same applies if a Δy-strip is rotated about the y-axis.

5. Discussion of the findings

From the results reported in the previous section, our earlier inkling that most students are struggling with this section is confirmed. The general performance in the entire instrument was poor; students clearly find this section hard to comprehend. In this diagnostic study five different categories were identified to establish where exactly things go wrong and to find out about was learned and how it was learned.

Evidence from this experiment seems to indicate that students found it slightly easier to translate from visual representations to an algebraic representation than the other way round. Even so, there were students who found it difficult to interpret a diagram or a graph, they lacked the rich mental representations of a concept [3] hence they failed to interpret diagrams appropriately. In cases where students managed, it was evident that it happened in cases where the conceptual knowledge (in form of diagrams or equations) had repeatedly been presented to them, hence ending up being procedural knowledge [6].

Students found moving between two and three dimensions problematic. In some instances, when asked to draw a diagram in 3D, students would draw a 2D-graph exposing ignorance of what is meant by two or three-dimensional situations. For this particular topic, this misconception must be considered as one of the most serious gaps in their mathematical knowledge.

Students performed fairly well in the category on general algebraic skills. This confirms our suspicion that students prefer procedural mathematics to conceptual thinking. It seems as if students might find conceptual thinking difficult. When presented with repeated problems, however, they would be reproducing what was learned before and seem to succeed mostly. Referring to Bernstein's recognition and realisation rules [1], it seems that most students have some recognition rules but do not apply them meaningfully, hence they fail to realise. Students seem to display some competencies wrongly when the context does not require these competencies and fail when the context does requires them. In Bernstein's terms, one can say that most students were unable to recognise the context they were in, hence failed to realise (giving the correct response).

6. Conclusions and recommendations

Our general feeling after this diagnostic investigation is that although students perform better in some of the categories, their poor overall performance indicates that this section of the syllabus is perhaps cognitively more demanding than many other topics. Students' approach is to rely on types of problems that they have been exposed to before, so they fail if the problems in exams differ from what they have seen before.

In presenting the topic, the study indicates that more emphasis should be on visual learning and the conceptual development of the formulae involved (disc/ring/shell). We recommend that this section should be taught more conceptually, where students focus on how the formulae are derived from the diagrams rather than only calculating areas or volumes. Students should be more exposed to translating between visual and algebraic representations. Attention should be given to conceptual understanding of graphical representation. Attention should also be given to moving between two and three dimensions and between continuous and discrete situations to make them conceptually more capable and to learn in depth, not just procedurally. It is recommended that attention be given to visualizing, and here software packages that can animate graphical representations by rotating diagrams to move between a plan view and a 3D view.

Learning procedures without reasoning is meaningless, as the procedures cannot be used when the context of the problem changes. Further research should be done as to how teaching could be improved so as to enable meaningful learning and better understanding.

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Using Maple to investigate the solution of partial differential equations

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Abstract

Many one-dimensional physical processes are easy for students to visualise, though two-dimensional cases present problems. The difficulty seems to be in the understanding of how each of the different parameters influences the phenomenon. Maple, via animation options, creates the opportunity to examine the total picture of various physical phenomena and thereby analysis of the influence of different parameters on evolution of physical process. This paper will demonstrate how an image of the numerical solution for partial differential equations can be created by the Maple code. This approach may be helpful for students studying courses concerning numerical methods.

Keywords: Partial differential equations, visualization of PDE's solution, Maple code.

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The mathematical education in the current scenario

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Abstract

The development of computer technology has affected university education along with other changes in the past few years. The difficulty of accommodating and retaining new students (tutorship system has helped this in both public and private universities) combined with lack of mathematical skills greatly hinder the teaching and learning of mathematics at this level.

Some notable differences between the average level of mathematics education and above, with reference to the purposes, objectives, methods and approaches to teaching lead to many problems. The mathematics teachers of the Universidad Tecnológica Nacional Facultad Regional Buenos Aires, in particular those teaching discrete mathematics, face a double challenge with students whose average level of preparation, knowledge and attitudes still are questionable as university students and other topics must have the level and quality of study in each subject deserves.

The dropout rate is a constant disappointment that impacts not only on the educational mission of the department but on the university institution. To attempt to rectify this situation, it is necessary to provide innovative teaching where the educators assist the growth of cognitive strategies to motivate students to become main players rather than spectators in their mathematical development.

The curricular activities of the course will encourage the potential of each student with the use of new technologies (computers) and traditional methods (tutor system) taking into account the individuality of each student. This research will show that respect of the individuality of each student and the uses of alternative methodologies, such as computer technologies, small integrated pieces of work, will assist in the development of how to surmise, build and design.

Keywords: university, permanency, tutor, teaching, mathematics, learning

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Maple Presentation

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Workshop abstract

Maple can help students visualize many results and, in many cases, reduce the difficulties of lengthy and tedious calculations so that students can concentrate on the solutions to their problems rather than the mechanics of how they obtained the solution.

I would like to present four topics:-

(1) The use and representation of characteristics in solving PDE's and the regions of attainable solutions. Students find these ideas very hard to visualize.

(2) Interesting Input-Output problems where different types of periodic, non-periodic or discontinuous input can produce surprising output.

Most students only consider the usual inputs of sinusoidal, exponential or polynomial – this demonstration examines other important types of input.

(3) The use of z transforms to solve difference equations. Here we will look at the types of solutions and show their connection to the Laplace transform.

These important ideas are poorly understood by many students.

(4) Problems in optimal control and switching curves.

Students have great difficulty envisaging these types of problems and Maple can help greatly with their understanding.

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Cellular Automata on a spreadsheet

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Abstract

Mathematicians have always dealt with objects, relationships between those objects, rules governing change and finally numbers. Theorems, Lemmas and models have traditionally been worked out with pen and paper.

Many structures are too complicated for people to do using traditional methods, however computers are ideally suited for some of these tasks. Our group of mathematics educators believes that 'doing maths' should include such structures, and we actively welcome technological tools which help is to do so. This view of Mathematics encourages students to use modern technology to tackle interesting tasks. In doing so, they also gain valuable skills which enhance their employability.

One particular example of this approach is that of Cellular Automata implemented on a spreadsheet. This paper describes how Applied Mathematics students in the second and final years at Sheffield Hallam University have been modelling such things as drug absorption, and game theory using CA and Microsoft Excel.

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Didactic analysis of a probability problem

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Solutions of a probability problem are analysed from the anthropological approach by Ives Chevallard of Didactics of Mathematics elements: the notions of task, technique, technology and theory. The analysed solutions include personal strategies, the construction of a tree of probabilities, the application of elementary probability results, a simulation process and the employment of a Markov chain with two possible states. Didactic analysis of the problem considered in this report can be useful for the training of mathematics teachers at university level. Moreover, it may contribute for students to integrate concepts and procedures studied through different subjects.

Keywords: Probability; Didactic analysis; Problem solving *AMS Subject Classification:* 97C30; 97C70

1. Introduction

Problem solving is a usual practise in Probability courses. From the point of view of Didactics of Mathematics this activity can be analysed from diverse perspectives. In this report some concepts that come from the anthropological approach by Ives Chevallard elements are employed: the notions of task, technique, technology and theory. In this approach, the institutions play an important role in the study of the didactic phenomenon [1–2].

Any activity consists in the realization of a task or a tasks system. For a person, a task is routine if he/she knows a way to do it, i.e. if he/she has an appropriate technique. Otherwise it is a problematic task. A problematic task could become in a routine with the acquisition of an appropriate technique. The employment of a technique can require to carry out sub–tasks, with theirs respective sub–techniques. Systems of institutional tasks exist in each institution (a culture, a family, a class, a teaching level, etc.), which are carried out from institutional techniques. These are legitimated within that institution.

The reasoning that allows to understand and justify a technique is called technology. One of the institutional activities consists in building and specifying the technologies corresponding to techniques employed in that institution. A technique which is accepted as valid in an institution may be invalid in another.

The reasoning that bases the technology is named theory. In certain institutions it can simply consist in the reference to another institution whose authority endorses it.

In this report it is shown how these concepts are applied to an analysis of a probability problem solution. It is part of a project that includes the analysis of other problems and their solutions. They are going to be proposed to a small group of future mathematics teachers. This problem, and its didactic analysis realized under a different approach, appears in [3]. This analysis scheme was used in [7] to analyse a combinatory problem.

The basic results about Probability Theory can be found, for example in [4] or [6].

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2. The Problem

Let us consider the following problem:

Twenty seven explorers are lost in a cave where three roads begin. One road leads to the exterior in one hour. The other two roads do not have exit: if the explorers enter for one of them, they return to the cave in two days; if they enter for the other one, return to the cave in three days. As the explorers do not take any light and the cave is dark and has many obstacles, they choose, every time that they make a trial to leave, one of the three roads at random. Let us suppose that:

(A) *Each explorer has provisions to survive at most six days.*

(B) *The provisions are available without limit of time.*

For each one of the previous assumptions:

1. Find the probability that an explorer comes out from the cave.

2. Find the expected number of explorers that reach the exterior.

The statement contains two parts. The first one describes a possible situation in the real world. The second one is directed to the reader, and it proposes him/her a group of tasks and the conditions to carry out them.

The employed techniques to performer the assigned tasks will depend on each individual and/or institutional knowledge that a person could have and on his/her ability to use them in this problem.

3. The problem solution procedures under the assumption (A)

The statement assigns two tasks to the reader. Also, it points out the basic condition to carry out the tasks: it is supposed that a road in each trial to leave the cave is randomly selected.

The first task consists in determining the probability that an explorer comes out from the cave. To determine this probability, three sub–tasks are carried out:

Sub–task 1: determine the possible paths that an explorer can select from the cave.

Sub–task 2: find the probability that an explorer comes out from the cave using each one of that paths.

Sub–task 3: determine the probability that an explorer comes out from the cave.

The second task consists in determining the expected number of explorers which come out from the cave.

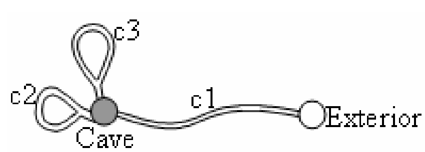

Figure 1. Graphic used to visualize the situation.

Figure 1 shows a graphic strategy which can be used to visualize the situation. Each road is selected at random, therefore the probability that an explorer chooses one of the three roads is equal to 1/3. The roads that an explorer can select from the cave are denoted by: c1, road which leads to the exterior in one hour; c2, road which returns to the cave in two days and c3, road which returns to the cave in three days. Next, four possible solution ways are shown.

3.1. Solution 1

3.1.1: First task: determine the probability that an explorer comes out from the cave. To carry out the first sub–task, the possible paths are represented using a tree diagram like it is shown in the Figure 2, considering that an explorer comes out from the cave in certain moment or he/she dies for lack of provisions.

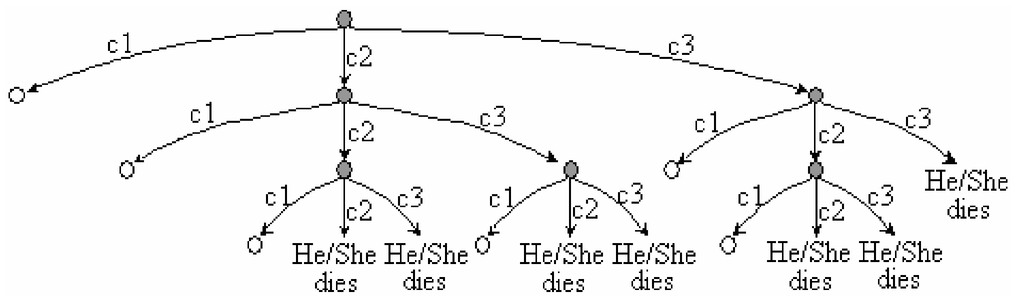

Figure 2. Tree diagram that represents the possible paths.

Looking at this picture, clearly arise that the paths are not equally likely, therefore the probability that an explorer leaves the cave cannot be calculated using the Laplace´s rule. Then, the paths that lead to the exterior of the cave are the only considered. Next, each one of these paths is enumerated indicating the succession of selected roads as follows, T_1: c1; T_2: c2 c1; T_3: c2 c2 c1; T_4: c2 c3 c1; T_5: c3 c1 and T_6: c3 c2 c1.

The second sub–task consists in determining the probability that an explorer uses the path T_i to come out from the cave, for each $i \in \{1,2,3,4,5,6\}$. Given the presence of obstacles and darkness inside the cave, the choice of one of the three possible roads in each trial is independent of the other choices. Therefore $P(T_1) = 1/3$, $P(T_2) = (1/3)^2$, $P(T_3) = (1/3)^3$, $P(T_4) = (1/3)^3$, $P(T_5) = (1/3)^2$ and $P(T_6) = (1/3)^3$, where $P(X)$ denotes the probability of an event *X*.

Each path represents an excluding way of leaving the cave with regard to the rest of the paths. So, the third sub–task, to determine the probability that an explorer comes out from the cave, consists in adding the probabilities that an explorer achieves it using the path T_i, for each $i \in \{1,2,3,4,5,6\}$. Then $P(T_1)+P(T_2)+P(T_3)+P(T_4)+P(T_5)+ P(T_6)=18/27$.

3.1.2. Second task: determining the expected number of explorers which come out from the cave. The probability that an explorer comes out from the cave is $18/27$, therefore it is expected that $27\times 18/27 = 18$ explorers come out from the cave.

The employed technique in the first sub–task is the construction of a tree diagram. The technology that supports the technique comes from the field of the combinatorial analysis. The employed techniques in the second and third sub–tasks belong to the field of Probability. The technologies that justify the techniques are the mutually exclusive events probability properties and the independent events probability properties, which belong to Probability Theory.

The corresponding technology to the technique used in the second task is based on the concept of direct proportion. The theory that justifies it is the theory of proportions.

3.2. Solution 2

3.2.1 First task: determine the probability that an explorer comes out from the cave. The quantity of hours used in the process is kept in mind to determine the possible paths. If an explorer chooses the road c1 he/she arrives to the exterior in one hour, if he/she selects the road c2 returns to the cave in 48 hours, and it takes 72 hours in returning to the cave if he/she chooses the road c3. Each explorer has provisions to survive at most six days, so the time used to come out from the cave should be at most 144 hours. An explorer can go through the road c3 at most two times because $144h/72h=2$. An explorer can not go through the road c2 more than three times because $144h/48h=3$. He/she can go one time through each one of the roads c2 and c3, since $144h-(72h+48h)=24h$, this is not enough time to go by some of the other two roads once again. Therefore, the possibilities of survival are the following: $72h+1h=73h$ (c3 c1); $72h+48h+1h=121h$ (c3 c2 c1, or c2 c3 c1); $48h+48h+1h=97h$ (c2 c2 c1); $48h+1h=49h$ (c2 c1) and $1h$ (c1).

The remaining sub–tasks and task are equal to those in the points 3.1.1 and 3.1.2.

In this case, a specific personal technique has been used for this sub–task, where the technology consists in the enumeration of the survival possibilities keeping in mind the time used in each case.

3.3. Solution 3

3.3.1 First task: **determine the probability that an explorer comes out from the cave.** Let us consider the events $C_{i,k}$: *An explorer chooses the ith road in the kth trial,* where $i, k \in \{1, 2, 3\}$. Because of the cave physical conditions the choice of a road is random, so $P(C_{i,k})=1/3$, for $i, k \in \{1, 2, 3\}$. The event of interest is S: *An explorer comes out from the cave in at most six days*. The possible paths which an explorer can choose to reach the exterior of the cave in at most six days are denoted as follows: $S_1=C_{1,1}$, $S_2=C_{2,1}\cap C_{1,2}$, $S_3=C_{3,1}\cap C_{1,2}$, $S_4=C_{2,1}\cap C_{2,2}\cap C_{1,3}$, $S_5=C_{2,1}\cap C_{3,2}\cap C_{1,3}$ and $S_6=C_{3,1}\cap C_{2,2}\cap C_{1,3}$. The independence property in the road choice is used to find $P(S_j)$, $j\in\{1, 2, ..., 6\}$. Therefore $P(S_1)=1/3$, $P(S_2)=1/9$, $P(S_3)=1/9$, $P(S_4)=1/27$, $P(S_5)=1/27$ and $P(S_6)=1/27$. The event of interest is $S=\bigcup_{j=1}^{6} S_j$. The events S_j, $j\in\{1, 2, ..., 6\}$ are mutually exclusive, then

$$P(S)=\sum_{j=1}^{6} P(S_j)=1/3+2\times 1/9+\ 3\times 1/27=2/3.$$

3.3.2. Second task: determining the expected number of explorers which come out from the cave. Let us define the random variable *X: Quantity of explorers that are able to come out from the cave in at most six days, among 27.* The random variable *X* is a binomial random variable with parameters $n=27$ and $p=2/3$. The expected number of explorers that come out from the cave is the expectation of the random variable X. Therefore $E(X)=n\times p=27\times 2/3=18$, where $E(X)$ denotes the expectation of a random variable *X*.

In this case a specific personal technique has been elaborated for the first sub–task and its corresponding technology. This technique is based on the enumeration of the possible paths in increasing order according to their lengths.

The corresponding technology to the techniques used in the remaining sub–tasks and in the second task belongs to the field of Probability. In the second and third sub–tasks, the technological elements are the concepts of independent events, mutually exclusive events and properties about the calculation of their probability. In the second task, the used technology corresponds to the concepts of binomial random variable and the expectation of a random variable. The theory that justifies this technology is Probability Theory.

3.4. Solution 4

A person that does not know the necessary mathematical tools to achieve a solution like those given previously, can appeal to a simulation experiment strategy and obtain an estimated result. This simulation can be made using balanced dices, charts of random numbers, an appropriate software, etc. It is necessary that a correspondence exists between the original problem and the simulated problem that makes them equivalent in terms of probability. Thus, the selection of a road in the simulated problem should be obtained with identical probability as in the original problem.

In this case the random numbers generator of the software Microsoft® Excel has been used, as well as some predetermined functions to create formulas, in order to guarantee the adjustment of the simulation process.

To represent the random experiment the following model is built: each one of the possible roads that an explorer can select from the interior of the cave is represented with an element of the set $\{1, 2, 3\}$, like this:

1: road which leads to the exterior in one hour.

2: road which returns to the cave in two days.

3: road which returns to the cave in three days.

The choice of a road at random is represented by the random generation of one of these three numbers. Each one of the N repetitions of the experiment concludes when an explorer reaches the exterior or he/she dies for lack of provisions, this happens in at most three trials. It is necessary to generate at most three numbers at random of the previously mentioned set to match the pattern to the conditions of the original problem. When the experiment concludes before the third trial, the symbol 0 is used instead of a new generation at random of a number in the set $\{1, 2, 3\}$. When the process concludes, each three–tuple will contain a 1 at most. The quantity of three–tuples that contain a 1 represents the quantity of times, n, that the event of interest was observed, i.e. the quantity of times that an explorer reaches the exterior.

3.4.1. First task: determine the probability that an explorer comes out from the cave. From a simulation, $n = 666$ three–tuples that contain a 1 are obtained in $N = 1000$ repetitions of the experiment. Therefore, the probability that an explorer comes out from the cave is $p = n/N = 0.666$.

3.4.2. Second task: determining the expected number of explorers which come out from the cave. If twenty seven explorers are inside the cave, it is expected that $27 \times 0.666 = 17.982 \cong 18$ explorers reach the exterior.

The task, proposed in the probabilistic frame, was performed in the statistical frame because a relative frequency was calculated. The frame change allowed to avoid the knowledge of the probability function properties that has been used in the previous solutions. Finally, the answer was given in the probabilistic frame.

To carry out this task it is necessary to build a particular model to represent the random experiment. In the pattern, the possible roads are represented by the first three natural numbers and the action of choosing a road by the selection of one of these three numbers at random.

The employed technique to perform the first task consists in repeating a random experiment a large number of times, *N*, then to count the number of times, n, that the event of interest was observed; and finally to calculate the relative frequency n/N and to assign the probability $p = n/N$ to the event.

The corresponding technology is the Frequency Interpretation of Probability. The theory associated to this technology is the Law of the Large Numbers of Probability Theory that is proved in advanced courses of Probability.

The technology corresponding to the technique used in the second task is based on the concept of direct proportion. The theory that justifies it is the theory of proportions.

4. The problem solution procedures under the assumption (B)

The statement assigns to the reader two tasks to carry out. The first task consists in determining the probability that an explorer comes out from the cave. The second task consists in determining the expected number of explorers that come out from the cave.

As in the previous assumption, one of the three roads is selected at random. In this case it is supposed that the provisions are available without limit of time. Next, three possible solution ways are shown.

4.1. Solution 1

4.1.1. First task: **determine the probability that an explorer comes out from the cave.** Let us consider the random variable *X: The minimal number of necessary trials in which an explorer comes out from the cave*. The image of the random variable *X* is the set of the positive integers. For each positive integer ***i***, let A_i be the event *An explorer chooses the road that leads to the exterior in the ith trial.* It is clear that $P(A_i) = 1/3$ and $P(\overline{A_i}) = 2/3$. The events A_i, for each positive integer *i,* are independent. So $P(X = i) = P\left(\overline{A_1} \cap \ldots \cap \overline{A_{i-1}} \cap A_i\right) = P(\overline{A_1}) \times \ldots \times P(\overline{A_{i-1}}) \times P(A_i) = (2/3)^{i-1} \times (1/3)$. Let *A* be the event *An explorer comes out from the cave.* An explorer can come out from the cave in ***i*** trials, for each positive integer *i*, then $P(A) = \sum_{i=1}^{\infty} P(X = i) = \sum_{i=1}^{\infty} (1/3) \times (2/3)^{i-1} = (1/3) \times \sum_{j=0}^{\infty} (2/3)^j = (1/3) \times \left[1/(1/3)\right] = 1$.

4.1.2. Second task: determining the expected number of explorers which come out from the cave. From the previous result it is deduced that an explorer, sooner or later, will come out from the cave (provided he/she lives enough!). Then, it is expected that 27 explorers come out from the cave.

The technology corresponding to the technique used in the first task consists in the use of the following concepts: random variable, independent events, probability calculation properties, geometric series and calculation of the convergent geometric series sum. The theory that justifies this technology is Probability Theory and Mathematical Calculus.

The technology corresponding to the technique used in the second task is based on the concept of direct proportion. The theory that justifies it is the theory of proportions.

4.2. Solution 2

4.2.1 First task: **determine the probability that an explorer comes out from the cave.** Let us consider the random variable *X: Number of trials necessary until choosing for the first time the road that leads to the exterior.* The random variable X is a geometric random variable with parameter $p = 1/3$. Therefore the probability that an explorer needs x trials to come out from the cave, for each positive integer x, is $P(X = x) = (1/3)\times(2/3)^{x-1}$. Let S be the event *An explorer comes out from the cave*, then $P(S) = P(X \geq 1) = \sum_{x=1}^{\infty} P(X = x) = 1$.

The second task is equal to that in the point 4.1.2.

The technology corresponding to the technique used in the first task consists in the employment of the geometric random variable concept. The theory that justifies this technology is Probability Theory.

4.3. Solution 3

4.3.1 First task: **determine the probability that an explorer comes out from the cave.** The problem can be model using Markov chains. The state of a system can be observed a finite number of times. In this case, there are two possible states

State 1: the explorer is in the cave.

State 2: the explorer is in the exterior.

In a given trial, the probability that an explorer who is in the state 1 remains in the same state is 2/3 and the probability that he/she changes to the state 2 is 1/3. The state of the system is observed for each time (in this case it corresponds with each selection of a road made by an explorer). At time *t*, which is random, the conditional probability of transition from a given state to any other does not depend on the way the state was reached; therefore the labelled directed graph (see [5]) that it is shown in Figure 3 represents a Markov chain.

For each positive integer k and each integer $j \in \{1, 2\}$, let $Z_k^{(j)}$ be the event *The system is in the state j at time k*. The event $Z_k^{(j)}$ in this context means *an explorer is in the state j after the kth trial*. The system is a Markov chain with two possible states, because for each positive integer k and integers $j_1, ..., j_k \in \{1, 2\}$, the events $Z_1^{(j_1)}, Z_2^{(j_2)}, ..., Z_k^{(j_k)}$ satisfy the condition

$$P\left(Z_k^{(j_k)} \middle| Z_{k-1}^{(j_{k-1})} \cap ... \cap Z_1^{(j_1)}\right) = P\left(Z_k^{(j_k)} \middle| Z_{k-1}^{(j_{k-1})}\right), \qquad (1)$$

where $P(X \mid Y)$ denotes the conditional probability of X given Y.

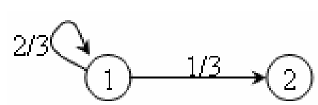

Figure 3. Graph corresponding to the Markov chain.

The equation (1), denominated 'Markov condition', it is satisfied in this case since the conditional probability of transition from a given state at time k only depends on the state reached in the previous immediate trial, i.e. at time $k-1$, and it is independent of how this state was arrived. Also, for any states i, j, the conditional probability that the Markov chain be in the state *j* at time *t*, given that it was in the state *i* at time $t-1$, denoted by $P(i, j)$, is independent of *t*. Then the Markov chain is homogeneous. The probabilities of transition $P(i, j)$ of the Markov chain with two states that it was defined can be represented by the following transition matrix:

$$P=\begin{bmatrix} P(1,1) & P(1,2) \\ P(2,1) & P(2,2) \end{bmatrix}=\begin{bmatrix} 2/3 & 1/3 \\ 0 & 1 \end{bmatrix}$$

When the state 2 is reached, it is impossible to leave it, i.e. $P(2,1)=0$. This means that the state 2 is absorbent. It interests to find the conditional probability of arriving to the absorbent state 2, given that it is left from state 1. This conditional probability is denoted by $u_2(1)$.

The following basic result is used (see [4], p. 166): If j is an absorbent state in a Markov chain with states $\{1, 2, ..., r\}$, then the probabilities $u_j(1)$, $u_j(2)$, ..., $u_j(r)$ are the only solution of the equations system

$u_j(j)=1$,

$u_j(i)=0$, if the state *j* cannot be reached from the state *i*,

$u_j(i)=\sum_{k=1}^{r} P(i,k)\times u_j(k)$, if the state *j* can be reached from the state *i*.

Therefore, $u_2(1)=\sum_{k=1}^{2} P(1,k)\times u_2(k)=P(1,1)\times u_2(1)+P(1,2)\times u_2(2)=(2/3)\times u_2(1)+ \ 1/3$, from it results $u_2(1)=1$. That is to say, the probability that an explorer comes out from the cave is equal to 1.

The second task is equal to that in the point 4.1.2.

The technology corresponding to the technique used in the first task is based on the concept and some properties of Markov chains. The theory that justifies it is Theory of Markov chains.

5. Final considerations

In this report some procedures have been analysed to solve a probability problem. In the finite case four solutions are shown that arise from four techniques to begin to solve the problem: possible paths are described through a tree diagram, in terms of time and in terms of events; the fourth solution implies a simulation process. In the infinite case three solutions are shown using: the description of events in terms of random variables, the identification of the geometric distribution, and the employment of Markov chains.

To propose this problem to future mathematics teachers allows them to find solutions, integrating knowledge that appear in diverse contexts and to analyse the didactic aspects of these solutions. This analysis allows them, for example, to decide to what group of students the problem will be proposed or what solution procedures are expected that they use according to the techniques and institutional technologies they have.

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Introducing Bayesian statistics to undergraduates

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This article is about the appropriateness, need and implementation of undergraduate Bayesian Statistics. The article shows how the Bayesian paradigm is increasingly applied in many and various fields yet there is not a corresponding number of undergraduate courses preparing students for this exciting and useful methodology. A method for introducing Bayes rule is suggested using a table of counts and then adapting it to show how a prior is updated to a posterior. The software package WinBUGS is used to show the Bayesian equivalent of a 2 sample t-test for comparing means with suggestions on how to bring out differences in the two paradigms both in terms of interpretation and data adjustment. Philosophical advantages of the Bayesian approach are discussed.

Keywords: Bayes; Bayesian; Undergraduate Statistics; Posterior; Prior.

AMS Subject Classification: 62C10, 62F15

Introduction

Bayesian methods are increasingly applied to real problems posed by science, yet this contrasts with the type of courses on offer at undergraduate level. Most courses are taught within a classical frequentist framework and very few with Bayesian methodology. Berger [1] in 2000 gives an overview of Bayesian application listing activity in diverse fields such as archaeology, atmospheric sciences, economics and econometrics, education, epidemiology, engineering, genetics, hydrology, law, measurement and assay, medicine, physical sciences, quality management through to social sciences. Areas of theoretical development abound, Berger [1] lists some 32 topics from biostatistics to time series. Press [2] in 2003 devotes most of appendix 5 to citing technical papers that apply the Bayesian paradigm in solving many and varied real problems across a range of 20 fields. Ashby [3] in 2006 gives a 25 year review of Bayesian statistics in medicine by looking at papers that appeared in Statistics in Medicine stating in her summary

> ...Bayesian statistics has now permeated all the major areas of medical statistics, including clinical trials, epidemiology, meta-analyses and evidence synthesis, spatial modelling, longitudinal modelling, survival modelling, molecular genetics and decision-making in respect of new technologies.

She also gives a brief discussion of what she thinks the future will hold and predicts that Bayesian statistics will increasingly be applied in newer and rapidly developing areas such as the human genome. Those areas already serviced by classical methods are likely to retain their dominance.

Books devoted to Bayesian theory now are common. The University of Auckland's voyager library search engine gave over 200 hits for the keyword "Bayes" most of which are classified under the subject of "Bayesian Statistical Decision theory".

With the growing importance of Bayesian methods it would seem reasonable to expect some courses to be offered in an undergraduate statistics curriculum. In the following sections the

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Bayesian paradigm will be reviewed by way of examples. These will be presented to show how the Bayesian methodology could be introduced to students, with examples of implementation and comparison at various levels of assumed student proficiency.

2 Introducing and applying the Bayesian paradigm in year two or three

Moore [4] in 1997 gave four reasons to hesitate teaching Bayesian ideas in a first course in working statistics.

(i) Bayesian ideas are rarely used in practice

(ii) There are no agreed standard methods to deal with standard statistical problems

(iii) Conditional probability is difficult to teach to students

(iv) Bayes for beginners tends to impede the move toward data analysis and design of data production

Some thoughtful and interesting replies to these points from a Bayesian stance have already been forwarded by Albert [5], Berry [6] and Lindley [7]. As for objection (i), it is true that frequentist methods are still the dominant paradigm, but as has been pointed out in the introduction, Bayesian methodology is increasingly applied as a look through the volumes of *Case Studies in Bayesian Statistics*, Springer-New York, will verify. Objection (ii) is not strictly true since Bayes formula is the method, what this criticism amounts to is that priors are largely subjective. As Berry points out the Statistician employing standard methods is acting more like a technician than a scientist. The Bayesian methodology allows the statistician to be a scientist building the prior and details of Bayes rule to answer the problem. Objection (iii) is possibly true but can be helped by showing the universality of conditional probability by way of tables of counts and conditional probability definitions. Moore's [8] statement in his reply is very telling and one cannot help but be sympathetic to Lindley and Albert since It seems reasonable that all probability is indeed conditional, Moore says

> It amounts to insisting that, to find the probability of three heads in ten tosses of a fair coin, a student must think of this as a conditional probability given $p = 0.5$.

In response to this one might ask – what does the assumption of "fair coin" mean when calculating the probability? Does it matter? With regard to hypothesis testing, the P-Value is made conditioned on the assumption that the NULL hypothesis is true. These are essential issues of material consequence, they comprise issues that a student must grasp in order to progress further in his or her understanding of Statistics. Objection (iv) is to some degree answered pragmatically by those who have actually implemented Bayes for beginners. Albert [9] and Bolstad [10] did not hesitate to teach Bayesian inference at an introductory level. Their courses included an emphasis on data analysis and having done things differently from each other, believe their courses are effective and serve their students well.

Having taught introductory first year statistics from a frequentist standpoint for a number of years, I tend to agree that the mathematical maturity of students needs to be higher than what could be reasonably expected of a large number of level one introductory students. Parameter recognition for various conjugate priors belonging to certain distributions, whose names alone would probably be enough to scare many stage one students off is probably too much. Keeping it simpler means only dealing with discrete distributions and this would limit what could be covered.

It may be better to introduce Bayesian statistics at a second or third year level so that the mathematical and statistical maturity of students is higher, enabling a more in depth and satisfying presentation of Bayesian methods. Some tentative efforts in trying Bayesian theory at first year level at the University of Auckland Statistics Department would seem to confirm this.

2.1 The plan

In order to teach a new topic in a statistics course, previous knowledge should be built on, commonalities repeated and linked and important differences emphasized so that the way forward be made easier and more intelligible to students. The definition of probability is clearly a difference. In the Bayesian setting probability is essentially expressing the knowledge one has of a parameter, hypothesis etc so that where in frequentist thinking a parameter is fixed and unknown, in the Bayesian paradigm a distribution is assigned.

In the case of building the Bayesian paradigm for students it may be thought that the foundations are so different that a new start be made with no attempt to reconnect with previously taught statistical and probabilistic theory. This I believe would be a mistake since there is a direct link to what would often be traditionally taught in many first year statistics courses, namely "Bayes rule". In some cases Bayes rule will not be explicitly taught and instead a table of counts incorporating the prior information is constructed so that the posterior distribution is found by calculating a simple conditional. Students may be quite unaware of labels such as "prior" and "posterior" these connections can be made and Bayes rule shown to be an equivalent way to find the desired conditional probability. As Albert [11] and Berry [12] have shown the Bayesian paradigm can most easily be introduced using discrete distributions in a tabular format where Bayes update paradigm can be made more evident.

Since all the students entering the course would already have been exposed to the standard frequentist tests it would make sense to revisit them again but this time re-analyze from a Bayesian perspective and compare the results. Two Bayesian tools would be helpful, the first is Bayesian multiple hypothesis testing and the second is posterior high density intervals. The first is suitably difficult for a level 2 or 3 course and the second is easily taught with some demonstrations in R or S-Plus and are generated automatically in WinBUGS.

2.2 Bayes theorem revisited

In the introductory text *Chance Encounters*, Wild and Seber [13] make use of the following Elisa HIV test problem in which prior HIV information is updated with an imperfect test to determine the probability of infection with HIV. For people who are HIV positive, 99.7% test positive and for people who are HIV negative, 0.3% test positive (false positive). It is estimated that 0.1% of the New Zealand population are HIV positive. Table 1 shows how the information can be tabulated, from this questions of interest can be formulated and answered.

	Test Result		
HIV Status	Test +ve	Test -ve	Total
HIV+	997	3	1000
HIV-	2997	996003	999000
Total	3994	996006	1000000

Table 1. Table of counts for HIV testing

This procedure will often be done in the context of teaching tables of counts and following the recommendations of Moore and others little in the way of formulaic expressions will be used, rather the table itself will be appealed to in solving conditional questions like establishing the values of $\Pr(HIV + ve \mid Test + ve)$ and $\Pr(HIV + ve \mid Test - ve)$. The table can be reformatted to bring out Bayes Rule and the Bayesian paradigm by forming the Bayes box which Albert [11] and Albert and Rossman [14] use effectively.

We are told that 0.1% of the population of NZ has HIV, which means that 99.9% of the NZ population does not have HIV. This can be used to define an individual's Prior distribution, we are interested in updating our knowledge of the distribution of an individual's HIV status given the observed test result.

Suppose the test result is positive then what will be the distribution of his or her HIV status given this latest information? We can use Table 1 to calculate it (we have used the prior information in constructing the table) $\Pr(HIV^{+} \mid Test^{+ve}) = \frac{997}{3994} \approx 0.25$ $\Pr(HIV^{-} \mid Test^{+ve}) = \frac{2997}{3994} \approx 0.75$ or we can construct Table 2 and implement Bayes rule

$$post \propto prior \times likelihood,$$
$$\Pr(HIV^{+} \mid Test^{+ve}) \propto \Pr(HIV^{+}) \Pr(Test^{+ve} \mid HIV^{+}).$$

HIV Status	Prior	Pr(Test+|Model)	Product	Posterior
HIV+	0.001	0.997	0.000997	0.25
HIV-	0.999	0.003	0.002997	0.75
Total	1	1	0.003994	1

Table 2. Bayes box

Where the likelihood is simply the probability of the observation given the model. Once the components and structure of equation 1 are understood, the methodology can then be extended to the continuous case by analogy.

2.3 Old problems new paradigm

A good theme for a third year course would be to compare frequentist and Bayesian methods, results and possible interpretations. To facilitate this, an equal tail 95% density region for the posterior will be calculated. The Bayesian hypothesis test technique of Neath and Cavanaugh [15] will be used to calculate $\Pr(H_0 \mid data)$.

As an example of what might be done, the following uses a two sample comparison of means. The student will be familiar with the 2-sample t-test with its confidence interval and p-value.

Possibly the easiest software to use in running Bayesian problems at a second or third year level is WinBUGS [16]. The students at an undergraduate level need not have to be familiar with the Gibbs sampler and MCMC, they can be told that the software merely produces a sample from the posterior after it has been coded in a WinBUGS model. The MCMC Gibbs sampler can be defined as a black box (interestingly the compiler used to make the WinBUGS executable is called Blackbox) which usually works well in producing posterior samples which a modeler can use to make conclusions about the parameter(s) of interest.

Basic theory on non-informative priors would need to be taught and this can be easily done using Jeffreys' theory for local and scale parameters as given by Robert [17]. As an example we shall examine the talk times (minutes) until recharge for different batteries used in cell phones taken from the University of Auckland's introductory stage 1 statistics workbook 2007 (see Table 3).

Table 3. Battery Talk times in minutes till recharge

Nickel-cadmium:				
62.4	102.7	89.2	93.0	99.2
83.6	105.3	88.9	88.2	72.1
80.6	89.1	78.0	95.4	92.4
102.3	97.0	88.3	67.7	98.7
Nickel-metal hydride:				
66.8	73.0	91.3	71.4	64.2
60.5	77.7	88.9	80.3	73.9
59.1	60.9	66.4	62.6	57.6
79.8	52.7	62.8	68.8	65.3

For both battery types we shall assume the time till recharge is normally distributed and for the Nickel Cadmium batteries,

$Time \sim N(\mu_1, \sigma_1^2)$.

Similarly for the Nickel-metal hydride batteries,

$Time \sim N(\mu_2, \sigma_2^2)$.

Since the data in this instance are written as a list with Battery taking 1 or 2 for distinguishing the type of battery, the following WinBUGS model code with a *FOR* loop was used to produce a sample from the posterior.

```
model
{
for( i in 1 : N ) {
Time[i] ~ dnorm(mu[Battery[i]], tau[Battery[i]]) # Likelihood
}
mu[1] ~ dnorm(0.0, 1.0E-6) # priors
mu[2] ~ dnorm(0.0, 1.0E-6)
tau[1] ~ dgamma(0.001, 0.001)# priors
tau[2] ~ dgamma(0.001, 0.001)
delta<-mu[1]-mu[2]
}
```

In this case the model does not assume the standard deviations are equal. Non informative priors are placed on the means and precisions so that the results will not be sensitive to them. A logical node is defined (delta) so that the distribution of the difference of means can be monitored and summarized. The posterior density plots of each parameter (mu[1] and mu[2]) and delta with their summary statistics are shown in Figure 1.

Node statistics

node	mean	sd	MC error	2.5%	median	97.5%	start	sample
mu[1]	88.69	2.767	0.008456	83.23	88.7	94.18	501	100000
mu[2]	69.19	2.439	0.007462	64.36	69.2	74.0	501	100000

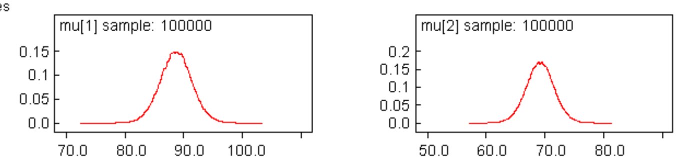
stats(delta)

Node statistics

node	mean	sd	MC error	2.5%	median	97.5%	start	sample
delta	19.5	3.682	0.01138	12.24	19.51	26.72	501	100000

density(mu)

Densities



density(delta)

Densities

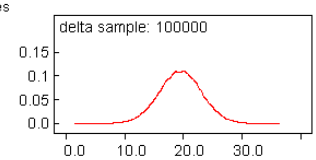


Figure 1. Posterior high density regions and posterior distributions for the Battery example as produced in WinBUGS

The theory on hypothesis testing yields the following result, $\Pr(H_0|\text{data}) = 0.00004$.

Students can refresh their knowledge of frequentist methods and perform a two sample t-test using SPSS or some other appropriate software to form a table such as shown in Figure 2.

The frequentist P-value and 95% confidence intervals are 0 and (12.4,26.6) respectively. These can easily be compared with the Bayesian output. Whereas the frequentist methods would require a different test (ANOVA when assuming equal variances) to deal with more groups, the Bayesian code would need very little amendment.

3 Interpretative advantages

The exciting challenge of teaching Statistics with regard to student perceptions is to make the subject relevant and interesting. The Bayesian intervals are intuitive and give straightforward answers to the kind of questions scientists, engineers and researchers in general would ask. What interval contains the difference in means (delta = mu[1] − mu[2]) with probability 0.95? One such answer is (12.24, 26.72). What is the probability that mu[1] exceeds mu[2] by a minimum of 12.24 and maximum of 26.72? The answer is 0.95. This contrasts markedly with the classical confidence interval which rests on multiple samples and long range relative frequencies. What realistic question would cause an individual to respond with a frequentist confidence interval as the answer?

Group Statistics

	Battery	N	Mean	Std. Deviation	Std. Error Mean
Time	Cadmium	20	88.7050	11.76066	2.62976
	Metal	20	69.1900	10.30528	2.30433

Independent Samples Test

		Levene's Test for Equality of Variances		t-test for Equality of Means						
									95% Confidence Interval of the Difference	
		F	Sig.	t	df	Sig. (2-tailed)	Mean Difference	Std. Error Difference	Lower	Upper
Time	Equal variances assumed	.079	.780	5.581	38	.000	19.51500	3.49651	12.43668	26.59332
	Equal variances not assumed			5.581	37.356	.000	19.51500	3.49651	12.43267	26.59733

Figure 2. Frequentist two sample t-test for the battery data as produced by SPSS

The 2.5% density posterior tails define a region that is comparable with the frequentist interval in size (see figure 2, this is due in part to the priors assigned), however the interpretation is very different. It can often be the case that the frequentist P-value gives misleading results especially when a point NULL hypotheis is tested. It can be shown that a NULL hypothesis is rejected when the posterior probability of the NULL is quite high.

An interesting exercise is to set up an excel worksheet that calculates simultaneously the probability of the NULL hypothesis given the data and the P-value consequent to adding a constant to the data for the Nickel-metal hydride group, this will move the two groups closer together and increase the P-Value. Play the game until the P value is just under the 5% level and then see what the Bayesian posterior probability of the NULL hypothesis is. If 12min. is added to the second group, Pr(H0|data) = 0.4 and the P-Value is 0.04. The NULL hypothesis would be rejected at the 5% level when its posterior probability is 40% with non-informative priors.

This kind of exercise is fun to do and is not difficult to set up, it highlights some important issues about interpretation and emphasizes warnings that have frequently sounded in the literature by such as Berger and Delampady [18], Berger and Selke [19] and others about P-values.

4 Philosophical advantages

The above battery example with unadjusted data gave similar results regardless of the paradigm. This is great news for both parties, the problem remains however that the frequentist and Bayesian interpretations are necessarily different and furthermore that some examples will not agree in their conclusions.

The philosophical advantages of the Bayesian approach are very clear. The likelihood principle says essentially that all experimental information must come through the data observed and expressed through the likelihood function. Two likelihoods proportional to one another bring the same information about the parameter. When a classical P-value is calculated more than the observed experimental data is used. The P-value is the probability that given the NULL is true, random variation would produce a sample estimate at least as extreme as the one obtained by experiment. But this breaks the likelihood principle since unobserved data is used in the likelihood to make inference. Maximum likelihood procedures as well as Bayesian methods preserve the likelihood principle. This means that there will always be a possible disparity in results between the frequentist and Bayesian paradigms.

The Bayesian paradigm is closer to the scientific investigative cycle, where a hypothesis or theory is formulated after which data is gathered and information updated. The prior expresses initial beliefs about a parameter, this is then updated by collecting experimental data which is expressed through the likelihood and combined with the prior to form the posterior. Just as science is cyclical so also is the Bayesian paradigm so that a previous posterior becomes the prior and updates to form the new posterior, see Figure 3. In this context the prior's subjectivity is closer to reality and can be presented positively.

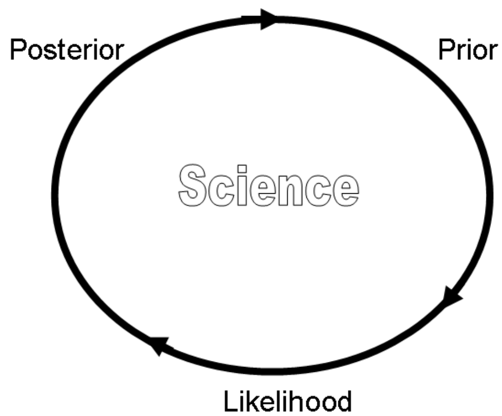


Figure 3. The scientific cycle using Bayesian notions

5 Conclusion

Bayesian methods are increasingly applied in real world problems. The paradigm rests on Bayes rule which once learnt is applied repeatedly. This has a unifying effect and ties all Bayesian applications together and contrasts sharply with the maize of frequentist methods which seem sometimes to be unrelated and ad. hoc. Inference is far easier to interpret and teach since a posterior interval will contain the parameter of interest with known probability. The posterior probability of a hypothesis is calculated and gives a more direct statement of its likelihood. The Bayesian approach would most likely be easily introduced using discrete distributions, generalizing afterward to the continuous case. A possible thematic undercurrent to undergraduate Bayesian statistics would be to resurrect all the main frequentist tests and give Bayesian alternatives with comparisons made on the results of the two paradigms. This would highlight the Bayesian and frequentist strengths and weaknesses. The course would likely be comfortably taught at the second or third year of undergraduate studies. However a well planned first Bayes course could be taught after due consideration was given to the student's mathematical and statistical pre-requisite knowledge.

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Learning and teaching probability: why?

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Abstract

We learn from experience. Actually, many times we learn badly from one single experience, since we often ignore the role that chance plays towards success or failure. How come that we make decisions under uncertainty every day, but at the same time we have a natural tendency to reject reasoning that involves probabilistic arguments? Why is our intuition so dissociated from "reality" when probabilities are involved? What tools can be used to help students (and us) bridge this gap?

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Bringing technology into universities of technology

Using spreadsheets in teaching numerical solutions of first-order differential equations

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Although South Africa is one of the largest users of the Internet in Africa it is still far behind First World countries such as America. In many parts of the world it is being assumed that students entering university are computer literate, but in the environment that this study was done this is not the case. This "digital divide" is evident at the Vaal University of Technology since many of our students had little or no exposure to computers at school level. Furthermore, a subject such as mathematics is traditionally taught by means of chalk and talk and this paper tries to bring together technology, in the form of spreadsheets and two very important concepts in engineering mathematics, namely differential equations and numerical methods. Students spend time in computer labs improving their technological literacy while at the same time mastering the cumbersome algorithms needed to solve first order differential equations with numerical methods. Our choice of spreadsheet is Excel, which in particular is simple, practical and widely available.

Key words: *First order differential equations, numerical methods, spreadsheets, Excel, digital divide.*

1. Introduction

A variety of scientific and engineering problems arise due to the fact that natural phenomena involve change and are best described by equations that relate changing quantities. This is the foundation of *differential equations*, and their study forms one of the most challenging branches of mathematics. A fairly simple example of an everyday problem is computing the position of a moving particle using its velocity and acceleration or the cooling or heating of a body dependent on the surrounding temperature. These quantities are dynamic and we therefore need to find a *function* from the prescribed information that describes the changing quantities. Judging from the perspective of a university of technology in South Africa, students are seldom comfortable with this concept when it is first introduced, usually in the second semester of their studies. Yet, as engineering students this must be one of the fundamental concepts for later understanding.

In conjunction with the difficulties encountered when trying to visualise the solutions of differential equations, numeric solutions of differential equations are found to be cumbersome to calculate. Add to this the inexperience of students on Computer Assisted Instruction (CAI), and the ideal opportunity arises to address all three problematic areas simultaneously and enhance the learning experience in the process.

The aim of the study is to show how Excel can be used to teach the principles of first-order differential equations to relatively under prepared students at a South African university of technology.

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2. The use of spreadsheets in education

The first electronic spreadsheets appeared in 1979 and since then more educators have been turning towards spreadsheets, especially where repetitive calculations are called for. John E Baker [1] gives a brief outline of the first 25 years of spreadsheets and provides arguments and motivation for further research in this area. Hsiao [2], makes the very valid point that while computers are clearly useful tools for education, one of the main disadvantages is having to program them. Although there are several sophisticated mathematics packages available, such as Mathematica, Matlab, Maple and Derive, there is seldom enough time when teaching mathematics to spend time on mastering the coding required for these packages in order to appreciate the benefits. The cost involved in purchasing these packages is also a prohibitive factor. Morishita *et al* [3] states that in their experience it took a long time to learn computer languages and it is hard to obtain proper results in a limited time. The spreadsheet offers an alternative to conventional programming and allows experimentation with numerical methods [4]. Furthermore, the spreadsheet is relatively simple to use and almost instantaneous numerical simulations are possible. With the first appearance of spreadsheets, they were welcomed as "a success story of making programming easier" [5]. Nardi and Miller [6] add that the biggest advantage of spreadsheets is not cognitive but motivational: '… after a few hours of work, spreadsheet users are rewarded by simple functioning programs'. Smith [7] predicts three positive outcomes for students when spreadsheets are used as a mathematical teaching tool.

- Reversal of the declining interest in mathematics
- Improvement of technological literacy and enhancement of career preparation
- Revitalization of mathematical skills through problem solving

Excel, in particular, is a spreadsheet program that is simple, practical and widely available. This makes it an excellent tool for Computer Assisted Instruction (CAI) and unlike the packaged learning programs mentioned above; students can learn mathematical concepts by actually writing formulas into the worksheets. However, no programming knowledge or skills are required.

3. The digital divide

The concept of "digital divide" is well established. It refers to the disparity between those who have use and have access to Information Communication and Technology (ICT) tools, and those who do not. Below are statistics [8], [9] relating specifically to the Internet and access to it, which demonstrates this "digital divide" between the First World and the Third World:

- In 2005 the number of Internet users passed the 1 billion mark
- 18.3% of users live in the USA (First World)
- Of just over 900 million African residents, about 2.5 percent (23.5 million) are online, compared to the worldwide average of 16 percent. (Third World)
- South Africa is one of the largest users in Africa at 3.6 million.

A survey done by Lim [10] at Deakin University in Australia found that the Faculty of Science and Technology students on the rural campus had the lowest overall level of ICT skills (both spreadsheets and other skills) of all enrolled first year students at the start of 2003. In Africa 89% of people live in rural areas, where the needs are the greatest. Also, the work of Hellwig and Lloyd [11] indicated that the "digital divide", between those who have and those who do not have computer access correlates with the difficult financial conditions in many of the rural communities.

Moreover, Hodge *et al* [12] suggests that information technology is rapidly changing the way individuals live, firms do business and governments interact. Yet, the pessimists paint a gloomy picture of a world split even further apart between the ICT have's and the have not's. In South African schools there is a very limited use of IT in the pedagogic process, except at well-endowed private schools.

Watkins [13] says: 'Today's students are completely at ease using a computer for everything from researching a term paper to synching data from their PDA to creating CAD drawings.' Teaching in a third world environment such as we do does not necessarily warrant the same reaction.

4. Universities of technology

The Ekuhuleni Campus of the Vaal University of Technology, where this study was conducted, is situated in the Gauteng province, one of the most densely populated regions in the country. Universities of technology attempt to provide greater learning opportunities in making the student more skilled, more competent and more employable by taking the institution into the workplace liaising with industry to ensure that prospective employees receive a relevant education.

Institutes of technology and polytechnics have existed at least since the 18^{th} century but became increasingly popular as the needs for industrialisation grew. In some cases, polytechnics or institutes of technology are engineering schools or technical colleges. Polytechnics and institutes of technology are considered universities when they have autonomy to offer masters and doctoral degrees as is the case with the Vaal University of Technology. At the same time these institutions must participate in independent research to be formally considered a university [14].

From a survey of a class of 49 engineering students in their third semester, the students in this study, it appeared that almost two thirds were from "rural" areas. Almost 60% did not have access to a computer at home and only about 40% had regular computer access. The majority of students seldom had access to computers at school. Of those with regular access, most had computer studies as a school subject and computer literacy was not taught in general.

At South African universities of technology an introductory computer skills subject is introduced in the first semester of the first year. For engineering students it would span over a six month course, which would cover the basics of Windows, Word and Excel. Unfortunately it is often the case that they do not have much exposure after this initial introduction, primarily due to the lack of hardware, time constraints and teaching resources and do not retain the knowledge sufficiently.

This brings about a serious dilemma. As the name "university of technology" suggests, our engineering students should be at least proficient in the use of computers and basic software. Yet, since computing facilities within schools depend largely on financial resources, many of our students had little or no exposure to computers at school level. At university level computer facilities are available, perhaps not always to the ideal extent but there is an added problem. A subject such as mathematics is traditionally taught by means of chalk and talk and it takes a concerted effort to shift this paradigm. So despite the digital divide resulting in a technological under preparedness it is still the task of the lecturer to find a way to see that technology is part and parcel of the universities of technology. We feel the answer lies in the use of spreadsheets.

5. Methodology

This study was conducted with a group of 49 students in their third semester of engineering mathematics. On completion of the section on numerical methods, which included lectures and laboratory sessions, students completed a questionnaire to determine their learning experiences. The questionnaire comprises of questions on

- previous exposure to technology at school level and at home
- preference of a technology-based learning environment to "chalk and talk"

The questionnaire was followed by informal student interviews, based on responses to the questionnaire. We report on the findings in the Findings section.

At the Vaal University of Technology students are in their third semester when first introduced to using numerical methods to solve differential equations of the form $\frac{dy}{dx} = f(x, y)$. The course starts with Euler's method moving on to Runge-Kutta of the second and fourth order. After introducing the concepts in a theoretical class the student spends the next three classes of one hour each putting the relevant formulas into a spreadsheet. Students can also use the computer lab on a voluntary basis, but these classes are generally reserved for Information Technology students and are thus not that freely available.

Starting with Euler's method the student is taught some of the basics of using Excel effectively along with some basic graphing techniques. By the second practical session Runge-Kutta order two is introduced, which has a "more complicated" algorithm. Time does not allow us to do Runge-Kutta order four in a practical class. Mathematics 3 consists of a full program and does not allow us more time on this particular subject matter. Unfortunately we have not yet moved to assessing the student practically so for examination purposes they still have to do the procedures by hand.

In our approach to using Excel we focus on simplicity. In [15], in a similar study, the lecturer touches on inserting the formulas but then quickly moves on to using the built-in macro language (visual basic) as the Runge-Kutta fourth order algorithm is cumbersome to work with. We try to avoid the use of macros as once a program is written it can often mask the mathematics that it is intended to represent while typing in the formulas ensures that the procedure is constantly exposed [16].

Shannon [17] maintains: "Calculating fourth order Runge-Kutta approximations with a calculator is so tedious that it is rarely instructive", and then turns to Lotus and Derive for calculating fourth order Runge-Kutta approximations and graphing them. Although these computer packages are easier to master than Matlab and Mathematica they are not available to students at our campus because of financial constraints.

6. The Excel procedure

We illustrate our use of Excel through an example.

Solve $\frac{dy}{dx} = xy$ with initial condition *y (1) =1* and step size 0.1 using Runge-Kutta of the second order.

- Enter the heading n into cell A1
- Enter the step size heading h into cell B1

- Form the next four columns with headings as indicated:

	A	B	C	D	E	F
1	*n*	*h*	*x*	*k1*	*k2*	*y*

- Enter the value n = 0, the actual step size $h = 0.1$ and initial values for x and y into row 2.

	A	B	C	D	E	F
1	*n*	*h*	*x*	*k1*	*k2*	*y*
2	0	0.1	1			1

To insert the Runge-Kutta formulae of the second order:

- ***For calculating*** $x+h$**:**

  Enter =C2+$B$2 into C3.

  Take note that using $B$2 will keep the value of h = 0.1 fixed when copying the cells down.

- ***For calculating*** $f_0 = hf(x, y)$:

  Enter =$B$2*(C2*F2) into D3.

- ***For calculating*** $f_1 = hf(x+h, y+hf_0)$:

  Enter =$B$2*((C2+$B$2)*(F2+D3)) into E3.

- ***For calculating*** $y(x+h) = y + hf_0 + hf_1$:

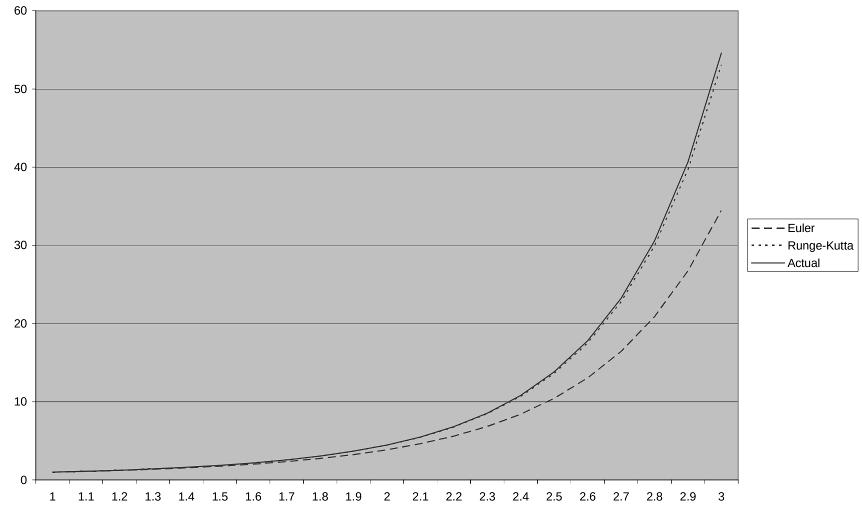
  Enter =F2+0.5*(D3+E3) into F3.

- Now copy cells C3 through to F3 down for as many lines as you like.

Finding the analytical solution to the equation is $y = e^{\frac{x^2-1}{2}}$ (it is not always possible to solve analytically) makes it possible to compare the actual solution to the Runge-Kutta solution. This comparison opens many doors to discussion. By varying the value of the step size h students discuss the relevancy of stepsize and the effect on the error. By inserting columns containing the actual values of y and the error in the Runge-Kutta solution we can proceed to plot these solutions. A typical table containing these results looks like this:

A	B	C	D	E	F	G	H
n	***h***	***x***	***k1***	***k2***	***y***	***Actual y***	***Absolute error***
0	0.1	1.0			1	1	0
1		1.1	0.100000	0.121000	1.110500	1.110711	0.000211
2		1.2	0.122155	0.147919	1.245537	1.246077	0.00054
3		1.3	0.149464	0.181350	1.410944	1.41199	0.001046
4		1.4	0.183423	0.223211	1.614261	1.616074	0.001813
5		1.5	0.225997	0.276039	1.865279	1.868246	0.002967
6		1.6	0.279792	0.343211	2.176780	2.181472	0.004692
7		1.7	0.348285	0.429261	2.565553	2.572813	0.00726
8		1.8	0.436144	0.540306	3.053778	3.064854	0.011076
9		1.9	0.549680	0.684657	3.670947	3.687689	0.016743
10		2.0	0.697480	0.873685	4.456529	4.481689	0.02516
↓		↓	↓	↓	↓	↓	↓

Using column F and G if available, the iterations using the Euler method the following graph is produced:



7. Findings

7.1 From the interviews and questionnaire

- Sixty percent of students seldom had access to computers during their school years. In the cases where students did have occasional access to computers, they were generally second hand, not in good working condition and incapable of hosting the latest software.
- Forty three percent of the students currently do not have access to a computer where they stay. Considering that these are engineering students in their second year of study, you would expect more of them to have computers where they stay. Here too, most of them that do have access to computers reported that their machines were second hand and the software out of date.
- Only students with Computer Studies as a school subject had done Excel before attending university. None of our students had done Excel before coming to university. In recent statistics supplied by the National Education Structure Management System (Neims), reported on by Rademeyer [18], it was found that 68% of schools in South Africa do not have computers at all. This study was carried out at each of the 28 742 schools in the country. It is therefore reasonable to assume that not many of the rural schools offer Computer Studies as a subject.
- Eighty percent felt that the introduction to Excel during the six month Computer Skills course was not enough to acquire any real skills.

7.2 Experiences with Excel

- The attendance figure for the laboratory sessions was significantly higher than the usual lectures.

- Students found it easy to understand the relevancy of step size since increasing it to 0.2 would immediate show up the increase in the absolute error, whilst reducing it would decrease the error.
- Seeing the graphs of both the actual solution as well as the Euler and RK solution helped students to 'see' how Runge-Kutta improved on the Euler method.
- After the initial difficulties of entering the formulae into the cells, students fared much better by the second and third practical session.
- Those students doing computer system engineering were markedly more comfortable using Excel than those doing other engineering courses.
- Students interacted during the practical sessions – stronger students assisting the weaker ones.
- It was evident that students enjoyed the sessions. Despite the inability of the students to access and use Excel, they displayed an encouraging eagerness to use and learn new technology.

8. Conclusion

This study was done to address the problem of the "digital divide" in a meaningful way, but more needs to be done to make a paradigm shift from mathematics as a talk and chalk subject to one that fully incorporates technology. For an institution constantly struggling with financial resources, turning to Excel instead of the more accepted mathematical packages makes it possible for all students to have easy access. They also do not need any prior programming skills in order to master the basics of Excel within a few hours.

The age old question of "why are we doing this?" is better addressed when students can "see" the results of solving differential equations. All the students interviewed were interested in doing more mathematics using Excel and felt that the learning experience was enjoyable. We have shown that meaningful use of technology is possible despite the digital divide. In only a few practical sessions the technological skills of the students increased and in so doing we have contributed to advancing the technology prowess of students at a university of technology as far as mathematics is concerned.

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The experience of Australian mathematics and statistics undergraduate students

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In the current Australian higher education context of high competition for students and steady decline in the enrolments in the basic sciences in general, improving the experience of mathematics and statistics undergraduate students is of strategic importance. This paper attempts to gain an insight into the quality of the experience of mathematics and statistics students and graduates in Australia. The analysis is made in the context of science teaching and courses, and is based on data from subject evaluations carried out at a traditional university and the Course Experience Questionnaire completed by bachelor graduates. The analysis highlights three areas that require greater attention, namely good teaching practice, social environment and generic graduate qualities.

Keywords: Student experience, student evaluation, graduates evaluation

1. Introduction

In recent years a greater focus has been placed on improving the experience of undergraduate students. In Australia, data collected from student surveys are now used to rank universities on their teaching performance. These rankings not only influence the student choice of university, but also provide access to additional significant funds. The Course Experience Questionnaire (CEQ) attempts to measure the experience of the bachelor graduates [1], and data from this survey contribute substantially to the Learning and Teaching Performance Fund formula [2]. Universities have responded to these external pressures by introducing institutional-wide mechanisms and programs for monitoring and improving the student experience while still enrolled.

Despite the large amount of data now available, to date no analysis has been made of what is the experience of mathematics and statistics students and graduates. Is it better or worse than that of other students and graduates? Are mathematical sciences students generally happy with what they get? Are there particular aspects of that experience that warrant special attention? These questions are timely and highly relevant given the state of mathematical sciences in Australia, where the pattern of declining enrolments in the basic sciences (mathematics, physics, chemistry) over the last fifteen to twenty years pose a cause for concern. While the number of science enrolments at university level roughly doubled since 1989, there was a decline of about 30% of full time students undertaking mathematics and statistics as a major study [3]. The recently completed review of Mathematical Sciences research in Australia confirms the seriousness of this critical decline, by highlighting that over the last decade mathematical sciences departments in the smaller universities have disappeared, and the number of permanent staff in the mathematical sciences departments of the large traditional universities fell by a third [4].

This paper attempts to look at two sets of data currently available to gain an insight into the experience of mathematics and statistics students and graduates over the last few years. The investigation tries to establish the best and worst aspects of the experience of mathematics

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and statistics students, and whether this experience is any different to the experience of other science students. The first set of data relates to the experience of undergraduate students as expressed through the regular subject evaluations at Monash University. The second set concerns the experience of bachelor graduates from all Australian universities as expressed a few months after graduation through the Course Experience Questionnaire. Each of these sets of data is analyzed separately, and in both cases a comparison is made between the mathematics and statistics group and the group that comprises all other sciences.

2. Mathematical sciences student satisfaction at Monash University

2.1 The evaluations

Since 2005 Monash University has been conducting a systematic evaluation of all subjects offered across the university using a common instrument [5]. The evaluation instrument consists of 8 common university items, up to ten faculty-specific items and two open ended items; the quantitative items of this instrument are included in Table 1. Students rate each statements in a Likert scale from 1 to 5 with 1='strongly disagree', 2='disagree', 3='neutral', 4='agree', and 5='strongly agree'.

Evaluation items	*Short version*
*The learning objectives of this unit were made clear to me	Clear objectives
*The unit enabled me to achieve the learning objectives	Subject design
The material in this unit is presented at an appropriate level	Appropriate level
The organisation and progression of the topics covered was coherent	Organisation/progression
The criteria for the assessment tasks were clear	Assessment criteria
Most of the material covered in this unit was either - new to me, or it was presented in more depth than in previous studies	New material
*I received constructive feedback on my work	Constructive feedback
*The feedback I received was provided in time to help me improve	Timely feedback
*The overall amount of work required of me for this unit was appropriate	Workload
*I found the unit to be intellectually stimulating	Intellectually stimulating
*I found the resources provided for the unit to be helpful	Helpful resources
The lectures helped me achieve the unit learning objectives	Lectures helpful
The tutorials/practical classes helped me achieve the unit learning objectives	Tutorial helpful
The lecturing staff motivated me to learn in the unit	Motivation: Lecturers
The tutors/demonstrators helped me learn in the unit	Motivation: tutors
Access to individual assistance (either face-to-face or online) was adequate	Individual assistance
*Overall I was satisfied with the quality of this unit	Overall satisfaction

Table 1: Quantitative items of the Faculty of Science evaluation instrument. (*) indicates university-wide items.

2.2. Findings

There is now sufficient aggregate data to make meaningful comparisons between the satisfaction level of students who evaluated mathematics and statistics subjects and those who evaluated subjects that belong to other areas of science. The data for four consecutive semesters (from semester 2 2005 to semester 1 2007) is summarised in table 2 and figure 1. The category 'mathematics' includes all evaluation data for mathematics and statistics subjects taught towards the Bachelor of Science. (It does not include specialist applied mathematics subjects in the areas of atmospheric science and astrophysics taught by the School of Mathematical Sciences, nor service mathematics subjects to other non-science disciplines such as engineering or business). Students enrol in these 'mathematics' subjects for the purpose of meeting the Bachelor of Science breadth requirement, or to complete a major or a minor in mathematics or statistics. The category 'other sciences' includes all other subjects taught towards the Bachelor of Science, including biological sciences, biomedical sciences, chemistry, geosciences and physics. The overall response rate for all science subjects over this period of time was 57%.

	Mathematics n=2806		**Other sciences** n=20,836		
Evaluation item	*mean*	*sd*	*mean*	*sd*	*p-value*
Clear objectives	3.759	0.865	**3.909**	0.802	0.000
Subject design	3.720	0.898	**3.818**	0.798	0.000
Appropriate level	3.795	0.879	**3.859**	0.818	0.000
Organisation/progression	3.706	0.987	**3.793**	0.889	0.000
Assessment criteria	**3.780**	0.922	3.744	0.918	0.028
New material	4.032	0.887	4.014	0.859	0.159
Constructive feedback	**3.663**	0.998	3.550	1.028	0.000
Timely feedback	**3.703**	0.983	3.494	1.047	0.000
Workload	**3.840**	0.885	3.745	0.926	0.000
Intellectually stimulating	3.718	1.032	**3.816**	0.964	0.000
Helpful resources	3.543	1.036	**3.716**	0.911	0.000
Lectures Helpful	3.591	1.123	**3.771**	0.918	0.000
Tutorial helpful	3.950	1.009	**3.825**	0.966	0.000
Motivation: Lecturers	3.552	1.096	**3.661**	0.986	0.000
Motivation: tutors	**3.938**	0.980	3.883	0.934	0.003
Individual assistance	**3.813**	0.951	3.746	0.950	0.000
Overall satisfaction	3.662	1.023	**3.767**	0.937	0.000

Table 2: Aggregated 2005 (semester 2), 2006 (semesters 1 and 2) and 2007 (semester 1) results of science subject evaluations at Monash University. Significant greater satisfaction appears in bold.

The summaries in table 2 and figure 1 show that there are significant differences for all but one evaluation item between the satisfaction of mathematics and statistics students and the students undertaking other sciences. The overall satisfaction item is a good indicator of the student satisfaction in general, and the data show that at Monash University, the overall satisfaction is lower for mathematics students.

However, although ‘other sciences’ students are overall happier than ‘mathematics’ students, this is not true for every single aspect of their subject experience. Mathematics and statistics students have higher levels of satisfaction than other sciences students in the areas of feedback, assessment criteria, tutorials and access to individual assistance. On the other hand, the opposite occurs in the areas of clear objectives, subject design, organization and progression, subject resources, intellectual stimulation and lectures. Furthermore, the most problematic aspects for mathematics and statistics students are appropriate resources and lectures, while for other science students the lowest ranked aspects are the effectiveness and timeliness of feedback.

It is interesting to note that the teaching format gives both the higher and the lower satisfaction to mathematics and statistics students. The effectiveness of lectures and the motivation from lecturers show two of the three lowest means, while tutorials and motivation from tutors correspond to the highest means. This is not the case for the ‘other sciences’ group, who also rate tutorials and practicals highly, but value the lectures more than mathematics and statistics students. This result is not surprising given that the only way to learn mathematics is by *doing* mathematics and that tutorials are usually structured around problem sets and other student centred activities.

These findings give food for thought and provide for mathematics and others sciences teaching staff at Monash an opportunity to learn from each other’s successes.

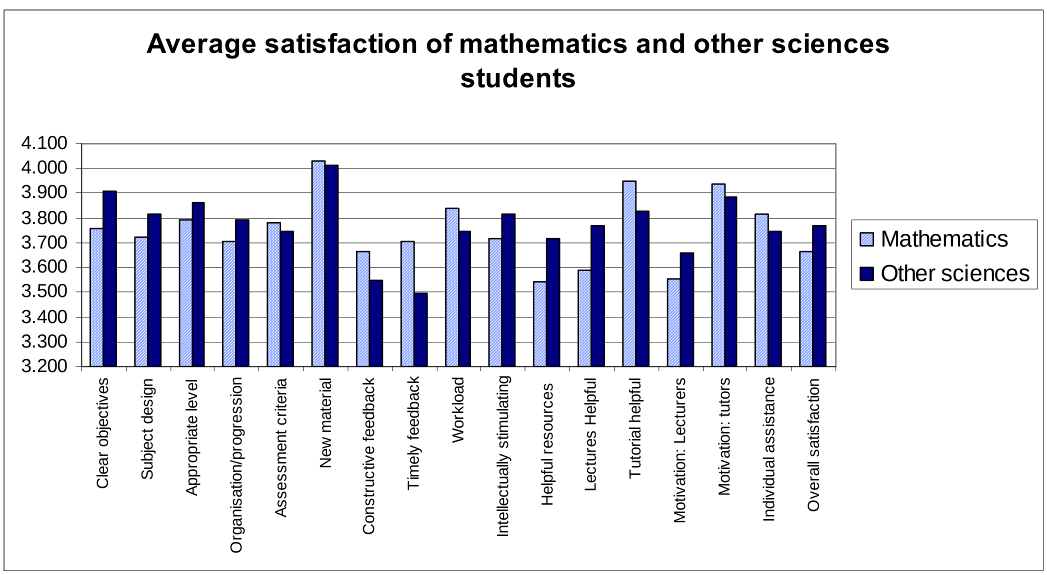

Figure 1: Aggregated 2005 (semester 2), 2006 (semesters 1 and 2) and 2007 (semester1) results of science subject evaluations at Monash University

3. Mathematics and statistics student satisfaction in Australia

3.1 The evaluations

The Course Experience Questionnaire (CEQ) is a national survey administered every year to graduates four months after graduation [1]. The survey items are grouped in ten clusters: Clear Goals and Standards, Intellectual Stimulation, Learning Resources, Good Teaching, Generic Skills, Graduate Qualities, Appropriate Assessment, Appropriate Workload, Student Support, and Learning Community (see appendix 1 for the evaluation items contributing to each cluster). The survey also includes an overall satisfaction item.

As with the Monash survey, graduates were asked to rate each of the item statements on a scale from 1 to 5 with 1='strongly disagree' and 5='strongly agree'. Due to the methodology used it is difficult to establish the precise response rate, and this varies by cluster with notable differences between the clusters that are used for the national ranking (Good Teaching, Generic Skills, and Overall Satisfaction) and all other clusters. However, comparison to national data on discipline graduates indicates that the number of graduate responses to the three main clusters represents between 50 and 60% of those who graduated between 2002 and 2006.

Since 2005 the clusters of Good Teaching and Generic Skills and the Overall Satisfaction item have been used as indicators for the national Learning and Teaching Performance Fund rankings of Australian Universities.

3.2. Findings

The summaries for the aggregated data covering the 2002–2006 period appear in table 3 and figure 2. The categorization of the groups 'mathematics' and 'other sciences' was made on the basis of the area of specialization indicated by the respondents. These summaries hence reflect the level of satisfaction of graduates who majored in a science area of study, and hence it does not include all students who undertook at least one mathematics or statistics subject as part of the (science or other) program of study. For each of the ten clusters above the percentage satisfaction indicates the percentage of students who answered the majority of the items in the cluster and have an overall mean of 3.5 or above for these items. The Overall Satisfaction statistics corresponds to the percentage of students who indicated that they either agree or strongly agree to the statement 'Overall I was satisfied with the quality of this course'.

These summaries show that there is no significant difference between the overall satisfaction of the graduates who completed a mathematics or statistics major and those who specialized in another area of science. The two cohorts also agreed that the university provided them with an intellectually stimulating experience; in fact it is pleasing to see that this was the aspect that attracted the highest level of satisfaction, with percentages above 80.

Despite this general similarity, there are differences between the two groups that are worth noting. On the one hand, the 'mathematics' group is more satisfied with good teaching, assessment, clear goals and standards, and workload. However, their level of satisfaction is not very high. In the area of Good Teaching and Clear Goals and Standards, only about 60% of the respondents who graduated with a mathematics or statistics major left the university with a recollection of a positive experience in the classroom, and with the feedback and support they received. Similarly, only 55% left with the impression that the assessment practices were appropriate; but it must be noted that the evaluation items contributing to this cluster are somewhat superficial and do not attempt to cover the whole range of aspects related to assessment, and focus only on whether assessment reflected students' understanding rather that their capacity to memorize facts. Finally, more than half of the mathematics and statistics graduates believed that the workload required to complete their courses was too demanding; even though compared to other science graduates they were less dissatisfied with this aspect of their study. This is not surprising as it is known that today' students are very likely to combine a full time study load with many hours of paid employment [6].

	Mathematics % satisfaction	n	**Other sciences** % satisfaction	n	p-value
Clear Goals and Standards	**60.38%**	253	55.77%	3023	0.032
Intellectual Stimulation	81.77%	314	84.50%	4803	0.090
Learning Resources	70.41%	295	68.51%	3713	0.206
Good Teaching	**59.46%**	1399	56.19%	14225	0.001
Generic Skills	68.10%	1601	**74.85%**	18950	0.000
Graduate Qualities	72.26%	719	**79.20%**	10476	0.000
Appropriate Assessment	**55.00%**	528	50.40%	4278	0.003
Appropriate Workload	**42.17%**	393	36.25%	2782	0.000
Student Support	75.19%	397	75.36%	5082	0.466
Learning Community	46.18%	326	**54.78%**	4677	0.000
Overall Satisfaction	75.21%	1769	75.51%	19062	0.375

Table 3: Aggregated 2002-2006 CEQ results for students who graduated with a science specialisation. Significant greater satisfaction appears in bold.

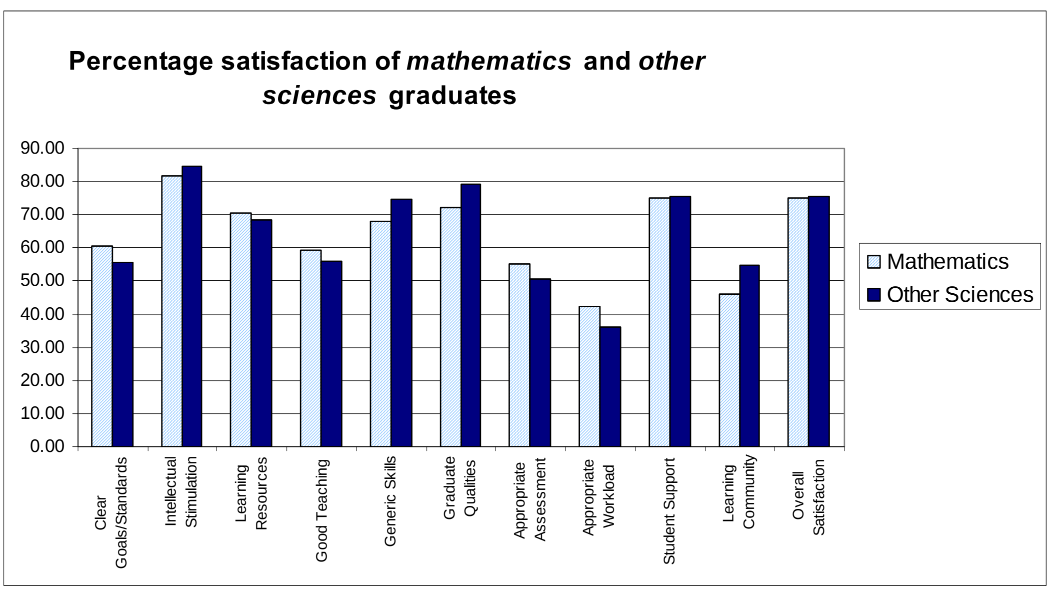

Figure 2: Aggregated 2002-2006 CEQ results for students who graduated with a science specialization.

On the other hand, mathematics and statistics graduates were less satisfied with Generic Skills, Graduate Qualities and with Learning Community. Only 68% of the mathematics graduates believed that their course helped them to develop analytic skills, problem solving skills, ability to work in a team, written communication, tackling unfamiliar problems and an ability to plan own work, and only 72% graduated with the sense that they have developed qualities that will enable them to tackle new situations, pursue new ideas and in general set them in the path of life long learning. Furthermore, the Learning Community cluster summaries indicate that many science graduates and more than 50% of the mathematics graduates respondents left the university feeling that they did not belong to a learning community.

In summary, the CEQ national data suggest that there is still much to do to improve the experience of Australian science students in general, and that any development of strategies to address the shortcomings would benefit from taking into account the discipline differences observed from what graduates are saying through this evaluation instrument.

4. Final discussion and conclusion

The aggregate data from the two surveys discussed above sheds some light into the experience of Australian mathematics students and graduates, as seen from their perspective. The results are not directly comparable as they involve different evaluation instruments, different timeframes and a different definition for the 'mathematics' group. However, it is still worth highlighting the areas that seem to be working well–taken in absolute terms, or relative to other sciences–and the areas that require fresh approaches.

The CEQ gives a picture of the different aspects of the university experience of Australian graduates with a specialization in mathematics or statistocs, while the Monash subject evaluations provide a more detailed insight into the teaching aspect in the context of this particular institution. The Monash experience cannot be directly extrapolated to other universities, but given the university's standing as the largest Australian university and a member of the Group of Eight [7], the conclusions may be applicable more widely to other traditional universities in Australia or around the world.

Analysis of the two sets of data highlights that if universities are to attract more students to do mathematics and statistics, there are three areas that need to be focused on. Firstly, serious thought should be put into improving the student experience with good teaching. In this context, good teaching includes interesting and effective explanations in lectures as well as helpful feedback and genuine interest in helping students to learn. Monash data suggest that the use of the lectures and tutorials format needs to be reconsidered. Lectures seem to be working for some students, but students value the tutorials and the interaction with tutors much more highly. In fact, this is the aspect of their teaching they are most satisfied with. Ironically, in large universities such as Monash, tutorials are usually conducted by inexperienced and poorly trained honours and postgraduate students.

Working in small groups seems to be particularly more important in mathematics teaching, and greater efforts should be made to use the small group interactive setting to teach mathematics. However, Monash data suggest that there are other aspects that require an improved emphasis. An investment must be made in developing and sharing good practices in mathematics and statistics teaching across and between the teaching departments, and in specialist training of academic and support staff.

Secondly, mathematical sciences departments should pay greater attention to their students' development not only within but also outside the classroom, and help them feel part of a learning community. Student engagement with the university could be fostered in many ways, but for some reason this area does not seem to have been explored widely. Team based assessment is a structured way to achieve a greater sense of belonging to a learning community. The importance of friendly and welcoming study spaces should not be underestimated. Students should be encouraged to participate in and feel welcome at departmental activities such as seminars, student barbeques, student camps, mentoring programs, and opportunities to interact with honours and postgraduate students.

Thirdly, more work needs to be done towards ensuring that mathematics and statistics students graduate with the confidence that they have a set of generic skills and graduate qualities that make them employable as well as ready for further studies. Although academics are strongly divided on the question whether it is the role of universities to develop skills sought by employers [8, 9, 10, 11], it is known that many prospective students shy away from mathematics because they fail to see clear career prospects. Departments of mathematics need to strengthen the message that a mathematics graduate is highly employable, and must place a greater emphasis on increasing awareness of the skills they learned by studying mathematics, and on addressing the gaps. A good starting point could be the graduate profiles developed in the United Kingdom [12].

Finally, it is hoped that the analysis of the data presented in this paper will provide both the background for further investigations on how mathematics students rate their undergraduate experience, and a stimulus for rethinking the teaching programs and the learning environment offered to these students.

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Appendix 1

Course Experience Questionnaire clusters

Generic Skills

- The course helped me develop my ability to work as a team member
- The course sharpened my analytic skills
- The course developed my problem solving skills
- The course improved my skills in written communication
- As a result of my course, I feel confident about tackling unfamiliar problems
- My course helped me to develop the ability to plan my own work

Clear Goals and Standards

- It was always easy to know the standard of work expected
- I usually had a clear idea of where I was going and what was expected of me in this course
- It was often hard to discover what was expected of me in this course (scale reversed before scale calculation)
- Staff made it clear right from the start what they expected from students.

Appropriate Workload cluster

- I was generally given enough time to understand the things I had to learn
- The sheer volume of work to be got through in this course meant it couldn't be all thoroughly comprehended (scale reversed before calculation)
- The workload was too heavy (scale reversed before calculation)
- There was a lot of pressure on me as a student in this course (scale reversed before calculation)

Appropriate Assessment

- To do well in this course all you really needed was a good memory (scale reversed before scale calculation)
- The staff seemed more interested in testing what I had memorized than what I had understood (scale reversed before scale calculation)
- Too many staff asked me questions just about facts (scale reversed before scale calculation)

Intellectual Motivation

- I found my studies intellectually stimulating
- I found the course motivating
- Overall, my university experience was worthwhile
- The course has stimulated my interest in the field of study

Student Support

- I was able to access information technology resources when I needed them
- Relevant learning resources were accessible when I needed them
- Health, welfare and counseling services met my requirements
- The library services were readily accessible
- I was satisfied with the course and careers advice provided

Graduate Qualities

- The course provided me with a broad overview of my field of knowledge
- The course developed my confidence to investigate new ideas
- University stimulated my enthusiasm for further learning
- I learned to apply principles from this course to new situations
- I consider what I learned valuable for my future
- My university experience encouraged me to value perspectives other than my own

Learning Resources

- The library resources were appropriate for my needs
- The study materials were clear and concise
- It was made clear what resources were available to help me learn
- Course materials were relevant and up to date
- Where it was used, the information technology in teaching and learning was effective

Learning Community

- I felt part of a group of students and staff committed to learning
- Students' ideas and suggestions were used during the course
- I learned to explore ideas confidently with other people
- I felt I belonged to the university community
- I was able to explore academic interests with staff and students

Inspiring mathematical thinking by processing a digital image using a spreadsheet

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Abstract

It is very well-known that student motivation can be enhanced by the use of relevant and familiar examples, contextualising theory and providing illustrations of areas in which it can be applied. This is particularly true of mathematics in which abstract theory can, for some students, be very de-motivational.

This paper is concerned with the use of digital images to illustrate mathematical concepts. Many (or most) students are familiar with the ways in which images captured by digital cameras can be post-processed, but are perhaps unaware that this requires a wide range of mathematics. Simple addition or subtraction is used to alter the image brightness; contrast involves a linear mapping of the pixel values; blurring involves averaging; sharpening requires convolution with, for example, a Laplacian filter. The are many, many more examples - it is perhaps surprising that almost any level of mathematics, from elementary to postgraduate, can find an area of application in digital image processing.

To assist students to implement these mathematical ideas with their own digital images, the author has written custom software, in the form of an add-in for Microsoft Excel. When loaded, this add-in allows students to import images. It decodes the individual image pixel values into three worksheets, one for each of the constituent red, green and blue components. Students can then process these values, using their knowledge of Excel functions and/or Visual Basic, to implement the mathematical techniques necessary to achieve the desired effect. The add-in then provides a way to regenerate a new jpeg image from these data.

This presentation will demonstrate the software, illustrate a range of mathematical techniques that can be carried out, and describe the experiences of a group of final year undergraduate students of mathematics who have used it here at SHU.

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The work of the SEFI Mathematics Working Group

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The paper describes the work and activities of the SEFI Mathematics Working Group. A group of academics from European universities, it advises upon the teaching of mathematics to engineering students and the development of the curriculum. Its current focus is engineering mathematics education within Bachelors and Masters Degree programmes following the Bologna Agreement, and examining the highly varied forms of assessment across institutions and countries.

Keywords: Mathematics; Engineering; Education; Curriculum. *AMS Subject Classification*: 97B10; 97B40; 97B70; 97C40

1. Introduction

The European Society for Engineering Education SEFI (Société Européenne pour la Formation des Ingenieurs) was established in Brussels in 1973 to monitor the formation of professional engineers within European institutions. It holds an annual conference but most of its work is undertaken by working groups dedicated to specific needs. The Mathematics Working Group (SEFI-MWG) was set up in 1982, initially drawing delegates from post-war Western Europe but from 1990 nearly all countries from Finisterre to the Urals came to be represented. Working 3-day seminars are held at approximately two-year intervals to progress the Group's work forward and a smaller executive sub-committee called the Steering Committee, meeting more regularly, coordinates the efforts meantime.

The SEFI-MWG was set up with the following aims:

- To provide a forum for the exchange of views and ideas among those interested in engineering mathematics
- To promote a fuller understanding of the role of mathematics in the engineering curriculum, and its relevance to industrial needs
- To foster cooperation in the development of courses and support material
- To recognise and promote the role of mathematics in the continuing education of engineers in collaboration with industry

To this one needs to add the following tasks, achievements and subsequent roles of the SEFI-MWG as these have evolved over the Group's 25-year existence:

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- The production of an advisory core curriculum in mathematics for the formation of the professional engineer within universities in the developed world (Curricula published in 1992/2002)

- The provision of an international forum for the exchange and development of ideas and practice in the teaching of mathematics to engineers
- The establishment of international initiatives to compare and contrast different practices in converting teaching into effective learning (Assessment Project)
- The monitoring and adaptation of widely varying curricular practice into a common internationally accepted form (Bologna Agreement)

2. The Evolution of the Core Curriculum

The Seminars in the 1980s enabled the widely different practices in engineer formation across Western Europe to be fully aired and possibly for the first time, to be mutually understood. There is general agreement that the specialisations of engineering; e.g. aeronautical, electrical, mechanical, marine, nuclear, etc have a distinct meaning to members of the public in the developed world but if one asked what a fully qualified professional engineer actually might do on a day-to-day basis there is a much less clear view than there might be over the duties of other professionals such as those in medicine or the law. Also, in the scientific community in many continental European countries some applied mathematicians and physicists are very often termed engineers. This has meant that in specifying a core curriculum of mathematical study the needs of more mathematically orientated professionals has been an issue.

Beginning with the 5th Seminar held in Plymouth, UK in 1988, sub-working groups of the SEFI-MWG set about defining an advisory curriculum of flexible length ranging from 220 to 320 hours aimed at appropriate institutions and countries. This was published in English, French and German by SEFI in 1992 [1]. The content comprised:

- Analysis and Calculus
- Linear Algebra
- Discrete Mathematics
- Probability and Statistics

in the proportions of about one half for the analysis and calculus and about one sixth each for the others, together with a proviso that numerical methods be infused within the curriculum. Prerequisite mathematical study, i.e. highschool mathematics, was specified and overlying the curriculum elective study in appropriate areas of mathematics was detailed for the differing specialisations of engineer. Introducing the curriculum was a discussion and commentary which recognised that so-called 'high technology' is really a mathematical technology and noted that advances in computation would limit the validity of the document to a 10-year lifespan.

A revised core curriculum document was published by SEFI in 2002 [2]. In many respects this retained the main features of the earlier document but the concept of the Core as well as the above four components was extended into the underlying and overlying material. New was the inclusion of Geometry as a specific topic and the special emphasis placed upon student learning outcomes. Also teaching practice in Central and Eastern Europe received more importance. The prerequisites are now termed Core Zero; the core itself is split into two hierarchical tiers, Core One and Core Two and the electives called Core Three. During the 1990s serious concern over the mathematical fitness and changing background of new entry engineering students emerged, firstly in the United Kingdom, but later on in many other countries. The underlying reasons appear to be cultural as well as educational but the impact on the curriculum has been that Core Zero material has encroached upon university mathematics. Also, the time allocated to teaching mathematics, i.e. the Core itself, was coming under pressure in many institutions and countries as engineering programmes evolved.

3. The Evolving Focus of the Seminars

The 6th Seminar in Balatonfured, Hungary in 1991 brought in Central and Eastern European delegates and further exchanges of the wide disposition in engineering mathematics education. Later in the 1990s focus came in special out-of-sequence seminars on areas of need, notably statistics at the Prague Seminar in 1994 and geometry at the Bratislava Seminar in 1997. By the 9th Seminar in Helsinki, Finland in 1998 the decline in entry competency was widely reported from within Europe and beyond. Emphasis too went to considering the study of the Core as the main component of lifelong learning. The widespread use of computer technology was emerging as an issue by this time, and whilst welcome in its role in the removal of drudgery in calculations delegates reported that considerable care was needed not to trivialise its use, notably with computer algebra. Interest too was taken in looking forward to a time when computational capacity would become almost optimal and whether or not it would be possible to define an irreducible core of mathematical knowledge that an engineer would need to have irrespective of computational advances. From the mid 1990s and onwards delegates came from outside Europe, notably from Argentina, Australia and the USA. The same themes were revisited at the 10th and 11th Seminars in Miskolc, Hungary in 2000 [3], and Gothenburg, Sweden in 2002 [4]. With learning outcomes becoming paramount emphasis was moving to assessment, a theme that would play a major part in the subsequent and present activities of the SEFI-MWG.

4. Recent Activities

The SEFI-MWG has a reputation for doing rather than talking and its early seminars in the 1980s placed strong emphasis on sub-working group discussions, such as the ones on analysis and calculus, discrete mathematics etc that led to the four main components of the Core Curriculum. Many of these sub-working groups made considerable progress, for example by writing questionnaires, and produced valuable interim reports that were carefully distilled as input into larger reports such as the Core Curricula. In the early 1990s there was a move towards more presented papers mainly to draw in Eastern European delegates who had to present a paper in order to receive financial support to attend. From 2000 onwards a way was found to move the balance back to the earlier model, i.e. round table working. Activities within the SEFI-MWG are currently concentrated on the Bologna Agreement and the Assessment Project and meanwhile valuable debates have taken place at the most recent seminars, namely the 12th Seminar in Vienna, Austria in 2004 [5], and the 13th Seminar at Kongsberg, Norway in 2006 [6].

4.1 The Bologna Agreement

The Bologna Agreement calls for Bachelor programmes of c.180 European Credits (ECTS), taken over about 3 years or 6/7 semesters, followed by c.120 ECTS up to masters level, or about 2 years or 4 semesters.

The decision to opt for this was dictated by the aim of transference in academic study between European institutions. This might be good in principle but might be inapplicable and inconsistent with the programmes that some institutions offer. However other institutions are moving this way and some countries such as Belgium are more committed than others. SEFI-MWG delegates however are worried about what Bachelor qualifications might come to mean noting that there could be a wide variety of levels and that mathematics might be put under pressure and reduced. It would also risk being undermined at the lower end of the curriculum with an increasing amount of 'levelling-up' Core Zero material. Some however have commented that Bologna could work if given time to bed in but there may be a need to distinguish between those Bachelor programmes that naturally lead on to Master programmes and research and others that constitute an exit pathway, i.e. an end of formal academic study.

Delegates were asked to go away from the 13th Seminar and address the mathematical curriculum requirements of a Bachelor programme. There are two considerations to be accounted for initially. Firstly, as the start point level of mathematical study seems to get lower with every passing year, this would need to be rationalised in terms of Core Zero and Core One of the Core Curriculum. Also, the endpoint of study would come before the end of Core Two: this too would need to be rationalised in terms of the type of programme and engineering discipline. What the Group might do, is define curricula for Bachelor Type A, (i.e. proceeding to masters) and Bachelor Type B (terminating). Less academically able students might go for Type B and it might be more open as to the start level of what such a mathematics programme should be.

4.2 The Assessment Project

In 2005 the SEFI-MWG began looking into the many different assessment models that operate across the many institutions and countries in Europe. The written examination appears internationally to be a well-tried, tested, and administratively proven model for assessment in many subjects. The UK prefers 3-hour exams whilst other countries have examinations up to 5 hours (Norway) or as little as 1.5 hours, perhaps supported by an oral examination (Central Europe). In many institutions large numbers of students dictate that oral exams are unaffordable at lower undergraduate level in terms of time and effort, as are the assessment, trustworthiness and reliability of coursework or take-away assignments. There is however much respect for assessment via assignment for project and other work at higher undergraduate levels. Learning outcomes spell out precisely what the student should be able to do having covered a particular element in a unit though the final examination cannot and maybe should not necessarily test all of these. Rather more specific learning outcomes can be assessed by formative testing. Delegates in Kongsberg spoke about the use of computerised tests to reinforce learning and gave excellent examples of the assessment of concepts with multiple-choice questions. There is a strong feeling that such precision testing be used carefully and proportionately in the formative phase and that students have full powers of expression and explanation in their final summative examination. Whatever form of assessment is adopted all assessors agree that the main aim of assessment is to measure the ability of students to communicate their mathematics most efficiently and effectively.

The next stage for the SEFI-MWG Assessment Project is to compare actual assessments from different countries and institutions across Europe. This is expected to start with a focus on first year calculus and linear algebra in programmes for engineers.

4.3 Future Investigations

In addition to discussing Assessment and the Bologna Agreement delegates at the recent seminars have also been looking at the following:

4.3.1 The key issues in teaching engineering mathematics for understanding. Students need to understand mathematics as a language of scientific communication and the understanding needs to be robust and versatile enough to cope with the unpredictable challenges of being a practising engineer. Good teaching facilitates learning, irons out misconception, and enables students of diverse mathematical background to reach a common minimal level of understanding. The goals of students are often short-term for a subject such as mathematics and maybe assessment orientated, i.e. they seek just enough knowledge and guidance to pass a specific examination. They can thus risk developing only a surface learning of the subject and need to deepen this by dealing with appropriately chosen mathematical challenges within an engineering context.

4.3.2 Innovative ways of teaching mathematics. Freshers can experience a considerable culture shock when reaching university and finding themselves in large classes for lectures. Engineering mathematics is delivered in this way in many institutions though some monitor the attendance, e.g. by using swipe cards coupled to following up and counselling poor attendees. All participating institutions in the SEFI-MWG use Information Technology (IT) to some extent and have moved forward from the view that its introduction is to reduce staff involvement. Rather IT, and Web-based technology have enhanced the quality of delivery, distance learning and power of communication. A new level of support has emerged in an environment of popular culture where e/mail is the most common form of academic communication. There is some evidence however that ill-considered computer aided teaching and assessment can cause frustration and anxiety so care needs to be taken as to measuring carefully the volume and quality of electronically available material.

5. Forward to the 14th Seminar in Loughborough, UK in April 2008

The next Seminar is due to take place at Loughborough University, United Kingdom on 6th to 9th April 2008. It will be an event held jointly with the Institute of Mathematics and its Applications (IMA) that will be holding its 6th Conference on the Mathematical Education of Engineers. The IMA conferences, held since 1994, have paralleled the SEFI-MWG seminars in many respects whilst focusing on UK related issues and in particular setting in train reports and measures to counterbalance the declining mathematical preparedness of engineering students. It promises to be an interesting and varied event.

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Overcoming the barrier of mathematics for university students–The beginning of engineering careers

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An entrance system has been initiated at the engineering faculty with the objective of decreasing the dropout rate of students in early stage of their careers. This system consists of the Bridging Course (30 days duration) and a program of Complementary Foundation (4 months duration).

The first part is an intensive course of required high school mathematics material for the course. The students that do not pass the initial evaluation must complete the program of Complementary Foundation.

This consists of two Modules: Mathematics and Study Techniques both oral and written works. These modules are designed to assist students with difficulties such as reasoning in a logical manner, lack of high school mathematics, mistakes made in reading and comprehension, study techniques, absence of study habits, etc… It was observed that to help students adapt to university life, they should be able to solve problems in an independent manner. A skill not taught at school but essential at university is self-management of time and problems.

Key words: conflict, Mathematics, entering, studying, engineering.

2000 Mathematics Subject Classification*: 97B02; 97D00*

1 Introduction

Situation

Of great concern to the engineering faculty, are the high dropout rate and the large number of students failing subjects in first year or who discontinue their studies. How do we retain these students in the university system? The dropout and failure rates are high in the first two years; this may largely be attributed to secondary school not developing study skills for an easy transition to university education [1].

This problem is not ignored by first year university educators and has been confirmed by numerous investigations on this subject, for example, an investigation carried out in the Basic Common Cycle (BCC) of the University of Buenos Aires (UBA), where lack of study skills are attributed to secondary school, as the author states that 86% of interviewed teachers state that students are unprepared for university study and 70% of students agreed that the preparation and development was "not enough" and "insufficient”. Due to this shortfall in the

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preparation of students, the order of learning subject is altered and often this leads to a restructuring of the university programme of study [2]

Problem of the first year student

To adapt to university life it is necessary to appreciate a student's point of view, in relation to demands and characteristics of social aspect. University is a place of learning, where students also need to learn to be a student. This learning requires adapting to new styles, teachers, and different institutional operations. So who are these people who are not able to learn and therefore don't stay at the institution as a student and in most of the cases, to abandon University or to remain a perpetual (failing) student of the university.

Studying at university is very different from the study culture in the high school and this difference produces problems. An example is the difficulty that the students have in reading and interpretation of concepts using the specific texts of the subject and the tendency to appeal to concise notes and simplistic explanations of the concepts. That is to say they avoid the use of the book sources, why? In student culture notes and portfolios replace the books.

At the university, students need to take control of the time dedicated to learning and integrate their own strategies to aid learning with illustrations, summaries, previous organisers, conceptual maps, textual structures, etc

To all of this, add problems associated when students are asked to write or talk on topics, as well as the absence of tools or study techniques, they need strategies of how to comprehend.

So if students recognise their difficulties in understanding, they will need to find some strategy that allows them to understand the study material. The reader is the one that should decide whether the observed difficulty is solved or not, and this decision has to do with the reason that he/she has for the task. If he/she decides to solve the lack of understanding he/she will choose some of the following options; to keep the current problem in mind hoping to clarify it with the reading of the text or to undertake an action immediately, it can stop and think of the reading, go back or jump ahead in the text, consult a dictionary, another bibliography, professor, etc… [3]

With regard to the retention strategies, there are diverse measures that can help them remember better. In reference to semantic memory, that is meanings, it supposes that the retention is richer and more durable; the more relationships are connected with the concepts. This supposes that different practices allow building of new relationships, as well as how much control of understanding and conscience are remembered of what has been studied. We make reference to the meta-memory, that is to say, the knowledge that we have of our memory, it’s limitations, resources, what allows us to analyse and to elaborate strategies to remember well: to take conscience of the problems and to look for possible solutions, to apply them and to decide which is the most beneficial according to each study situation.

2. Entrance system to the Engineering faculty

There have been many different entrance methods into the Engineering faculty over the years. Before the current system, all the students entered together, first year classes would be have numbers between 1000 and 1200 students though after three months the class size may be halved due to dropouts even before the first tests.

In the year 2005 an entrance system was implemented in the Engineering faculty that has the general objective of preventing this rate of departure of students so early. It was proposed in March to run a Bridging Course (lasting 30 days) and of the programme of complementary

Foundation (duration 4 months) whose contents have been established by the Academic Council of the faculty.

3. Bridging course

3.1 Organisation

The Bridging Course is a prerequisite for all the courses of the faculty; it is conducted in February- March. The aim is to develop average knowledge of Mathematics and students must pass this course in order to proceed to first year.

The students have a note of activities to work in the classroom, and they must attend to theoretical practical daily classes of three hours of duration. In this course the following contents are developed [4]:

Numeric groups. Intervals. Operations

Real function. Lineal function. Quadratic function

Algebraic expressions. Theorem of the Rest. Divisibility. Operations. Factoring

System of two lineal equations with two unknown quantities. Application problems.

Trigonometry: Systems of mensuration of angles. Trigonometric Work. Trigonometric identities and equations. Application problems.

The development of the program activities was coordinated and organized in all the commissions. This course involves two tests at the end of the course. The students that do not pass must complete the programme of Complementary Foundation.

4. Itinerary of Complementary Formation

4.1 Organisation

The programme of Complementary Foundation consists of two Modules:

Module 1: Mathematics

Module2: Study techniques with oral and written expression

These modules are taught together, and are specifically designed to attempt to correct difficulties in the logical reasoning, basic high school mathematics, flaws in reading comprehension and in the study techniques, absence of study habits, to mention some, since the list is wider. Since this course lasts four moths it gives the student time and the opportunity to really understand and learn the topics.

Each module is examined and students must pass both modules of the programme.

The course is repeated in the second semester for students who:

did not attend or didn't pass the first semester.

are studying final year of secondary school.

graduated secondary school or but have failed to attain university entrance requirements (Law 24521 of Superior Education).

The students that pass the Programme here, and fulfil the entrance requirements to the faculty will be able to enter the first year next year. It was observed that to help students adapt to university life, they should be able to solve problems in an independent manner. A skill not taught at school but essential at university is self management of time and problems.

4.2 Mathematics module

The Mathematical knowledge involves the demonstration of certain skills and abilities that will help the integration of new knowledge when passing from one level of learning to another. In particular in the transition from secondary school to University is fundamental that the key competencies are incorporated into the program [5].

In Mathematics the intellectual aptitudes are shown when operating with symbols, representations, images, use of languages (colloquial, mathematical, etc.) and include the analytic capacities, the creative ones and reflection on the thought processes. The practical aptitude or procedural content, correspond to the procedures related with the resolution of problems. Some are those that require the application of some algorithm, as in counting, calculating, graphing and measuring. Strategies refer to estimating, gathering, organising, comparing classifying, and analysing. There are always debates about what mathematics is needed by a high school graduate in order to be fully prepared for challenges and advances of technology and science. How do we prepare students for university life and studies?

Mathematics in some form or another is required in many disciplines at university level either specific mathematical content or the ability to reason logically. So one of the main objectives at primary and secondary level is to teach students how to logically reason and solve new situations using previous ideas. Thus effortlessly building new knowledge on this knowledge [6,7]. This should be kept in mind when proposing the activities and contents of this module.

There are 3 practical classes of 2 hours and one theoretical class of 2 hours each week. There were three different streams or groups of students, two at morning and one in the afternoon.

The teaching of this module was organised into theoretical classes and practical classes similar to the format of university

The assessment process consists of 2 tests, each one with its re evaluation test and at the end of the semester there was a further opportunity given to students who failed one of the tests. Although the mathematical content was comparable to that in the bridging course, in the practical classes different worksheets were used since there was extra time to explore the topics more deeply. For example, a practical on calculator handling in trigonometry was included

4.3 Techniques of study and oral and written expression module

This module is organised as workshops where the emphasis is on the student to be able to acquire information. Some activities are specifically designed to increase the knowledge of the students about the written (correction, grammar, adaptation, coherence and cohesion) language particularly composition, styles and values of the writing.

On the other hand, students are encouraged to share their reflections on writing process, the analysis of the different steps it takes, the blockages encountered by verbalizing their feelings in front of their peers. It is equally important that students understand social structure, the university institution and the student's place in this institution, than consider just the methods and techniques of academic work and practice and manipulate the tools of the academic university work [7,8]

4.3.1 Contents

The context of the program included:

Analysis of text: Its legibility. The plain language style.. The process of breaking down the text. Coherence and cohesion. Paragraphs: definition, more frequent errors. The size of the sentence. Punctuation: importance, the hierarchy of the signs. setting in use of the language: the exhibition. The application, an administrative necessary gender. [9]

Society, University and the student: Society and state in modern Argentina. The National University of Jujuy. The student commitment.

Culture, science and profession: scientific knowledge and professional practice. The intellectual work: the occupation of thinking [10].

Methods and technical of work intellectual: conceptual rationalization and intellectual procedures for the study in the university. [11]

The tools of the intellectual work: reading, notes, summaries, reports, exams, monographs

5 Development of the entrance system of teaching cycle 2005-2006

5.1 Inscribed applicant

In the year 2005 (2006) in the faculty of Engineering of the University of Jujuy had 1137 (1096) students. The faculty services the following courses: Chemical engineering, Mining Engineering, Computer Engineering and Industrial Engineering, Degree in Information Technology, Degree in Food Technology, Degree in Geology, Programmer Analyst, Biochemistry and Pharmacy through agreement with the National University of Tucumán and it also possesses the Common Articulate (CCA) Cycle, first year cycle in all the engineering of the five National University of Tucumán, it Jumps, Santiago of the Tideland and Catamarca.

5.2 Bridging course 2005-2006

The structure of the course was 6 tutorials, 3 in the morning and 3 in the afternoon, and it involved 1006 (906) students.

Each stram had daily classes of 3 hours duration from Monday to Friday, and the students had worksheets to develop in class.

The course lasted 3 weeks and it was run by a tutor. Two evaluation instances were given for this course.

In the first opportunity to sit the test, 713 (681) students of the 1137 (906) listed were present.

So at this stage we already observed natural an attrition. In this date first opportunity to sit the test, 229 (222) passed, that is, 32,1%. In the second opportunity to sit the test 462 (419) students turned up, of whom 125 (225) passed, that is, 27% approximately (51%).

After each test a schedule was prepared so that students could view their test.

A pass was awarded to students who obtained at least 50 out of a total of 100 points. In total, 364 (447) passed. See Table 1

The students who didn't pass the bridging course continued to study the Programme of Complementary Foundation (TFC). This TFC program was free for faculty student, but it was fee-based for outsiders.

6 General observations

It was observed that students adapted well to the demands of being a university students. This was a gradual process; students were doing only two modules which enabled them to adapt smoothly and steadily. These conclusions show the benefits experienced by the institution.

This experience was the result of feedback where: the advice to adopt an entrance system that would lower withdrawals and at the same time the bridging course provides the mechanism to prepare students for first year university studies. In short, problems, obstacles, and difficulties of the first year of study have always been a discussion topic and concern for new students. The observations were positive. This system allows the retention of those students that take longer to adapt to the transition, and TFC allows this. There will always be some students who can cope with the transition from school to university. Some students, who are quite capable, need longer to adapt to university life have a better chance of staying in the system.

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Visualisation of time series by the visually impaired

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In this paper challenges and solutions to the mathematical and statistical community when dealing with visual impairment are presented. In particular, the focus is on the accessibility of graphically displayed information in the field of time series analysis. We illustrate through an example how high resolution Braille can be used to visualize two important graphical representations for an observed time series – the time plot and the correlogram. It is shown how the time plot and correlogram are used in conjunction with the Phillips-Perron test to determine whether the observed series is stationary or non-stationary and, if non-stationary, whether the series has a deterministic or stochastic trend. Finally a collection of technological solutions that the first author is using in his undergraduate studies and research, are given.

Keywords: *Visual impairment; Graphically displayed information; High resolution Braille; TactIle Graphics EmbosseR (TIGER); Time series analysis; Time plot; Correlogram; Stationarity; Phillips-Perron test*

AMS Subject Classification: *97C80; 62M10*

1 Technological Innovation

Technological innovations have greatly impacted upon the mathematical world of the visually impaired over the last twenty years. For example, a report in 1983 on a case study in the UK, see [1], pointed out these two technological prospects:

‘Firstly, more sophisticated devices are being developed to convert print-face into electronic voice speech...’

‘Secondly, means of converting information stored by computer need to be more readily available.’

Both these and many other challenges have been overcome and are still being refined today.

The key feature of access technology is communication. Technological innovation should therefore not only strive to increase accessibility, but also seek solutions which enable the user to be independent. In order to communicate independently, there should exist systems for reading, writing and dealing with graphics. Furthermore, it is important that these solutions can be interpreted by all members of society, since the undergraduate student will probably pursue a professional career – hence the strong emphasis on independence.

The period of time consumed in order to apply a solution should also be taken into account. The transcription of materials into Braille, recording of textbooks and making graphs accessible can take up enormous amounts of time, which is not efficient in a productive world.

Another barrier to technological advance is the financial implications incurred by implementing a proposed solution. This has always been a challenge and especially hinders development and limits access to technology in developing countries.

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There are individual challenges, e.g. the visual incapacity of especially those being blind from birth, to whom technology might not have an answer, yet...!

2 Basics of Braille

Braille was invented by Louis Braille in 1829, originally for soldiers to enable them to read at night without the use of light – see [2]. Each Braille cell is a 3×2 matrix which can be thought of as [2]:

'binary-coded numbers that use filled positions for ones and empty positions for zeroes.'

There exist many Braille systems comprising of all 63 combinations of 'dots' which form the various symbols of the underlying system. The interpretation of the respective symbols therefore depends on the system being used. There are for example many different Braille systems for mathematics, chemistry and music. In addition, the interpretation of different symbols is also language dependent.

Although Braille enables good access to the Braille user, it is clear that it does not serve as a universal communication tool. However, the use of Braille wherever possible for personal use is strongly encouraged.

3 High resolution Braille

As explained in [3]:

'"Normal" Braille standards define the dot spacing within a Braille cell to be between 2.3 and 2.5 mm, the cell to cell spacing to be 6.0 to 6.2 mm and the dot height to be 0.25 to 0.53 mm.'

Therefore the Braille cell (the six dot blocks) would not be sufficient for creating graphical material, since blocks cannot be made more compact or be placed closer to one another without re-engineering the structure of the Braille cell. In 1996, Peter Langner developed a method to emboss 20 dots per inch, so-called 'high resolution Braille', which has inter-cell dot spacing of 2.54 mm and inter-cell spacing of 6.25 mm – see again [3]. This technology is known as TactIle Graphics EmbosseR (TIGER) and, according to [3]:

'... was patented by Oregon State University, licensed to the spin-off company ViewPlus Technologies (http://www.ViewPlus.com)...' – see reference [4].

As discussed in [5], it is possible with the TIGER Braille embosser to emboss graphics and text from any Windows based application.

4 Application to time series analysis

To illustrate the use of the TIGER Braille embosser, we look at a financial time series application in which SAS 9.1.3 for Windows is used to plot graphs relating to an observed time series. We will use these graphs, together with the Phillips-Perron test, to determine whether the series is stationary or non-stationary, and if non-stationary, whether the series has a deterministic or stochastic trend.

A time series is said to be covariance stationary[1] if its first two moments, that is, its mean and autocovariance function, are both independent of the time period. In effect, a stationary series

[1] See [6] or [7] for the difference between covariance (or weak) stationarity and strict stationarity. Since covariance stationarity is usually sufficient in time series modeling, we will use the term 'stationarity' in the paper to refer to 'covariance stationarity'. These texts,

has a constant mean and its autocovariances only depend on the lag (or distance) between the observations and not on time itself.

Let $z_1, z_2, ..., z_n$ denote the n observations of a time series. The time plot of the series is a graph of z_t against t for $t = 1, 2, ..., n$. The sample autocorrelation function of $z_1, z_2, ..., z_n$ is given by:

$$r_k = \frac{\sum_{t=k+1}^{n} (z_t - \bar{z})(z_{t-k} - \bar{z})}{\sum_{t=1}^{n} (z_t - \bar{z})^2} \tag{1}$$

where $\bar{z} = \frac{1}{n}\sum_{t=1}^{n} z_t$ is the sample mean and k is the lag between observations. The correlogram is then a plot of r_k against k for $k = 0, 1, 2, ...$.

Consider the daily South African Rand – Argentine Peso exchange rate from 2004/03/01 to 2005/02/28. This series has 255 observations (in effect, $n = 255$). The time plot of the series given below is obtained using the SAS procedure '**`proc gplot`**'.

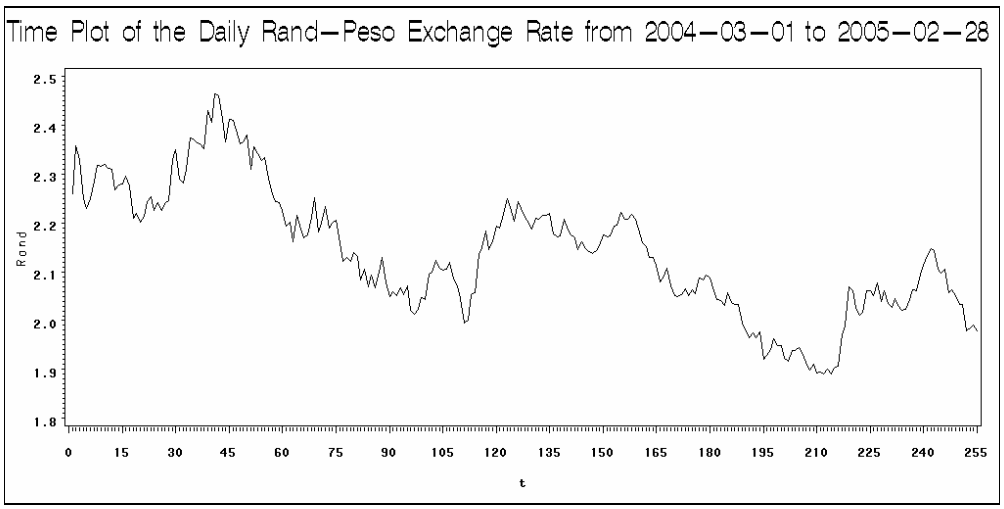

Figure 1 – Time plot of observed series taken from the SAS Graph Window

The corresponding plot using the TIGER Braille embosser is as follow:

[6] or [7], can also be consulted for more information regarding any other time series concepts dealt with in the paper.

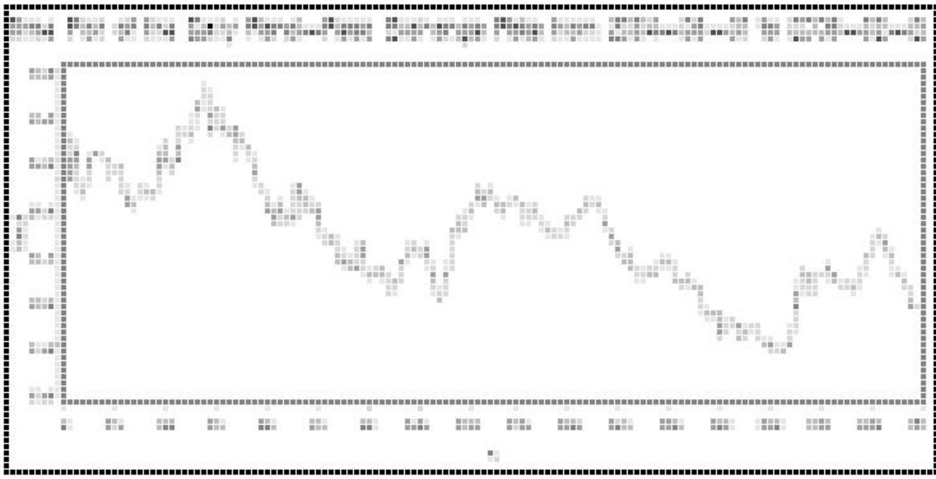

Figure 2 – Time plot of series as depicted by the TIGER Braille embosser

It is clear from the time plot that the Rand-Peso exchange rate has a downward trend over time. In effect, the mean level of the series decreases over time and hence the series is non-stationary.

The correlogram for the Rand-Peso exchange rate is obtained in SAS with '**`proc arima`**' and given below:

```
        Correlogram of the Daily Rand-Peso Exchange Rate from 2004-03-01 to 2005-02-28        1

                                   The ARIMA Procedure

                                  Name of Variable = zt

                             Mean of Working Series    2.139158
                             Standard Deviation        0.128086
                             Number of Observations         255

                                     Autocorrelations

Lag    Covariance    Correlation    -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1      Std Error

  0      0.016406        1.00000    |                    |********************|             0
  1      0.015952        0.97229    |                 .  |******************* |      0.062622
  2      0.015495        0.94445    |                .   |******************* |      0.106471
  3      0.015077        0.91897    |               .    |******************  |      0.135396
  4      0.014736        0.89822    |              .     |******************  |      0.157974
  5      0.014445        0.88044    |             .      |******************  |      0.176871
  6      0.014089        0.85874    |            .       |*****************   |      0.193296
  7      0.013695        0.83474    |            .       |*****************   |      0.207719
  8      0.013225        0.80610    |           .        |****************    |      0.220482
  9      0.012755        0.77746    |           .        |****************    |      0.231751
 10      0.012387        0.75502    |          .         |***************     |      0.241763
 11      0.012059        0.73505    |          .         |***************     |      0.250839
 12      0.011819        0.72039    |          .         |**************      |      0.259149

                           "." marks two standard errors
```

Figure 3 – Correlogram of observed series taken from the SAS Output Window

Note that the output in the SAS Output Window is in terms of text. Therefore the correlogram is a 'text plot' with the magnitude of each sample autocorrelation depicted using stars, '*'. The corresponding plot obtained with the TIGER Braille embosser is as follow:

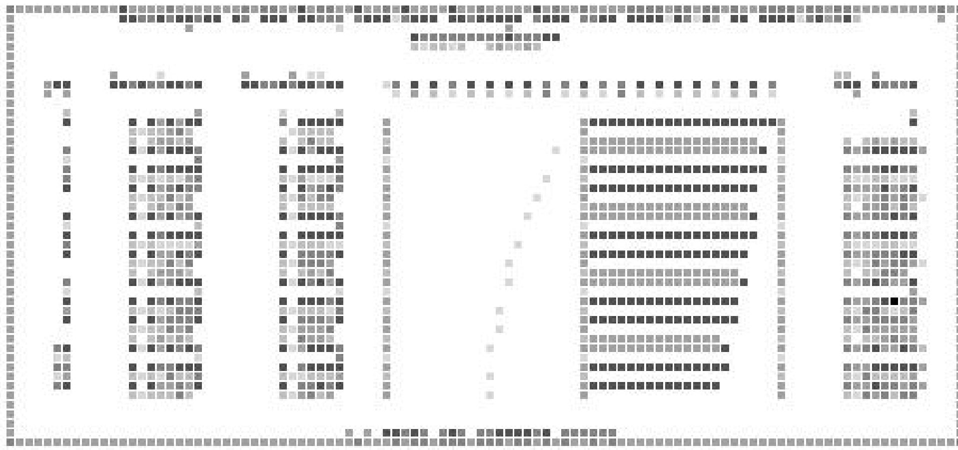

Figure 4 – Correlogram of series as depicted by the TIGER Braille embosser

The correlogram shows that the sample autocorrelations decrease at a very slow rate as the lag, *k*, increases. This is indicative of non-stationarity.

To determine whether the trend is stochastic or deterministic, the Phillips-Perron test, [8-9], can be used. The Phillips-Perron test has three specifications:

Case 1 (Zero Mean) – to be used if the observed series fluctuates around zero.

Case 2 (Single Mean) – to be used if the observed series fluctuates around a non-zero mean.

Case 3 (Trend) – to be used if the observed series has an upward or downward trend.

Since the Rand-Peso exchange rate has a downward trend – see again Figure 1 (or Figure 2), Case 3 is applicable. It is then assumed that the process that generated the series is the following AR(1) process:

$$z_t = \alpha + \rho z_{t-1} + \varepsilon_t \tag{2}$$

with ε_t an independent and identically distributed error (innovation) sequence. The null hypothesis, $H_0 : \rho = 1$, is tested against the alternative $H_1 : \rho < 1$. It follows that under the null hypothesis the series is a random walk with drift given by:

$$z_t = \alpha + z_{t-1} + \varepsilon_t \tag{3}$$

The random walk with drift is a series with a stochastic trend.

To perform the Phillips-Perron test, the parameter ρ is estimated using ordinary least squares (OLS) regression. The regression model is given by:

$$z_t = \alpha + \delta t + \rho z_{t-1} + \varepsilon_t \tag{4}$$

Note that a time trend, δt, is added to the model. Therefore, if the null hypothesis is rejected, it is concluded that the series has a deterministic trend.

One of two test statistics can be used for the Phillips-Perron test:

$$Z_\rho = \nu - \tfrac{1}{2}\left(\frac{n\hat{\sigma}_{\hat{\rho}}}{\hat{\sigma}_{OLS}}\right)^2 \left(\hat{\sigma}^2_{NW} - \hat{\gamma}_{0,\,n}\right) \tag{5}$$

or

$$Z_\tau = \frac{\tau\sqrt{\hat{\gamma}_{0,\,n}}}{\hat{\sigma}_{NW}} - \tfrac{1}{2}\left(\frac{n\hat{\sigma}_{\hat{\rho}}}{\hat{\sigma}_{OLS}\hat{\sigma}_{NW}}\right)\left(\hat{\sigma}^2_{NW} - \hat{\gamma}_{0,\,n}\right) \tag{6}$$

where:

$\hat{\rho}$ is the OLS estimate for ρ

$\hat{\sigma}_{\hat{\rho}}$ is the OLS standard error for $\hat{\rho}$

$\nu = n(\hat{\rho} - 1)$

$\tau = \dfrac{\hat{\rho} - 1}{\hat{\sigma}_{\hat{\rho}}}$

$\hat{\varepsilon}_t$ is the OLS residual from the estimated regression model

$\hat{\sigma}^2_{OLS} = \dfrac{1}{n-3}\sum_{t=1}^{n}\hat{\varepsilon}_t^2$ is the OLS estimate of σ^2_ε (note that there are 3 parameters, α, δ and ρ, in the regression model)

$\hat{\gamma}_{j,\,n} = \dfrac{1}{n}\sum_{t=j+1}^{n}\hat{\varepsilon}_t\hat{\varepsilon}_{t-j}$ (note that $\hat{\gamma}_{0,\,n} = \dfrac{1}{n}\sum_{t=1}^{n}\hat{\varepsilon}_t^2$ is a consistent estimate of σ^2_ε)

$\hat{\sigma}^2_{NW} = \hat{\gamma}_{0,\,n} + 2\sum_{j=1}^{\ell}\left(1 - \dfrac{j}{\ell+1}\right)\hat{\gamma}_{j,\,n}$ is the Newey-West estimate of σ^2_z, see [10], with ℓ the truncation lag number

The test statistic in equation (5) is referred to as a regression coefficient-based test statistic, whereas the test statistic in equation (6) is referred to as a studentized test statistic.

In SAS the Phillips-Perron test is requested with the statement '`stationarity=(pp=(0,1,2,3,4))`' in the procedure '`proc arima`' where, in this example, $\ell = 0, 1, 2, 3, 4$ is used. The corresponding SAS output is given in Figure 5. Note that all three specifications of the Phillips-Perron test are automatically performed. The user has to pick the correct specification based upon the time plot of the series. The Rand-Peso exchange rate has a downward trend, so Case 3 (Trend) is applicable.

The *p*-values with respect to Z_ρ (given under the heading '`Pr < Rho`') and with respect to Z_τ (given under the heading '`Pr < Tau`') are all greater than 0.05, so $H_0 : \rho = 1$ cannot be rejected and we conclude that the trend of the Rand-Peso exchange rate is stochastic.

Phillips-Perron Unit Root Tests

Type	Lags	Rho	Pr < Rho	Tau	Pr < Tau
Zero Mean	0	-0.1496	0.6483	-0.73	0.4001
	1	-0.1485	0.6486	-0.75	0.3938
	2	-0.1476	0.6488	-0.76	0.3881
	3	-0.1465	0.6490	-0.78	0.3806
	4	-0.1458	0.6492	-0.79	0.3752
Single Mean	0	-5.5192	0.3846	-1.61	0.4741
	1	-5.2782	0.4063	-1.58	0.4934
	2	-5.0833	0.4245	-1.55	0.5094
	3	-4.8454	0.4476	-1.51	0.5293
	4	-4.7052	0.4616	-1.48	0.5412
Trend	0	-14.1990	0.2060	-2.70	0.2360
	1	-13.8988	0.2179	-2.68	0.2476
	2	-13.6870	0.2267	-2.66	0.2560
	3	-13.3469	0.2413	-2.62	0.2701
	4	-13.2187	0.2470	-2.61	0.2755

Figure 5 – Phillips-Perron test for observed series

5 Various Technologies

This is by no means an exhaustive list of products and solutions available, and won't necessarily fit the individual needs of all the visually impaired. It is rather a collection of solutions that work for the first author. It is also assumed that computers are readily available and that the user is either computer competent or might become competent through training.

5.1 Screen reading software

Screen-reading software provides access to a computer in much the same way that a computer screen would. Information that appears on the screen is spoken by the computer. The company Freedom Scientific (http://www.freedomscientific.com) offers a product, JAWS, which enables Windows users to access their computer.

5.2 LaTeX

Communication is vitally important for any scholar or professional. The LaTeX system provides a linear way for the visually impaired to engage mathematics, which are often spatially arranged. Since a wide audience makes use of LaTeX, it provides a very effective way of reading and writing mathematics. Various tools exist to convert LaTeX to and from other formats – see http://www.latex-project.com.

5.3 Braille translation software

Braille might not be the best solution to communication, but it provides a solid basis, especially in cases where new concepts have to be learned. Duxbury for Windows (http://www.duxburysystems.com) provides translation from LaTeX to Braille.

5.4 MathML

As stated by [11]:

'It provides a much needed foundation for the inclusion of mathematical expressions in Web pages.'

Many mathematical texts are also created using MS Word and MathType (http://www.dessci.com). Using MathType these documents can be exported as MathML files and accessed by MathPlayer (from the same company). The mathematics is viewed in your web browser and spoken in words by the computer.

5.5 Optical Character Recognition (OCR)

In many cases the access to electronic material is denied, or limited. In such cases the only way to gain access to these materials is through scanning and the use of software that enables character recognition. Because of the spatial nature of mathematics, ordinary OCR software does not provide a solution to this problem. Science Accessibility Net, offers a product InftyReader (http://www.sciaccess.net) that enables OCR of mathematical texts.

5.6 ViewPlus Technologies

This company provides many solutions to the visually impaired, including the TIGER Braille embosser that was mentioned earlier – see http://viewplus.com.

To conclude, the blind and deaf educator Helen Keller (1880 – 1968) said:

'Not the senses I have but what I do with them is my kingdom.'

In our new millennium, with its as yet uncharted possibilities, these words are probably even truer than ever before.

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Mathematical learning webs and APOS theory

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We examine the concept of a mathematical learning web, and we propose basic principles for this concept. We then compare mathematical learning webs and APOS theory. The similarities include development levels and the importance of applying mathematics to solving problems. The similarities are interesting, especially, when one considers that the developer of the learning web concept was a practicing elementary school teacher and the developers of APOS theory included research mathematicians who were using educational research results obtained by Piaget. By comparing mathematical learning webs and APOS theory, we are able to make modifications or suggestions for each model. We also introduce a mathematical instruction web model.

Keywords: APOS theory; constructivism; learning web

2000 Mathematics Subject Classification: 97C50; 97C70; 97D20

Mathematical learning web -- informal introduction

In *Mathematical Reasoning in the Elementary Grades* [1], Susan Jo Russell introduces the concept of a mathematical learning web, which for her means the learning of mathematical concepts in a mathematical context which supports the new ideas. She introduces the concept to help explain why elementary school children learn mathematics better when the ideas are presented in an interrelated manner as opposed to isolated concepts or facts and to support a style of teaching which promotes mathematical reasoning. She uses the terms `memory' and `mathematical memory' to make a distinction between learning in isolation and learning in a 'web' of a mathematical context.

To help clarify the learning web concept and to support its validity, Russell tells about some of her experiences with the formula $m(A) = 180 - (360/n)$ for finding the measure of an internal angle A of a regular polygon with n sides [1]. At various times during her educational and professional careers, she needed to use specific instances of this general formula. When she needed to use the formulas regularly, she could usually remember them. However, when she did not use them for a while, then she had trouble remembering them. Through some mathematical experiences, she developed a context into which the formulas fit, and now, whenever she needs the general formula or an instance of it, she is able to easily recreate it.

Think about standing on the midpoint of a side of an n-sided regular polygon so that the adjacent vertex in the clockwise orientation is directly in front of you. Picture yourself walking around the polygon so that whenever you reach a vertex you turn clockwise to face the `next' adjacent vertex. When you reach your starting midpoint, you will have traced a Hamilton circuit, and importantly, for this context, you will have made n turns for a total of

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360 degrees. Thus, each external angle of a regular polygon has measure *360/n*, and the measure of each internal angle is *180 - 360/n*.

As described in [1], in a mathematical learning web it is as if multiple tentacles or threads support a mathematical idea or concept so that the idea or concept is connected to and understood relative to other mathematical concepts. Thinking of the interior-angle example, it may even be that the idea or concept is derivable from the related ideas and concepts.

Underlying principles for a mathematical learning web theory

Russell does not try to develop a theory for mathematical learning webs. She does, however, give properties or characteristics of her learning webs. The principles in this section come from her characteristics. To help in our development of learning web principles, we give a formal definition of the term mathematical memory.

Mathematical memory is the knowing of a mathematical idea or concept within a context so that the new idea or concept and the context are interwoven. The new idea or concept is understood in terms of the context, and in fact, the new idea or concept may be derivable from the context.

Principles of mathematical learning webs include the following.

- When mathematical ideas are learned without the simultaneous development of mathematical memory, then the ideas are learned in isolation, and these ideas are not related, at least, in the student's mind to other concepts.
- Mathematical reasoning necessarily involves a mathematical context.
- Mathematical reasoning and mathematical memory are themselves interrelated. Mathematical reasoning creates the `strings' that help form a mathematical learning web or mathematical memory. Also, mathematical learning webs facilitate abstract thinking and mathematical reasoning.
- Mathematical webs with their interconnections enable students to understand mathematical problems and to formulate solutions.
- Mathematical webs with their interconnections enable students to formulate interesting mathematical problems.

As indicated in the next to last principle, Russell is interested in students being able to solve mathematical problems, and she emphasizes the importance of mathematical learning webs in problem solving.

In the next section, we give some basics for APOS theory, a theory of learning and teaching mathematics at the college level. Then we will look at mathematical learning webs and APOS theory together.

APOS theory

This introduction to APOS theory is taken from [2]. APOS theory is built on the research in learning done by Jean Piaget. See, for example, [3] and [4]. The goals of APOS theory include developing a model for how college students learn mathematics, using the model to develop instructional methods which will assist students in becoming active learners, and developing methods to determine the effectiveness of APOS-based teaching methods. For this current paper, we are mainly interested in the learning model. The model has four basic stages or levels, from which the term `APOS' is derived. According to APOS theory, students learn mathematics in four progressive levels; they are the action, process, object, and schema levels. The authors of [2] claim that when solving a mathematical problem a student may move from one level to another. However, when learning an idea or concept, a student must move from action to process to object to schema. This is not to imply that each student reaches the schema or even the process level for ideas or concepts, but for example, a student may not be at an object level with respect to a concept without having already been at the process level for that concept.

Students understand a concept at the action level when they are able to follow explicit directions to create an outcome. A common example is students understand the concept of a function at the action level when they can manually follow an explicit procedure for converting arguments or inputs into answers or outputs. Another example is students' being able to plot order pairs on a Cartesian coordinate plane when given ordered pairs and explicit directions for plotting points.

After following explicit procedural steps a number of times, a student may begin to gain a process understanding of a concept. A student has a process understanding of a concept when she or he can internalize the procedural activities so that she or he does not need to physically do the steps but can mentally execute the procedure. For the function example, a student with a process understanding of functions is able mentally to carry out the steps needed to go from arguments to answers. For the graphing example, a student with a process understanding of graphing points can picture ordered pairs on a Cartesian plane without actually plotting the points. The student is able to think through the process.

Once a student internalizes a concept and reflects on that concept, then the student may move to the object level which is being able to conceive of the concept as an object which can itself be manipulated. With the function example, a student with an object level understanding is able to begin with a function and manipulate the function as a whole by, for example, finding an inverse, composing the function with another, or letting the function itself be an argument to another function. With the graphing example, a student with an object level understanding could manipulate a graph by, for example, reflecting it around the line $x=y$ or by taking its intersection or union with another graph.

The final level in the development of learning a mathematical concept is the schema level. A student is at the schema level of understanding a concept when she or he is able to bring together in a unified and usable whole all that she or he knows about the concept. This does not mean that the student knows all there is to know about the concept. Interestingly, a student at a process level for a concept may know more about the concept than another student who is at the object level of the same concept. These levels are not about how much a student knows about a concept but about how a student is able to use what she or he knows. When a student who is at the schema level of a concept is given a problem which can be at least partially solved using the concept, she or he is able to pull together her or his ideas related to the concept and use them to try to solve the problem.

To quote from [2], `An individual's mathematical knowledge is her or his tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing or reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations.'

Again, quoting from [2], `The purpose … is to propose a model of cognition: that is, a description of specific mental constructions that a learner might make in order to develop her or his understanding of the concept.' Thus, APOS theory is concerned with trying to determine the thinking steps and processes which a student goes through in reasoning about, understanding, and using a mathematical concept.

Mathematical learning webs and APOS theory

Though the developments of these two learning models were very different, they have some similarities, and when considered simultaneously, interesting questions and possible modifications of the models themselves arise.

They both emphasize the importance of a person's being able to bring together mathematical knowledge to solve a problem. Interestingly, the mathematical learning web model, the model developed for elementary students, takes this motivation one step further to place importance on being able to develop mathematical problems.

There are similarities in the web model's learning in isolation and APOS theory's action level. When one is at an action level so that she or he can only work with an idea or concept by physically following procedural steps without any internalization, then the person truly seems only to know the idea in isolation. In fact, when one is only at an action level with respect to a given concept, then the person is only a machine or robot with respect to using that concept. When teaching a beginning computer science course, one idea which is often mentioned is that a computer or a robot can not create; it simply follows instructions. This idea is significant in understanding the action level and in understanding differences in the action and process levels. A person with just an action level understanding of a concept is not able to do anything creative with the concept. In particular, the person can not use the concept in problem solving because the person has to mechanically follow procedural steps when applying the concept.

The concept is not internalized. Thus, though the person can mechanically follow steps when the concept is applied, she or he does not know when or how to apply the concept. Thus, the differences between action and process levels parallel the differences between mechanical abilities and creative abilities.

In APOS theory, it is thought that when a student faces a complex mathematical problem, then the student may combine action level, process level, object level, and schema level knowledge of different mathematical concepts to understand and and solve the problem. However, when it is realized that action level understanding only implies a robotic ability to carry out the procedural steps but no understanding or when or how to apply the concept, then it seems clear that merely action level understanding of a concept is not useful in problem solving. The authors of [2] do point out that when solving mathematical problems a person may move between levels. Thus, if a person has an object level understanding of a concept, then when solving a problem involving that concept the person may combine action, process, and object level understanding of the concept to solve the problem.

Since a person with only action level understanding of a concept can not use that concept in problem solving, one method which may be useful in helping students move from an action level to process level is to help the students learn how to apply the concept.

There also seem to be similarities between learning in isolation and process level understanding because a process level understanding does not need to include a context; it deals with internalizing a concept.

However, upon more reflection, one sees that internalizing a concept does give the concept a context though not necessarily a mathematical context; it is a mental context. Thus, to more fully develop mathematical learning webs, one could incorporate two distinct ideas of context. One is a mathematical context, and the other is a mental context. This mental context is a mental web which allows the person to mentally manipulate and work with the concept. This mental web moves the concept from robotic only use to possible creative uses.

In the next section, we consider a mathematical instruction web. However, before going there, we ask the question of whether teaching could be useful in helping students move from one APOS level to the next. More exactly, could having students teach others help those who are teaching move from one level to the next. One would think there is potential for this idea because to teach something one needs to internalize the concept unless one simply reads a lesson to others. Further, teaching involves being able to encapulate ideas which is similar to forming objects. Finally, effective teaching often involves pulling together diverse views and uses of an idea or concept, and this seems similar to schema level knowledge. Also, recall the APOS levels are not about how much one knows about a concept but about how one can use what one knows.

Mathematical instruction webs

A mathematical instruction web is a more complicated web than a mathematical learning web in that there are mathematical nodes and there are instruction nodes with connections between both types of nodes. One of the problems that some mathematics teachers have is they have learned mathematics or pedagogy in isolation, and they do not know how to relate the different nodes.

When considering mathematical instruction webs, we might think that they need not be complex because for mathematics teachers the mathematical nodes should be trivial in the sense that the mathematics should be well understood by the teacher. However, most of us who teach have had experiences when the material we are teaching is new for us, and thus, the mathematical nodes may not be trivial. Also, for some K-12 school teachers the mathematical nodes are not just nontrivial, they are intimidating. Also, though many teachers are comfortable talking in front of a class, the research of APOS theory implies that teachers often know little of what their students are thinking as they are trying to learn mathematical concepts. Thus, mathematical instruction webs may be anything but simple and clearly defined for mathematics teachers.

Building on the principles of mathematical learning webs and our discussions in this paper, we propose principles for mathematical instruction webs. However, before presenting the principles, we define instruction memory and application memory. Application memory is related to mental webs introduced earlier.

Instructional memory is the knowing of an idea or concept in a context which allows the idea or concept to be effectively explained to others. This may imply that the context includes an understanding of how the idea or concept is effectively learned and used.

Application memory is the knowing of an idea or concept in a context which allows the idea or concept to be effectively used in problem solving and problem creating situations.

Of course, mathematical, instructional, and application memories of a concept need not be and are probably not disjoint.

Principles of mathematical instruction webs

- When a teacher knows a mathematical concept on the action level only, she or he can only direct students to the same procedural rules which she or he has been shown.
- An effective mathematical instruction web is composed of mathematical memory, instruction memory, and application memory. Each of these memories is, in fact, a

subweb of the mathematical instruction web, and the subwebs are interconnected in the instruction web.

- Mathematics instruction and mathematical instruction webs are themselves interconnected. Mathematics instruction creates the `strings' that help form a mathematical instruction web, and a mathematical instruction web enables effective mathematics instruction. In effective instruction, mathematical memory, instruction memory, and application memory all work together.
- Mathematical instruction webs enable teachers to solve the mathematical learning problems of their students.
- Mathematical instruction webs allow teachers to create mathematical learning problems and situations for their students.

In this paper, we will not further develop the mathematical instruction web. We simply note that the interplay of the three subwebs: the mathematics learning web, the mathematics instruction web, and the application web, presents many interesting questions and challenges.

Conclusion

We have begun to formalize the mathematics learning web model. We have compared and contrasted the mathematics learning web model and APOS theory, and in the process we have made suggestions regarding each model. In particular, for the learning model, we have proposed the inclusion of application memory. For the APOS model, we have made suggestions for helping students move from one level to the next, and we have clarified the action level by comparing it to robotic or mechanical activities. Further, we have introduced the mathematical teaching web model. We have not given attention to the facts that the mathematics learning web model was developed for elementary mathematics students and that APOS theory was developed for college mathematics students. Experiments with the web models would need to address this issue.

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Developing maths-identities of overseas trained teachers from developing countries: two voices

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This paper reports on teacher-students' developing mathematics-teacher identities as they participated in a mathematics education module which they took as an option as part of their BEd in-service degree. Multiple qualitative methods have been used to gather data on the 14 teacher-students' mathematical development, attitudes and reflections on their professional practice over the course of the year. In this paper, in-depth accounts of two of the students'- (one Caribbean and one African) - developing identities as mathematics teachers are presented using socio-cultural, discursive and defended-subject theoretical frameworks for the analysis. The paper exemplifies discourses producing maths teacher identities while simultaneously defending against interrogation of mathematical and pedagogical practices.

Keywords*: teacher education; undergraduate mathematics education; social identity*

Introduction

This research concerns the developing maths-identities of teachers of mathematics who have come mostly from Caribbean or African countries to work in London and who were enrolled on a mathematics education module as part of their undergraduate BEd course. A shortage of teachers in London, has prompted the practice of teachers being recruited from abroad to fill vacancies, often in challenging schools [1, 2]. As some of these recruits were trained to certificate rather than degree level they have to study part-time after arriving in the UK: firstly, to get a bachelors degree, and then to gain Qualified Teacher Status (QTS), QTS being necessary for their continued right to UK residency (Teachernet 2007, webref). These 'teacher-students', (this term is used as they were practicing teachers at the same time as studying for a further qualification), already had some teaching qualifications, usually from their home country, and, as part of the entry requirements, were expected to have had some teaching experience. The range of teaching experience was 15 years to home country teaching practice placement only; some, but not all, were currently teaching in local schools.

This report is based on data from the "School Teachers Reconceptualising Mathematics" research project. This project was sponsored by the English National Centre for Excellence in Teaching Mathematics (NCETM) in order to investigate how this mathematics education module that the teachers were enrolled on impacted on their conception of mathematics as a school subject they were involved in teaching. The mathematics education module has been available for two years. Last academic year (2005-06) there were some notable comments from the teacher-students' reflective reviews, for example:

> I realise that maths is not a set of problems with a definite answer but a way of looking at problems and arriving at satisfactory solutions;

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my active involvement has enabled me to transfer to my classroom the atmosphere I was exposed to that fostered collaboration and interaction with materials and ideas.

This research was set up initially to find out what the maths education module was offering that enabled this espoused change of attitude on both their view of mathematics and of their teaching approaches. However, after starting the research it was apparent that this was rather a simplistic question, as the analysis presented below confirms. In this paper, the analysis of the qualitative data gathered has used a multiple-lens approach based on theoretical frameworks loosely corresponding to: socio-cultural theory, discourse analysis and a psychoanalytic conception of the defended subject. These were the frameworks, or 'lenses', which were used in a seminar series [A]on maths-identities which I was involved with during the data-gathering year 2006-07.

This paper is organised as follows: a perspective on the background of the study is given, followed by some methodological remarks and a brief introduction to the 'lenses' used to interpret data. Then interpretative pen portraits of Grace and of Charles are presented followed by a discussion and themes from other data prior to concluding.

Background

The 'black subject' [3] is at the centre of this research, despite its being ostensibly about overseas trained teachers, and I am not in a position, either by culture, experience or study, to have a special insight into these specific issues of cultural identity. To address this, I have found Stuart Hall's essay 'Cultural Identity and Diaspora' (Hall, *op. cit.*) helpful. He writes there about 'identity' as a production that is "constituted within, not outside, representation" (p222) so how the teacher-students, who are the subjects of the research, see themselves as well as how they are seen, constitute their ever-evolving identities. Hall's notion of 'cultural identity' – with reference to African Caribbean cultural identities particularly - is not "a sort of collective 'one true self' … underlying all other superficial differences", even though, he observes, this notion did "play a critical role in all the post-colonial struggles". His sense of 'cultural identity' is "a matter of 'becoming' as well as of 'being' … subject to the 'play' of history, culture and power. … identities are the names we give to the different ways we are positioned by, and position ourselves within, the different narratives of the past." (ibid. 223-225) And these teacher-students have many stories, of 'love of maths', of being a teacher, of being needed in London, that are creating and fabricating their maths teacher identities.

Hall writes that 'there is always a politics of identity, a politics of position' as cultural identities are made "within the discourses of history and culture" (p226). The subjects of this research are in the centre of a big tussle to do with teacher supply, recruitment, the problem of inner city youngsters (read 'Black youngsters') as well as the depletion of teachers from their home countries. In Jamaica, Britain's teacher shortage has brought a 'wholesale teacher recruitment by commercial agencies' (BBC, 2002, webref). Identity issues figure here as Jamaican teachers are attracted to Britain because of historical-colonial traces like school uniform, structures of public examinations and, of course, English as the language of instruction. Another political dimension is the fairly recent tightening of regulations for overseas teachers which demands that these teacher-students must gain QTS within four years of arriving in the UK. In practice, this does not give much time to upgrade qualifications, to settle into a new environment as well as to gain QTS. (Teachernet 2007, webref). The NASUWT-commissioned report [1] found that employing headteachers assumed that teachers who shared an African Caribbean heritage with pupils would "find it easy to manage African Caribbean pupils, which was not the case" (p38). They also reported on the difficulties such teachers had in adjusting to supply teaching and the humiliation they felt when told that they were ‘unqualified’ despite, in many cases, being considered first-rate in their home country and having even as much as 15 years teaching experience.

Hall discusses the ‘presences’: of the African, the European and the American New World on the Caribbean identity “[To the] Africa, which is a necessary part of the Caribbean imaginary, we can’t literally go home again …‘Europe’ belongs irrevocably to the ‘play’ of power, to the lines of force and consent, to the role of the *dominant* in Caribbean culture.” (p232) and the American presence ‘stands for the endless ways in which Caribbean people have been destined to “migrate”; it is the signifier of migration itself - of travelling, voyaging and returning as fate, as destiny; of the Antillean as the prototype of the modern or post-modern New World nomad, continually moving between centre and periphery.’ (p234) In terms of this project, this image of voyaging as destiny is apparent from study by Miller, Mulveney and Ochs, [2] of the Commonwealth teachers’ protocol. They remark that as ‘the global teaching profession is in a scramble to find teachers … rich countries, .. including the UK, continue to recruit teachers from poorer, less-developed countries …[and] some teachers in developing countries have voluntarily and forcibly migrated to industrialised countries’ (*op. cit.* : 154). They report that over the period January 2001 to July 2004, 20,610 ‘teacher’ work permits were issued by the UK and high recruitment over this period came from South Africa (6722), Australia (4484), New Zealand (2515), Jamaica (1671) and Canada (1591). (These figures need to be read with caution as there are non-work permit routes available too.) Jamaican representatives raised the matter of their ‘brain drain to the UK’ at the Commonwealth heads of government meeting in 2002 (subsequently The Commonwealth Teacher Recruitment Protocol was agreed in late 2004 that includes further monitoring but no legal authority). Yet as long ago as 1960 Elsa Walters [4] wrote ‘the demand for trained teachers in the West Indies is as old as the struggle to establish a popular system of education’. Hall’s words link the various journeys of teachers to and from the West Indies: ‘Diaspora identities are those which are constantly producing and reproducing themselves anew, through transformation and difference.’ (Hall, *op. Cit.* p235).

Methodology

During the academic year just past (2006-07), I was course leader for the mathematics education module in question and taught about half of the 20 scheduled classes which were distributed over the academic year, the other classes being taught by colleagues. The teacher-students were introduced to the research at the beginning of the academic year and only methods of collecting data that were considered beneficial to their development on the course were used. For example, towards the end of the course, each of the teacher-student was invited to come for an interview, it was emphasised that this was voluntary, but hopefully would be helpful, not only for the research project of which they were aware, but also for them to write their assignment on their developing reflective practice as they would have the audio-recording of their spoken narrative.

The data-sets that were collected throughout the year included: initial questionnaire on expectations, biographical data and short attitude survey, mathematical tasks with ‘affective’ personal response notes; presentations to the teacher-student class on three ways to teach a topic; short essays reflecting on their professional development and the role of the course, and the individual interviews (for which all of the students did volunteer).

Lenses, data and interpretations

Each of the shortcut terms ‘sociocultural’, ‘discourse’ and ‘defended-subject/psychoanalytic’ are markers for significant intellectual movements from the twentieth century that are still developing. These theories give rise to ways of seeing the world generally, identity specifically and can be applied to maths-identities in mathematics education in particular. Indicative source concepts related to these lenses and relevant to this paper are, respectively, theories of social learning through practices in communities [5], analysis of discourse [6] and theories of the defended subject [7]. In this report, the focus is on the teacher-students' maths-teacher identities. A maths-teacher identity an agglomeration of practices, positions, and feelings related to being a teacher of mathematics; it is both self-constructed and informed by others, (e.g. other teachers and representatives of regulatory bodies), in this sense it is analogous to Hall's cultural identity, mentioned above.

I present data in the form of pen portraits of two of the teacher-students called here Grace and Charles. These stories of Grace's and Charles's respective developing maths-teacher identities are constructed from their interviews, their short papers on their reflective practice and other recorded data such as quick response surveys in class. These two have been chosen as their data represents a good range of the issues: Grace is from Jamaica and has been in the UK two Years. She is certificate trained for primary school, though had not had a permanent post before she came to the UK. Since arrival in England, she has done a little teaching assistant work in primary schools in London and a little supply teaching in secondary schools. She has the least teaching experience of all the teacher-students. Charles is from Ghana and has been in the UK 4 years. He taught secondary mathematics and science in Ghana from 1988 to 2003 thus giving him 15 years professional experience before he arrived in London which was the most experience of all the teacher-students. Since coming to England he has not been able to secure a teaching post at a school but does teach maths and science as a private tutor.

Grace

'all discourse is 'placed', and the heart has its reasons'

Stuart Hall, *ibid.*, p223

What is Grace like? Cheerful, with shoulder-length straightened hair, still in her twenties so younger than most of the class, absent more than most (she missed four out of 20 sessions). Her teacherly qualities of communication skills (she has good eye contact and uses her hands when explaining) and interest in others' perceptions and reasoning more visible to me than her mathematical ones (of choosing to do mathematics and being accurate/aligned in her mathematical skills and reasoning); she is more confident socially than mathematically. From quick-response data, adjectives she uses about herself are 'excited' and 'pleased' and that 'understanding' maths is important and she wants to improve her understanding.

To get something of a picture of Grace's developing maths-teacher identity, firstly I present extracts from her interview and interpret the initial chunk of her interview (which is on maths) with reference to each of the lenses in turn. Then I offer further interpretation (on teaching and on maths teaching) without explicitly mentioning 'lenses' unless appropriate.

Starting at the beginning of her interview, after being asked what maths was like for her at school, Grace replies:

G It's a long time. At secondary school I had to do maths twice. The first time I took it I didn't pass because I really thought I couldn't do maths but then I got a different teacher and then I passed. From then I have a love for maths. --✂ three lines --

I *You said you have a love for maths?*

G I still have a love for maths.

I *What do you really like?*

> G I don't know, I think there is something special about doing maths and applying it. And I think most of this is practical and you don't have to do a lot of reading in the one sense. Yes there's theory and everything but there's a practical side to it. I think some subjects, like you get involved; it becomes a part of you like you're doing it and you're finding out, whether it's equations or just doing simple maths or where you have to use reasoning, it's like you're getting involved. So it's this feeling of accomplishment that it gives me.

The focus in this part of the interview was on maths. What sense of Grace's developing maths-identity come through from this short piece?

Using a 'socio-cultural' lens, what comes through to me is her sense of maths being part of the social and material world. At school, passing or not passing, is very much the way of things and people help people to get through. Doing maths is an active enterprise that gives personal satisfaction, does not require so much reading and has social purpose.

From my narrative/language 'discourse perspective' Grace tells her story, positioning her attitude as more enduring than her results and presents her notion of embodiment. She moves from the position of 'couldn't do maths' almost as if through the agency of another person. Her use of 'practical' validates maths as an enterprise, yet her 'I don't know' signals lack of desire to be pressed. Her exemplification of aspects of mathematics (e.g. 'equations') indicate what she feels comfortable with. Her spontaneous use of 'love' is remarkable, about which more below.

My understanding of 'defended subject/emotion' draws to my attention Grace's need to put emotion right up front. By using the word 'love', a pinnacle of words, it feels like her worth (as a maths person) cannot be challenged as her feelings for mathematics are intense and positive. Her ostensive positivity thus can be construed as defensive. She also avoids, defends herself against, possible difficulties of 'reading'-based subjects. Her use of 'special' reinforces the position of mathematics, which is then internalised, embodied, made hers through the personalisation of positive feeling and satisfaction. Being hers, she aims to defend herself against the possibilities of future 'not passing'.

Other aspects of Grace's developing maths teacher identity include her beliefs about learning, (which she describes in the current Visual-Auditory-Kinesthetic terms), her awareness of listening to learners to aide teaching, her delight in being challenged (e.g. she did not conceptualise fraction as numbers but as visual ratios, parts of whole-ones). Yet for Grace, becoming a teacher seems to have happened as a result of a serendipitous experience and comes across as a great shift in her self conception. When asked about what had drawn her to teaching she says:

> Wow! At first I never thought I would go into teaching. Ever. Ever. Ever. And then I started doing some voluntary work in a primary school teaching reading. And then I had a love for it because when I realised that I was working with kids and how much they've improved I felt like I've got a lot to contribute to the teaching. That's when my love started developing, that's when I did a lot for teaching, it's just like an inspiring moment I would say.

I was not able to tell whether teaching was too high status to have been on her adolescent career agenda or too staid a profession. Nevertheless, Grace's potential-teacher identity is changed by experience in a classroom and by her perception of the progress of the children she has worked with. The social and emotional context of the classroom has re-positioned her and generated her desire to get qualified. She can see herself in the society of the classroom; it is worthwhile. Interestingly, it seems that Grace feels that love justifies as well as drives her plans. Is it her sense of self that comes from what she loves? This emotion-driven theme is continued, with Grace, despite being 'tired' and having had warnings about 'behaviour', is set on teaching in a secondary school

> that's where my heart really is … I like the maths or science at that level. That's the level I like. And the difference is that you're a maths teacher or science teacher where you're not teaching all these different subjects, like art.

However, her next few sentences on only recently understanding the formula for the area of a triangle as well as other evidence (e.g., 'I always thought probability was just one in two chance of doing something. It's like basically yes or no, when you did that probability [now I have a] different perspective on probability.') indicate that she is not (speaking with a teacher trainer's hat on) at 'that [secondary maths] level'. Her confidence is buoyant, her pleasure in her progress towards her espoused goal of understanding admirable, but her mathematical training is not yet enough for 'the level' she says she likes; (why does she say she 'likes' secondary maths?) And this mis-match between her language of interest and her enthusiasm and her available mathematical knowledge, skills and understanding is not within her awareness, for if it were she would surely defend her self against being seen as so unknowledgeable? From the perspective of the English secondary maths teacher training community, Grace's identity as a secondary maths teacher is not yet viable; will she find this out and continue her studies or will a school-teaching opportunity come first?

Charles

'getting to the professional touch now'

Charles, from his March 2007 interview.

Charles is modest, helpful and had 100% attendance. Charles seemed to me, at least at the beginning of the course, more confident with his mathematics than he was socially. He had taken A levels in maths and chemistry and a three-year maths (teaching) diploma in Ghana. Over the year, other students got to know that his reserved manner did not mean that he was not always willing to help and he was pleased when fellow students turned to him. From quick-response data, adjectives he uses about himself are 'excited', 'interested' and 'pleased' and that as an important principle for teaching mathematics he wants 'to assist students/pupils develop knowledge in solving problems'.

Starting with analysing the first part of his interview, I'll analyse Charles's responses from the three perspectives and then draw some other points together concerning his mathematics teacher identity.

> I tell me a little bit about what maths was like at school for you when you were a child?
>
> C Yes I had an interest in mathematics. Initially the teacher who started with me was very good and I developed an interest in the subject. So I decided to carry it forward. Basically that's where my interest lies. And I've progressed and found it very well.
>
> *I When you talked about your first teacher, how old were you at that stage?*
>
> C I was 15 years. It's in the secondary.
>
> *I Yes. So your teacher was* (interrupted)
>
> C Very knowledgeable, yes, very good.
>
> *I So before you were 15 can you remember what it was like, maths?*
>
> C I was doing very well. In fact in all the subjects I was doing well. Mathematics especially I was doing very well. But my interest actually when I decided that I would carry it forward is when I got the teacher and he did very well.
>
> *I Can you tell me about the teacher?*

C I can't remember his name. Actually he was one of the US Peace Corps who came to the school, she (inaudible)

I Say that again?

C A Chinese lady. (sounds like: Shi-e Wu)…taught us the mathematics very, very well and I developed an interest in it. But she told us that mathematics is the basis for all academic work and if you want to study it very well to the highest level then you will need to do very well in mathematics. As you move ahead you come across mathematics in every aspect of your learning. And I took that even further, I also experienced it if you have the mathematics as the basics you can do almost everything at the highest level.

Reading this transcript was quite salutary. I didn't think of Charles as being other than fluent in English, but this text suggests to me that I did not always pick up on when there had been a linguistic misunderstanding. In terms of his maths-identity, the three perspectives again illuminate different facets:

Starting with a socio-cultural lens, what I pick up on here is Charles's recognition of the importance of the skill of his remembered teacher. His image of her, is of a 'Chinese lady' who came via the US peace corps, suggests this 'sent' teacher had a certain international status. From the text, he has a memory of her view of mathematics as both foundational and door opening which he seems to have adopted.

Looking at his narrative and the language, Charles uses the term 'very well' repeatedly to refer both to the teacher's skill and his success. He also uses the term 'interest' several times in the interview. The story Charles seems to be telling is that he has chosen mathematics, having 'decided to carry it forward' - maths as important baggage for his life-journey – having been oriented to the importance of mathematics by the Peace Corps teacher and that his 'interest' and as he'd done 'very well' suggest that maths chose him too. In this extract, he positions himself as having 'progressed' and thus it is a suitable, if not natural, that he continues with mathematics.

From the defended subject and emotion-orientated point of view, his defence seems to be solidly constructed around the twin virtues of achieving 'very well' and having 'interest'. And with these virtues established, his maths-identity should be undeniable! I am surprised that he gets the gender of the teacher confused (but I cannot be sure that I have misheard); is this gender slip to do with identification with himself and the subject or just a linguistic slip?

Other aspects of Charles's developing maths teacher identity include his newly taken-on notions of mathematics teaching, his pride in being a student at London University (and his identifying of me as his current teacher), and his "one problem" of not being able to get a classroom teacher's job in London, despite his experience in Ghana.

Like Grace, teaching was not Charles's first choice; he was going to do engineering, following his father. However, the 'unfortunate' incident of the theft of school fees (see handout), thwarted his engineering ambitions and he used his less-good-than-required-for-engineering qualifications to get on a three year teaching diploma. And he 'decided to stay…[as] teaching is also a very good profession'. Charles's tone is almost apologetic and his story of the theft does not occur until half way through the interview. It seems that Charles's maths teacher identity is quite at risk at present - he cannot get a classroom post in London and he has to return to study after many years a professional, so this admission that he did not originally intend to teach is quite uncomfortable. Nevertheless, when asked specifically about mathematics, Charles's reply is completely about his 'being with students' and how he 'did his best' for them which made them 'happy'. Even when pushed, Charles does not talk about mathematics, even as Grace had done, though, when asked about effects of the course, he does mention a geometric puzzle that I had presented to the class on the previous week He gives this as an exemplar of his new views on mathematics teaching which he sums up by saying '[maths teaching is] more than just delivery but trying to give something special to the students'. This is what he calls 'the professional touch' and he relates it to my practice of offering task-based learning; he says: 'I love to come to your class … [I get] a lot of ideas'. He also insists on saying at the end of the interview that he 'always dreamt…of getting a qualification from here' [University of London].

The picture so far is that Charles, a dignified, middle-aged professional man is returning to study at an institution he respects. He is rather defensive about his professional journey, and while he is 'very happy' to be doing some private tutoring, he would rather be a classroom teacher. With Grace, she is seeking out a potential identity as a maths teacher, but Charles has a maths teacher identity already: so he must defend this aspect of self. Now Grace is naïve enough to expose her lack of mathematical knowledge, what about Charles and mathematics? Information from other data sources show that, relative to the English QTS, Charles has both a lack of mathematical and of pedagogical skill. For example, he writes confidently, but incorrectly, about the area-comparison puzzle, and in his presentation of three ways to teach a topic (his choice was long multiplication) to his peers and teachers, he confuses his methods of multiplication and stands and delivers rather than offering tasks for the other teacher-students. His maths-teacher identity is tenuous.

Discussion

Grace and Charles were students of mine on the mathematics education module, so my investment in them, and theirs in me, will surely impact on my reading of their maths-identities. I found myself surprised when Grace went on about her 'love for maths', as I had seen her in class as someone who seemed to avoid mathematical talk and other activity and whose mathematical performances (class presentation, problem solving done in class, written mathematics) were not rich, deep or accurate. So what is Grace doing when she expounds with such enthusiasm? Is she trying to create an identity through talking herself up or is trying to impress her maths-teacher identity on me? Charles has a life history that includes identification with teaching mathematics yet his practices did not change sufficiently within the period of the course to come within my conception of a potential mathematics teacher. In his interview Charles was aligning himself with the practices I had been offering, but either had not developed the skills to make them his own or did not actually desire to adopt them; (as I don't think Charles was other than a very straightforward person, a lack of desire to adopt these practices would have to be operating at a subconscious level, as well they might).

Themes or issues from analysing the data from the whole class of teacher-students also relate to the identities as teacher of mathematics. So, there was general talk and writing about shifting from didactic teaching to presenting mathematics in different ways. However, at their presentations, which was, to be sure, only half way through the course, only three out of the 14 gave task-based ways of teaching a topic. The notion of assessment for learning has been introduced on more than one of their BEd courses and this was mentioned by the majority as being influential on their practice. The teacher-students as a class were quick to pick up on and use language that expressed their feelings about mathematics learning. Quick written responses after mathematical tasks had been worked on were rich with expressions of feeling: elation, surprise, fear and depression. They also, like the teachers Bibby [8]worked with, expressed shame, e.g. 'what if I was the only one who couldn't do it'. Another theme was professionalism as mathematics teachers and settling in to working in London/English schools where the issues of pupil behaviour were ever-present.

Concluding

This research shows some of the tensions overseas trained teachers experience and the ways they adapt by adopting positive discourses that defend their mathematics teacher identities. Specifically, they have a need to position themselves as mathematics teachers because they have found, or think they might be able to find, employment in teaching mathematics, yet their relationship with mathematics is fragile as their mathematical knowledge base and mathematics-related pedagogical skills are limited; Their experience of mathematics pedagogy comes from their home country and is not the same as that expected in London schools; to protect their fragility, a defensive positivity is espoused.

Acknowledgement

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Flow diagrams to model steady-state of Markovian queues

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Markovian queues are frequently present in diverse engineering fields. Their steady state model is useful in solving problems related to dimensions of installations, equipment and human resources as well as to obtain system parameters representing the system performance. To model these steady states, systems of differential equations could be used to represent the transient states and later, once equilibrium has been reached, to obtain the equations for the steady state. When presenting these types of problems to students with a weaker mathematics background, it may be easier to use a flow chart method. This paper intends to show how these flow charts may be utilised to represent simple Markovian queues.

Keywords: Flow diagrams for stochastic queues; modelling Markovian queues.

1 Introduction

Stochastic processes are those whose nature is governed by probabilistic laws [2]. A Markovian process is a very simple stochastic problem where in order to establish the behaviour of a given parameter at a future time, the only information needed is the one related to the value of the parameter at the present time, independent on the past behaviour. That is, it is a memoryless process.

Stochastic queue is a process where clients look for some service provided by servers units under certain pre-established policy and where the demand is larger than the offer. Clients can originate from finite or infinite populations, they can arrive by units or in groups which size could be constant or a stochastic variable, there could be as many servers as management decides to provide. The services can be performed simultaneously or sequentially and following certain policy of attendance (FIFO, LIFO, SIRO, and PRI) and the two important processes present in the system, namely, arrivals and services can follow different probability distributions.

It has been shown that the exponential distribution is the only continuous distribution that has this memoryless property and the inter-arrival times are exponentially distributed whilst the number of arrivals in a given period of times follows a Poisson distribution.

These properties allow the modelling of the simplest Markovian queue, namely birth and death process (where the only allowed transitions from a given state are to their adjacent states, that is if the system at a given time is in state n, in a very near future, it will be either in state $n-1$ or $n+1$).

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The parameters in which we are interested at are those useful for obtaining the dimensions of installations, equipments and human resources as well as those used to control the operational processes performances. Some of these parameters are: the expected number of clients in the system (or in the queue), the expected waiting time any client will spend in the system (or in the queue), the expected time servers are busy, the percentage of clients served immediately after arrival, and so on.

To obtain these parameters, the studied system has to be in its steady state where the mean values of stochastic variables represent the system behaviour.

2 Objective

The purpose of this paper is to present the flow diagram representation of the steady state of a stochastic queue as a short cut to obtain the probabilities associated to each of the system states. These, in turn, will be used to determine the process performance parameters.

3 Flow diagrams

Flow diagrams are composed of nodes representing the process states (number of elements in the process at a given time) and by arcs representing transitions between states. Assuming that the system under study is at an equilibrium state (steady state) the flow balance law can be applied. This law states that the average of the total flow into any state has to equal to the average of the total flow out of the state. As an example, assume we are working with the steady state of birth and death processes where individuals are born at a constant rate λ and they die at a constant rate μ, the correspondent flow diagram for states *n*-1, *n* and *n*+1 is given by the diagram of figure 1.

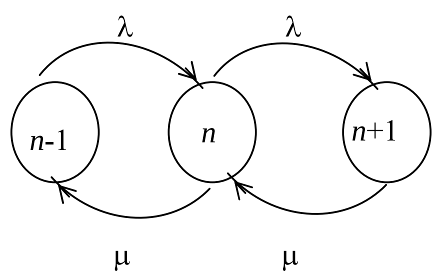

Figure 1: Flow Diagram

Using the flow balance law,, the correspondent balance equation for the *n*-state, is

$$\lambda P_n + \mu P_n = \lambda P_{n-1} + \mu P_{n+1} \qquad \text{(Equation 1)}$$

where P_n is the probability the system is at state *n* for all possible *n*.

Equations like these are useful in obtaining system performance parameters.

4 Performance Parameters

There are several parameters that express the system performance and that are useful to measure installations, equipments and human resources. Among then, the parameter *L,* which represents the expected number of elements or clients in the system at any time, is useful one as this leads to other parameters. One of these parameters is *N* the stochastic variable representing the number of elements in the system, its mean value, *L,* is defined as follows:

$$L = \mathrm{E}[N] = \sum_{n} nP_n \qquad \text{(Equation 2)}$$

where P_n is the probability the system is at state *n*, for all possible state.

All other parameters representing efficiency or performance of a given system depend on the set of states' probabilities which will be obtained from flow diagrams and corresponding balance equations as found in figure 1.

The following three scenarios are examples for the simplest birth and death systems where set P_n and the parameter *L* will be obtained using flow diagrams;

- M/M/1/∞/FIFO
- M/M/c/K/FIFO
- $M^{[X]}$/M/1/∞/FIFO (a complicated Markovian process where arrivals are in groups but servicing is individual)

The last example is beyond the scope of undergraduate studies, but is useful for graduate students in diverse areas like Applied Mathematics, Operations Research and Production Engineering Programs.

4.1 M/M/1/∞/FIFO

This model, according to Kendall notation, is a stochastic Markovian model characterised as a birth and death problem where the elements arrive at inter-arrivals times following an exponential distribution with constant rate λ, the services times are exponentially distributed with constant rate μ; and the services are performed by an unique server [3]. There is no limitation for the number of elements in the system and the adopted policy for attendances is the FIFO one, that is, first element in the queue waiting for service is the first element to be attended as soon as server is free.

The flow diagram that represents this process is the one of figure 2:

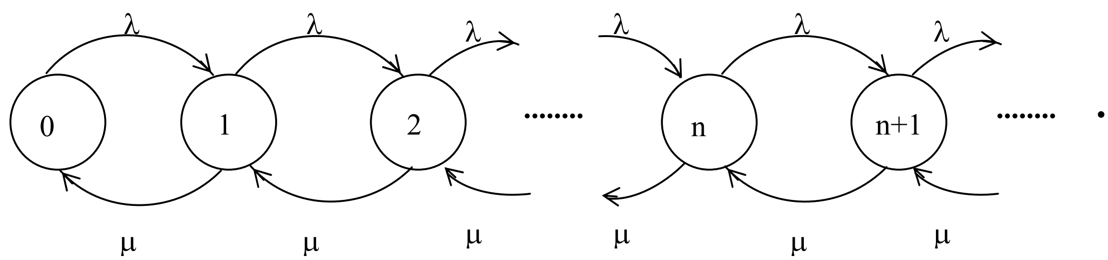

Figure 2: Flow Diagram M/M/1/∞/FIFO

And the correspondent balance equations are

$$\lambda P_n + \mu P_n = \lambda P_{n-1} + \mu P_{n+1} \qquad n \geq 1 \tag{3}$$

$$\lambda P_0 = \mu P_1$$

From the set of recurrent equations, the P_n are obtained for every n:

$$P_n = \left(\frac{\lambda}{\mu}\right)^r P_0 \qquad n \geq 1 \tag{4}$$

Since

$$1 = \sum_{n=o}^{\infty} P_n$$

then

$$P_0 = 1 - \frac{\lambda}{\mu} \tag{5}$$

From (4) and (5) and from the L definition, equation (6) is obtained:

$$L = \left(\frac{\lambda}{\mu - \lambda}\right) \tag{6}$$

where the infinite sum converges if $\lambda < \mu$, condition for the system to reach the steady state.

Other performance parameters are obtained from L, P_n and from Little's formulae [1].

4.2 M/M/c/K/FIFO

This model, according to Kendall notation, is an stochastic Markovian model characterized as a birth and death problem where the elements arrive to the system at inter-arrivals times following an exponential distribution with constant rate λ, the services times of each of the servers are identically exponentially distributed with constant rate μ; the services are performed by c servers; there is limitation for the number of elements in the system, K, and the adopted policy for attendances is the FIFO one, that is, first element in the queue waiting for service is the first element to be attended as soon as one server is free.

The flow diagram that represents this process is the one of figure 3:

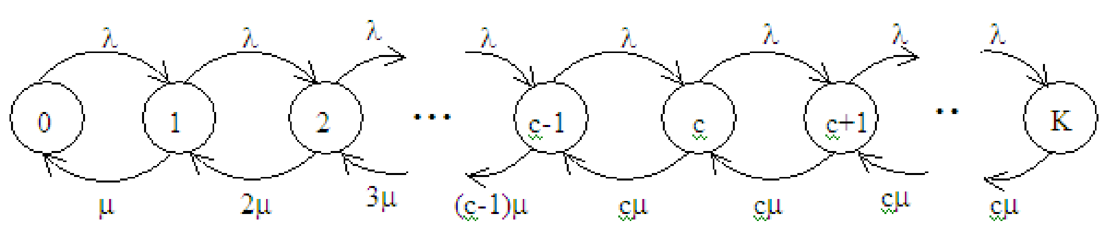

Figure 3: Flow Diagram M/M/c/K/FIFO

And the correspondent balance equations are

$$\begin{aligned}
&\lambda\, P_n + n\, \mu P_n = \lambda\, P_{n-1} + (n+1)\, \mu P_{n+1} \qquad 0 < n < c \\
&n\lambda\, P_n + c\, \mu P_n = \lambda\, P_{n-1} + c\, \mu P_{n+1} \qquad c < n < K-1 \\
&\lambda P_0 = \mu P_1 \\
&\lambda P_{K-1} = c\mu P_K
\end{aligned} \tag{7}$$

From the set of recurrent equations the set of P_n is obtained:

$$P_n = \begin{cases} \dfrac{r^n}{n!} P_0 & 1 \le n \le c-1 \\ \dfrac{r^n}{c!c^{n-c}} & c \le n \le K \end{cases} \tag{8}$$

where $r = \dfrac{\lambda}{\mu}$.

Since $1 = \sum_1^k P_n$, then

$$P_0 = \begin{cases} \left[\sum_0^{c-1} \dfrac{r^n}{n!} + \dfrac{r^c}{c!}(K-c+1) \right]^{-1} & if \quad \dfrac{r}{c} = 1 \\ \left[\sum_0^{c-1} \dfrac{r^n}{n!} + \dfrac{r^c}{c!(1-\frac{r}{c})} \left(1 - \left[\dfrac{r}{c}\right]^{K-c+1} \right) \right]^{-1} & if \quad \dfrac{r}{c} \ne 1 \end{cases} \tag{9}$$

In this kind of system, one with limited space for queue , there is an important parameter of performance that have to be taking care of, namely, the rate of lost clients given by equation (10)

$$\lambda' = \lambda\,(1 - P_K) \tag{10}$$

The parameter, L, may be found by the substitution of the equations (8) and (9) in equation (2). Other performance parameters are obtained from L, P_n and Little's formulae modified due to the limitation in the system capacity using λ` instead λ [1].

4.3 $M^{[X]}$/M/1/∞/FIFO

This model, according to Kendall notation, is a stochastic Markovian model where the elements arrive in system in groups of k elements at inter-arrivals times following exponential distribution with rates λ_k. The services times are exponentially distributed with constant rate μ and the services are performed by a unique server; there is no limitation for the number of elements in the system. The adopted policy for attendances is the FIFO one, that is, one element on the first group in the queue waiting for service is the first element to be attended as soon as the server is free. This is not a simple birth and death process since the arrivals bring k clients at a time into the system. The flow diagram, figure 4, shows this process:

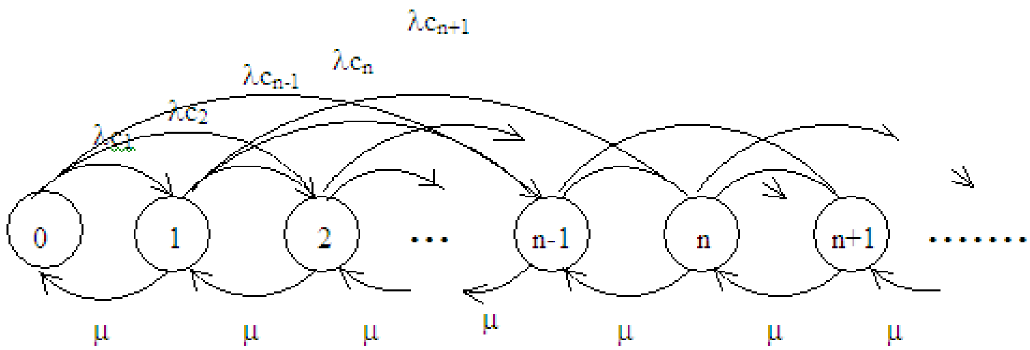

Figure 4 Flow Diagram $M^{[X]}/M/1/\infty/FIFO$

From this diagram the following equations represent the steady state of the system:

$$\lambda P_0 = \mu P_1$$

$$\lambda P_n + \mu P_n = \lambda \sum_{k=1}^{n} P_{n-k} C_k + \mu P_{n+1} \quad n \geq 1 \tag{11}$$

where $\lambda = \sum_k \lambda_k$ and

$$C_k = P[x = k \qquad \forall k \tag{12}$$

Due to the complexity of this process the correspondent states probabilities are obtained from the probability generating functions given by:

$$P_n = \frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n} P(z) \tag{13}$$

where

$$P(z) = \sum_{0}^{\infty} P_n z^n \quad |z| \leq 1 \tag{14}$$

and

$$C(z) = \sum_{k} C_k \; z^k \quad |z| \leq 1 \tag{15}$$

5 Conclusions

When working with undergraduate students, appropriated teaching techniques should be chosen taking into account the background they already have. The use of flow diagrams proved to be very useful in the representation of the steady state of stochastic processes. These diagrams provide a practical way of finding the balance equations that represents the steady state of the process under consideration. With this technique the students easily grasp the main ideas without dealing with more advanced techniques of mathematical and probability concepts.

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An interactive online calculus text

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We describe the development and use of an online textbook for first-year calculus, the second edition of a book that first appeared in print. The principles on which the book is based are the same as those the first two authors developed in Project CALC more than a decade ago, but the online version has many interactive features that could not be in a print-based text. We expect the book to be published by the Mathematical Association of America, but for the time being (while development continues), it is freely available to all at http://www.math.duke.edu/education/calculustext. We are eager to have teachers and students use any or all of it and provide feedback.

Keywords: Calculus, single-variable, differential, integral, online, interactive, computer algebra system

AMS Subject Classification: 97-01, 97D30, 97D40, 97D80, 97U20, 97U70

1. Background: Project CALC

The textbook we describe here is the second edition of *Calculus: Modeling and Application*, a book that was first developed as part of Project CALC: Calculus As a Laboratory Course. This project was funded by the National Science Foundation (NSF) from 1988 to 1995, and the first edition textbook and lab materials were published in 1996 by Houghton Mifflin, shortly after acquiring the intended publisher, D. C. Heath. Some online lab modules were later developed as part of the Connected Curriculum Project, also funded by NSF from 1993 to 2001. These print and online materials are still in use at some colleges.

Project CALC had (and still has) the following characteristics:

- hands-on activities
- discovery learning
- real-world applications
- writing and revision of writing
- high expectations of students
- teamwork
- intelligent use of available tools
- emphasis on students checking their own work

These features were selected on the basis of research on what educational strategies lead to durable and transferable learning, as well as on modeling what students could be expected to encounter once they leave the academic world.

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Early on we established goals for what we expected Project CALC students to learn and do, while in each semester of the course, as well as after the course was over. For example, here are some of our in-course goals:

Students in the Project CALC sequence should be able to

- understand concepts rather than merely mimic techniques;
- demonstrate understanding by explaining in written or oral form the meanings and important applications of concepts;
- construct and analyze mathematical models of real-world phenomena, including both discrete and continuous models;
- distinguish between discrete and continuous models and make judgments about the appropriateness of the choice for a given problem;
- understand the relationship between a process and the corresponding inverse process;
- select between formal and approximate methods for solution of a problem and make judgments about the appropriateness of the choice;
- select the proper tool or tools for the task at hand.

And at the end of the course:

Students who complete the Project CALC sequence should be able to

- use mathematics to structure their understanding of and investigate questions in the world around them,
- use calculus to formulate problems, to solve problems, and to communicate their solutions of problems to others,
- use technology as an integral part of this process of formulation, solution, and communication,
- work and learn cooperatively.

In 1991, Project CALC won the EDUCOM Higher Education Software Award as *Best Mathematics Curriculum Innovation*, and in 1993 it was cited by Project Kaleidoscope as *A Program that Works*.

2. The Second Edition

Our desires for the second edition are to

a) Make the text flexible, hyperlinked, interactive, richly illustrated, and available at low cost, and

b) Demonstrate the feasibility of an online textbook.

We are supported in this effort by an NSF grant (NSF-0231083) to the Mathematical Association of America (MAA) for the MathDL Books Online Project, which is also supporting development of a Mathematical Modeling text by Frank Wattenberg and his colleagues at the US Military Academy, as well as two demonstration chapters of David Bressoud's *A Radical Approach to Real Analysis*.

The first step in our process has been to redesign and redevelop the textbook entirely online, and that step is essentially complete. Some of the issues we encountered in this phase were page design, navigation, directory and file structure, sources for illustrations, nature and implementation of the built-in interactivity, technical requirements for the user, and effective presentation of mathematics online. Here are brief comments on how we dealt with those issues.

The basic page design and a cascading style sheet (CSS) for implementing it were provided for us by an in-house designer at MAA. We constructed our own hierarchical directory structure, based on chapters and sections of the book, with consistent naming of file types across chapters, and relative internal links, so that the entire structure could be moved without breaking any links.

Our navigation is based on a pop-up Table of Contents window for each chapter that remains open beside or overlapping the main text window. Supplementary notes, comments, and checkpoints for students also open pop-up windows that can be closed when they are no longer needed. Each main page has forward and back buttons, as well as a link to Contents page.

Our illustrations are of two kinds – mathematical graphs or diagrams and photographs or other illustrations. For the first kind, we construct a mathematically correct graph in a computer algebra system (usually Maple®) and then edit it in a graphic tool (usually Paint Shop Pro®). For the second, we find public domain or otherwise free photos (e.g., from US government sites or one of several free stock photo sites) or we take our own digital pictures, and then we crop, resize, etc. (again in Paint Shop Pro®).

We have a variety of interactive features, ranging from low to high on a technology scale. For example, we have a built-in pop-up numeric calculator, as well as files from which a student can print simple graph layouts, detailed graph paper, or slope fields for sketching solutions of differential equations. We have some embedded applets for carrying out certain experiments, and we have many prepared computer algebra (CAS) files in Maple® and Mathcad® that each get students started on an assigned task, but that will not completely solve the problem without student thought and inputs.

Our main text pages are constructed in XHTML, with most of the formulas presented in MathML, using code constructed in WebEQ®. We also use ASCIIMathML (a free product available under the GNU General Public License) for some of our formulas, including all the ones in pop-up pages, which are ordinary HTML.

To support our use of MathML across platforms, we require that the user have or install the Firefox browser and Mozilla's MathML fonts. To support ASCIIMathML and our control code for pop-ups and other interactions, we require that the user enable javascript. And to support our CAS activities, we require (at present) either Maple® (version 9.5 or later) or Mathcad® (version 13 or later). Our embedded applets require the Flash® player, which is included with all modern browsers, but it's also available as a free download.

3. What's in the Book?

In this section we present a brief summary of each chapter, including some of our reasons for structuring the book as we have.

Chapter 1: Relationships. The central question of the introductory chapter – which contains no calculus – is "What is a function?". Our objective is to separate this concept from other relationships between varying quantities and especially to separate it from "formula". Our purpose is to replace some of the inappropriate ideas students typically bring from secondary mathematics with a healthy regard for the mathematical concept that will be the foundation for the rest of the course. We also take up the algebra of functions and pose some problems for which the solutions are functions or classes of functions (e.g., symmetry, additivity).

Chapter 2: Models of Growth: Rates of Change. Here we establish some basic reasons for studying calculus, especially to be to able solve differential equations. Our primary example is the natural population growth equation, the simplest ODE to solve, and an immediate reason for moving beyond polynomials. We introduce difference quotients, derivatives, slope fields, initial value problems, solutions (which are, of course, functions or families of functions), exponential and logarithmic functions, and logarithmic plotting. The primary tool for understanding the derivative is zooming in on locally linear functions, and the primary formula is "slope equals rise over run."

Chapter 3: Initial Value Problems. This short chapter builds on Chapter 2, introducing Newton's Law of Cooling (exponential decay) to solve a murder mystery, then studying falling objects without air resistance (polynomial solutions).

Chapter 4: Differential Calculus and its Uses. This is the heart of the first-semester course, consolidating what has been learned about derivatives to take up optimization, concavity, Newton's Method (as an exercise in local linearity), and the basic formulas for differentiation. The product rule is introduced to study the growth rate of energy consumption, the chain rule to study reflection and refraction, and implicit differentiation to calculate derivatives of the logarithmic functions and of general powers. Zooming in is related to differentials and Leibniz notation.

Chapter 5: Modeling with Differential Equations. Here we return to falling bodies (e.g., raindrops, skydivers) and introduce air resistance proportional to the velocity or its square. The latter requires (for now) numerical solutions, and we take up Euler's Method as another "slope equals rise over run" application. We introduce periodic motion (with second-order ODE's, harking back to Chapter 3, where we derived position from constant acceleration), along with the basic trigonometric functions and their derivatives. This chapter concludes the first semester, and at the end of the chapter we summarize the derivative calculations.

Chapter 6: Antidifferentiation. At the start of the second semester, we turn our derivative summary inside out and catalog the functions for which we can now find antiderivatives – a necessary step if we're going to solve differential equations. We expand our tool kit with the simplest case of partial fractions to solve the Verhulst (logistic) model of population growth and explore how Verhulst, writing in 1840, could predict the US population in 1940.

Chapter 7: The Fundamental Theorem of Calculus. The big moment everyone has been waiting for – we introduce the integral as an averaging process, e.g., finding average temperature over a day or a year, and then relate that to area under a curve. We approach the FTC by exploring the linkage between speedometer and odometer, and then we "derive" the theorem by solving a differential equation – given the derivative, what's the function? – a question for which we already know one kind of answer. The partial sums of the left-hand rectangular approximations to area are, in fact, the Euler approximations to the solution of the differential equation, and this establishes the connection between antidifferentiation and area. Given this connection, it makes sense to introduce the indefinite integral as a notation for antidifferentiation.

Chapter 8: Integral Calculus and its Uses. This is the second-semester analog of Chapter 4. We start with a problem of fundamental physical importance, moments and centers of mass, to reinforce the idea of integration as averaging. We develop numerical methods through Simpson's Rule (as a weighted average of the trapezoidal and midpoint rules), so that no definite integral need remain unevaluated when one is working at a computer. Then we address the basic rules for integration by hand: algebraic and trigonometric substitutions and integration by parts. We close with an elementary look at Fourier analysis, using an electrocardiogram as an example.

Chapter 9: Probability and Integration. Our model problem in this chapter is reliability theory – how long do things last? The simplest model is the exponential distribution, which leads naturally into improper integrals. Now that students have experienced eight chapters of limiting behaviors, it is appropriate to introduce the standard notation for limits (but not the ε-δ definition, which belongs in a later course). We also take up other probability distributions (e.g., the normal) for which finding a mean or standard deviation may involve proper or improper integrals that can't be evaluated in closed form. This leads to defining some functions (e.g., the error function) by their integral representations.

Before we describe the last chapter, a brief polemic. It has become traditional in the US (and perhaps elsewhere) to end Calculus II with a chapter called "Sequences and Series", usually a compendium of everything we know short of a real analysis course, and invariably at a much higher level of sophistication than the rest of the course. The argument for doing this is that it's single-variable calculus, and the unstated argument is that it doesn't fit neatly anywhere else. There doesn't seem to be an argument that learning all possible convergence tests for series of constants is a skill needed by our future scientists and engineers. In our last chapter, we attempt to make the content flow naturally from what preceded it and to focus on skills that are or could be useful.

Chapter 10: Polynomial and Series Representations of Functions. As the title suggests, our emphasis is on representation of important functions, whether approximately by polynomials (perhaps very long polynomials) or by "infinitely long" polynomials. We start with the easy ones – exponential and trigonometric – and work up to the error function, using substitutions, differentiations, integrations. As a practical application, we note that it would take too long to evaluate the error function by integration, say, on a calculator or in a CAS, but it can be evaluated fast enough to graph it by using a relatively short polynomial. The primary tools for testing convergence are the alternating series test (AST) and the ratio test (RT) – and often they are the only tools needed. The first is geometrically obvious, and the second we obtain by comparing the tail of a series to that of a geometric series. Both come with error estimates. For power series, only the RT is needed unless there is a finite radius of convergence – and then the AST or comparison with, say, a harmonic series will usually do the trick.

4. Classroom Testing

Our textbook has been and is being classroom-tested at Hood College in Frederick, MD (USA) under the guidance of the third author, who is the lead teacher for calculus. Hood is a private, coeducational, liberal arts college with about 1200 undergraduate students, including a significant number of commuter students. The Project CALC materials have been used at Hood, under the leadership of department chair Betty Mayfield, almost from the beginning of their development, so it was natural for Hood to try the second edition.

In the Fall Term of 2006, the online text was used by 70 students in three sections with two teachers and four undergraduate teaching assistants (TA's). In Spring 2007, 45 of those students continued in Calculus II in two sections with two teachers and five TA's, and another 20 students started Calculus I with one instructor and one TA. In the current term (Fall 2007), Calculus I will again have three sections and Calculus II will have one (with both continuing and incoming students), and all sections will use the online text.

The challenges of using this book include getting students to accept an online text, learning how to use the text in class, convincing students to actually read the book, developing new versions of labs and projects, and coping with editing that was ongoing while the course was in progress. On the asset side of the ledger are the direct links to technology and to outside information, the checkpoints and activities with (slightly) "hidden" answers, and the opportunity to have a direct impact on the emerging edition.

At Duke our most recent teaching environment is the Interactive Computer Classroom [1], which functions as both laboratory and classroom as needed. This would be the ideal environment for use of an online text. The teaching environment at Hood is almost as good – each calculus section is scheduled simultaneously in a classroom and a lab that are next door to each other, allowing the instructor to move the class back and forth as necessary for the activity at hand. The classroom seats 24 at non-movable tables (plus an extra table for TA's), and the lab has 12 computers with two chairs each. Each room has an instructor's station with computer, document camera, VCR/DVD player, and ceiling-mounted projector. Each section is scheduled for three periods per week of an hour and 45 minutes.

A typical class day at Hood includes varied activities that might be any mix of discussion of the text (sometimes via lecture), working on a lab in pairs or on a project in somewhat larger groups, working on individual worksheets (possibly consulting with a neighbor), or writing a group report. Over the course of the year, the Hood classes covered most of the text but omitted periodic motion and circular functions in the first semester, as well as Fourier representations and some probability theory in the second semester. Parts of the omitted material were covered instead in labs, projects, or worksheets, either from the first edition (and eventually to be in the second edition) or of local design.

Here is our advice to instructors considering adoption of this online text:

- Be ready and willing to talk with your students about why your class is structured this way.
- Have the students read a bit about math education research, e.g., [2].
- Know what you want from your course: what's the focus?
- You can't be all things to all people.
- Exploration takes time, and there's no substitute for experience – the students'.
- Letting the class explore means you don't have complete control over what will happen next. Be flexible.
- Find ways to find out what your students really know.
- Work the projects ahead of time!
- As you teach, keep track of what you do when. (Some topics in the text are introduced at a surface level and revisited as tools become available.)
- If possible, find and hire TA's, especially for lab activities.

5. Previews of Coming Attractions

Over the course of the 2007-2008 academic year, we will complete the following additions and enhancements.

- Implementation of routine exercises in WeBWorK, a free online homework system that features randomly generated and individualized assignments, many ways to ask and answer questions, instructor-controlled numbers of repetitions and levels of hints, and automatic grading.

- An Instructor's Guide, with inputs from classroom teachers who have used and are using the text.
- Additional CAS options (e.g., Mathematica)
- Sections on the use and misuse of CAS integration tools (Ch. 8, to replace a first edition section on use of tables) and on convergence of a series to the right function (Ch. 10).
- More projects drawn from our first edition text and lab manuals and from the Connected Curriculum Project (http://www.math.duke.edu/education/ccp/). So far, we have projects on the spread of AIDS (Ch. 2), on air traffic control (Ch. 4, develops ideas of related rates and optimization), and on the area of Crater Lake (Ch. 7).
- Enrichment material – more applications (e.g., the SIR model for spread of epidemics, pendulum motion, discrete logistic growth and chaos, the gamma distribution) and theory (e.g., the Mean Value Theorem, the logical underpinnings of the calculus)
- Search/Browse/Index capabilities.

References

[1] Smith, D. A., 2001, The Active/Interactive Classroom, in D. Holton (Ed.), *The Teaching and Learning of Mathematics at University Level: An ICMI Study* (Dordrecht: Kluwer Academic Publishers). Available online at http://www.math.duke.edu/~das/essays/classroom/.

[2] Smith, D. A., 1996, Thinking about Learning, Learning about Thinking, in A. W. Roberts (Ed.), *Calculus: The Dynamics of Change*, MAA Notes No. 39 (Washington, DC: Mathematical Association of America). Available online at http://www.math.duke.edu/~das/essays/thinking/.

[A] I am referring to the UK Economic and Social Science Research Council (ESRC)-funded seminar series: Mathematical Relationships Identities and Participation, http://www.lancs.ac.uk/fass/events/mathematicalrelationships/.