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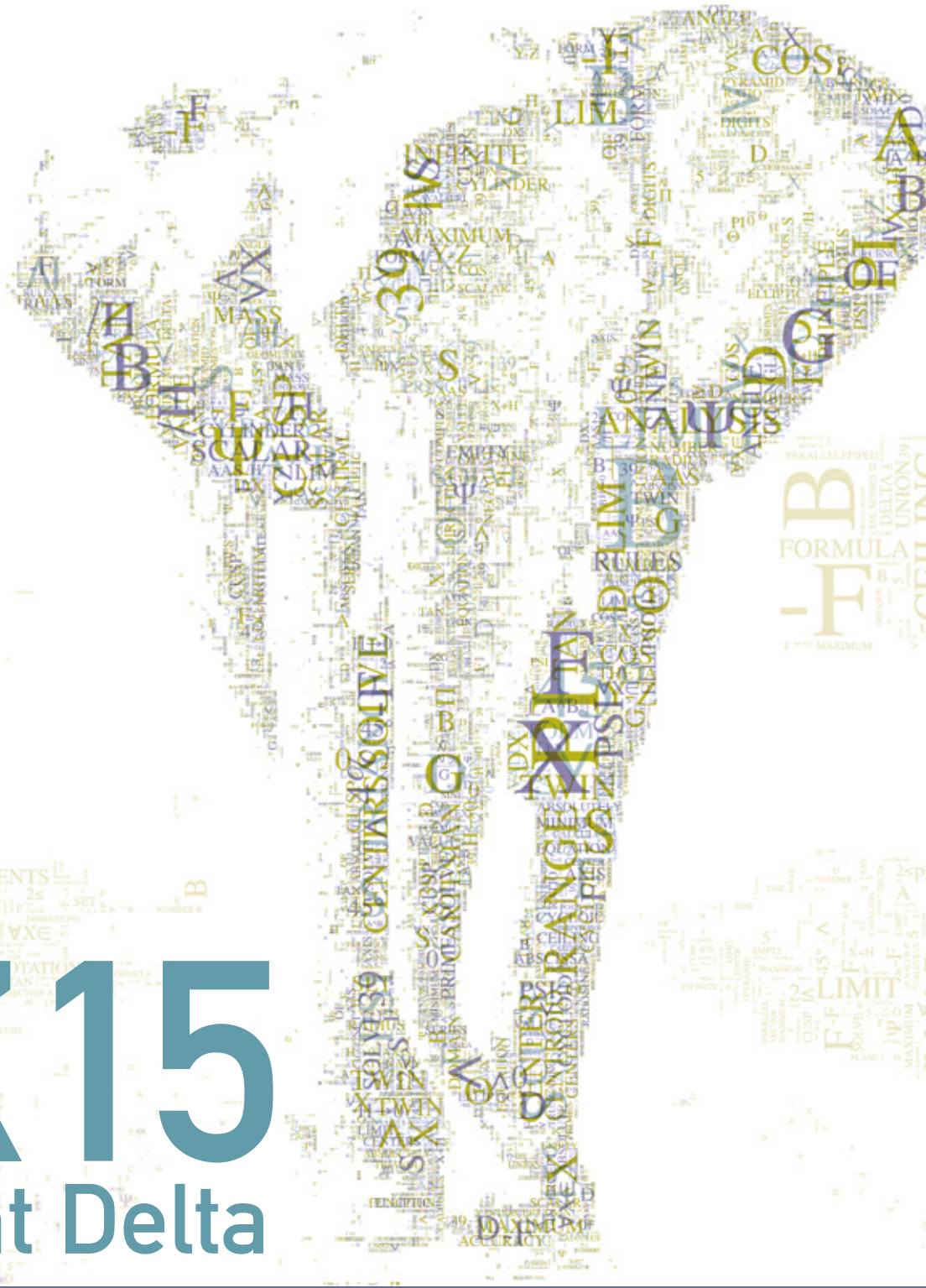

ε∆'15

Elephant Delta

10TH Southern Hemisphere Conference on the Teaching and Learning of Undergraduate Mathematics and Statistics

22 – 27 NOVEMBER 2015 | Port Elizabeth, South Africa

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Editors' Foreword

Proceedings of the 10th Southern Hemisphere Conference on the Teaching and Learning of Undergraduate Mathematics and Statistics

The first of the Delta series of conferences on the teaching and learning of undergraduate mathematics and statistics was held in Brisbane, Australia, in 1997. The Delta conferences have been held every second year in various southern hemisphere countries. The Elephant Delta is the tenth conference in the Delta series and held from 22 to 27 November 2015 at the Board Walk Hotel in Port Elizabeth, South Africa.

At this Elephant Delta conference presentations will include papers accepted for a Special Delta Issue of the International Journal of Mathematical Education in Science and Technology, full papers accepted for the conference proceedings and accepted abstracts for oral or poster presentations.

The papers contained in the conference proceedings underwent a double-blind peer-review process with at least two reviewers. During this process the identities of the authors were concealed to the reviewers and vice versa. We received 11 papers of which 7 were selected for inclusion into the proceedings after the review process.

We give our sincere thanks to the reviewers who assisted us in the review process, we thank all those involved in making this conference and its proceedings happen.

Proceedings Editorial Team: Rénette J Blignaut[1] and Rita Kizito[2]

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Welcome

Welcome to Elephant Delta, the Tenth Southern Hemisphere Conference on the Teaching and Learning of Undergraduate Mathematics and Statistics.

We are very pleased indeed to welcome you to Port Elizabeth, the Friendly City and a gateway to the Eastern Cape. Our programme, within the bounds of the topic of the conference, is varied and promises much opportunity for learning and for networking with peers across the Southern Hemisphere, with our Northern Hemisphere colleagues joining us under the African sun. The publications for this conference include the special issue 46(7) of the International Journal of Mathematical Education in Science and Technology, the Proceedings, the Communications and the Programme, all drawn up and edited by the Elephant Delta committee. The iJMEST issue and the Proceedings were double blind peer reviewed by at least two reviewers per paper.

We have worked hard to ensure that this conference is worthy of the name of "Delta", living up to the high standards of the conferences which have preceded this one. We hope that you will enjoy the conference, the city and your experiences in the Eastern Cape.

In pursuit of excellence in teaching and learning,

Tracy Craig, *on behalf of the organising committee*

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An Analysis of the Reasoning Abilities of Students in the Transition Period from Secondary to Tertiary Mathematics

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Abstract

This article reports the results of an empirical study to determine the reasoning abilities of a group of first year mathematics students at entrance level at a tertiary institution. In the empirical study the questions in the first mathematics test written on tertiary level, as well as the students' performances in the test, were analysed using Lithner's framework. Questions in the test were categorised as imitative (memorised, guided and algorithmic reasoning) and creative (local and global) reasoning. Student performances were evaluated in accordance with these categories to give an indication of their reasoning abilities. The results indicate that these students experienced difficulties when answering questions that required creative elements, in contrast with questions that were based mainly on memorized or algorithmic reasoning.

Keywords: mathematical reasoning; creative reasoning; transition from secondary to tertiary mathematics; reasoning skills

Introduction

The introduction of a new school curriculum in South Africa and the subsequent problems that students who followed this curriculum encountered at first year university level, refocused attention on the transition from secondary to tertiary mathematics.[1] This transition forms part of a student's overall journey from elementary to advanced mathematics. Mathematical reasoning is an important process through which mathematical understanding develops. At the entrance level to first year mathematics the formal deductive

reasoning of advanced mathematics is not yet required from students. Learners would be expected to display mastery in reasoning abilities when they tackle mathematical tasks at tertiary level. First year lecturers expect their students to be able to reason creatively; implying that they must be able to transfer mathematical knowledge from familiar to unfamiliar mathematical contexts. In general mathematical reasoning is not restricted to formal proof, but is a way of thinking to produce assertions and reach conclusions and includes the thinking done when confronted with ordinary mathematics tasks.[2] Reasoning is developed when students are given opportunities to explore on their own and are then expected to verify the results of their explorations.[3] This process leads to creative reasoning that goes beyond just following strict algorithmic paths or recalling ideas provided by others.[2] Creative reasoning implies the application of existing procedural knowledge to new situations, and the creation of a new sequence of reasoning, or the recreation of a forgotten sequence, to solve novel problems.[4]

Elementary or school mathematics is rich in opportunities to develop a repertoire of reasoning skills and to build connections before entering advanced mathematics courses.[5] The ability to reason creatively is one of the critical outcomes of the outcomes-based school curriculum in South Africa. Learners emerging from the school system should 'demonstrate an ability to think logically and analytically' and 'be able to transfer skills from familiar to unfamiliar situations'.[6,p.5] In the Curriculum and Assessment Policy Statement (CAPS) of Basic Education in South Africa (DoE, 2011) it is stated that the curriculum is based 'on active and critical learning, encouraging an active and critical approach to learning rather than rote and uncritical learning of given truths'. It is clear that the compilers of the new school curriculum aimed to prepare learners for a smooth transition to tertiary level.

The development of creative reasoning at school level is realised by the creation of learning environments where learners can use their intuitions to experiment, search for

patterns, reason logically, generalise and make conjectures on their own or with others.[6,7] However, teachers struggled to come to terms with these changes and when they experienced difficulties they tended to fall back on their old ways of teaching.[8] In reality the implemented curriculum differs from the intended curriculum.[9] The intention of the school curriculum is to encourage reasoning, but in practice the addition of more topics to the curriculum has had a negative impact on the type of learning that takes place at school level. There is less time to focus on mathematical reasoning and thinking, hence learners focus on surface features and neglect the structural features of mathematical problems.[10] The quest for good Grade 12(the last year in secondary school) pass rates has contributed to this negative trend as the focus on good results resulted in quick-fix approaches where problems are solved using a standard method imitated from previous work in order to prepare learners for the examination.[11,1]

The nature of mathematics teaching in secondary schools will influence the ability of learners to successfully make the transition to tertiary mathematics courses.[12] Articles on the preparedness of first-year students from the new curriculum for tertiary mathematics tend to focus on mathematical content.[13,14] Another approach to this problem would be to inquire whether first-year students from the new school curriculum are able to solve problems that require creative reasoning. To investigate the latter question we searched literature to find a scientific framework or model with which to evaluate the reasoning abilities of students in a rigorous manner. In this article we discuss the framework of Lithner [2] and report on an empirical study in which we have applied this framework to evaluate the creative reasoning abilities of first-year mathematics students.

Conceptual Framework for Evaluating Reasoning Skills

In general it is difficult to measure reasoning skills, but employing Lithner's framework

enabled us to evaluate reasoning skills based on scientific research. In the framework of Lithner [2] mathematical items are classified in terms of creative and imitative reasoning. Creative reasoning (CR) indicates that the problem should not be familiar to the students from previous encounters. Creativity is associated with the creation of new and well-founded task solutions. The characters of creative reasoning are novelty, flexibility, plausibility and mathematical foundation. Novelty is where a new sequence of reasoning is created or a forgotten sequence is re-created. There are no templates to follow. Flexibility means to apply different approaches and adaptations to the situation. Plausibility is used to describe reasoning that is supported by arguments that are not necessarily as strict as in proofs. These are arguments supporting the strategy choice and motivating why the conclusions are true. Guesses, vague intuitions and affective reasons are not considered. There must be a mathematical foundation for the reasoning. It means that the argumentation is founded on mathematical properties of the components involved in the reasoning. There are two subcategories for creative reasoning, namely *local creative reasoning* (LCR), which includes minor local elements of creative reasoning, and *global creative reasoning* (GCR), which includes larger elements of creativity. Problems featuring LCR have no complete solution schemes and the reasoning has to be constructed by the students themselves by connecting new ideas with existing ideas.

Imitative reasoning [2] implies copying or following a model or an example from a textbook or from earlier task solutions without any attempt at originality. It encompasses all reasoning that is based on previous experiences. If teaching is limited to copying or following a model or an example, without any attempt at originality, it leads to imitative reasoning. [15] Imitative reasoning can be subdivided into *memorised reasoning* (MR) and *algorithmic reasoning* (AR). The difference between the latter two categories is that memorised reasoning is a complete or literal memorisation of the solution, while algorithmic reasoning is the

recalling of a prescribed solution procedure and its application to a new data set. MR is limited to questions founded on recalling a complete answer, for example to recollect a fact or a proof. There are three different versions of algorithmic reasoning [2]: *familiar algorithmic reasoning* (FAR), *guided algorithmic reasoning* (GAR) and *delimiting algorithmic reasoning* (DAR). The basic notion of each subcategory is the recalling of an appropriate algorithm that will guarantee the attainment of a correct answer. An example of FAR is when a student/learner connects a differentiation algorithm to a task asking for the maximum value of a function without considering the inherent meaning of differentiation. GAR occurs when a person's reasoning is guided by a source external to the task – through text or another person. An example is test conditions which include a textbook or a formula sheet that can be used to copy a described procedure. DAR is when a student chooses an algorithm from a set of algorithms that seemed (correctly or wrongly) to correspond with the surface property related to the task. For example: A function $f(x)$ is given and the question is to evaluate $\frac{f(x+h)-f(x)}{h}$. DAR will occur if the student chose to determine the derivative $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

The classification of a problem as creative or imitative in nature is not entirely neutral and objective, as it is dependent on knowledge of students' previous history regarding the solution of the problem. The main classification involves a distinction between algorithmic or memorized reasoning on the one hand, and if creative reasoning is required on the other hand. Algorithmic and memorized reasoning is sufficient for solving a task if students can identify the task type and carry out the imitation. A question can be classified as requiring memorised reasoning if there were at least three tasks or examples in the study material that were exactly the same as the question in the test. This specific number, i.e., of at least three examples and/ or exercises, was validated by Palm et al. [16] in an earlier study on the basis of research in the Cognitive Psychology. A question can be classified as requiring FAR if at least three very

similar examples or exercises were given in the study material. Questions that could not be classified as either algorithmic or memorised reasoning were classified as LCR or GCR, provided that the conceptual knowledge needed to solve them appeared in the study material and that it can be reasonably expected that a typical student at a similar level would be able to solve the problem.

Empirical Study

Purpose

The purpose of the empirical study was:

- to use the framework of Lithner to classify the questions of a mathematics test in terms of memorized-, algorithmic- and creative reasoning; and
- to investigate the abilities of a group of students to answer the questions in the mathematics test regarding memorized-, algorithmic- and creative reasoning skills.

Research Design

The research was undertaken to examine a phenomenon at a specific time without the intention to change or modify the situation under investigation. The research methodology was descriptive in nature and was implemented in the form of a cross-sectional survey.[17]

Sample

The sample for the study was conveniently selected and consisted of all the first year mathematics students (n=647) enrolled for the mainstream mathematics module at a university in South Africa in a particular year. These students were all schooled according to an outcomes-based curriculum.

Research Instrument

The research instrument was the first mathematics test written by the students early in March of their first year. The test formed part of the normal assessment of the module and was compiled by the lecturers responsible for the module. The questions in the test were based on new content which built on work done in secondary school.

Research Procedure

The lecturers of the module marked the answer sheets according to a set memorandum. The marks that the students obtained for each question and sub-question were recorded in an Excel file. The average marks obtained for each question and sub-question were determined.

Analysis of Data

The analysis of the questions were based on the method employed by Palm, Boesen and Lithner.[16] They used the textbook prescribed at school level as a reference to what students have had an opportunity to learn. They acknowledged that one can never access students' complete thinking processes, but the textbook may represent the curriculum, which gives an indication of the students' anticipated learning experience. In the present study we included the textbook, manuals, study guides and class work prescribed for the module to get an indication of how familiar the problems were to the students.

After an initial analysis of the test written by the group we identified three constructs, namely memorised reasoning (MR), familiar algorithmic reasoning (FAR) and local creative reasoning (LCR). The former two, i.e., MR and FAR, are examples of imitative reasoning, while LCR is an example of creative reasoning. No items requiring global creative reasoning were identified. No guided algorithmic reasoning occurred, as it was not an open book test and no formula sheet was provided. These three constructs (MR, FAR and LCR) were

employed to analyse the written responses of the students. Marks were recorded and descriptive statistics were applied to compare the performances of the students according to the three constructs.

Validity and Reliability

Validity was established by involving five mathematics lecturers with experience in the students' transition from secondary to tertiary mathematics, in the classification process. The classification framework of Palm et al. [16] was explained in detail to all the lecturers taking part in the classification process. They first classified the questions individually and then discussed their classifications in a group to reach consensus. If consensus on the classification of a question could not be reached, the decision of the majority was accepted.

The statistical reliability of the constructs was measured using Cronbach's alpha coefficient. The results for the three constructs were compared using a t-test to determine if there were significant differences in performance on the questions requiring the three types of reasoning. A small p–value (<0.05) is considered as sufficient evidence that the result is statistically significant. Statistical significance does not necessarily imply that the result is important in practice, as these tests are dependent on sample size and have a tendency to yield smaller p-values as the size of the data set increases. A better measure is the effect size (so-called d value), which is independent of the sample size and is a measure of practical significance. It can be understood as a large enough effect to be important in practice and is described for differences in means.[18] Cohen (cited in [18]) gives the following guidelines for the interpretation of the effect size: small effect $0.2 \leq d < 0.5$; medium effect $0.5 \leq d < 0.8$ and large effect $d \geq 0.8$.

Discussion of Results

This section presents the tables with the questions in the test as well as the average percentages per question, as well as the average percentage achieved for each construct. The implications of the results of each construct are also discussed.

Memorised reasoning (MR)

Questions requiring memorised reasoning are listed in Table 1. These questions required students to complete a definition (1.1), define concepts (3.1, 4.1.1 and 4.1.2) and to draw an inverse trigonometric graph (6.1). For all these questions there were at least three instances in the study material where either the definitions were written out or graphs with the precise values were given, hence their designation as memorised reasoning. The Cronbach alpha value for this construct was 0.4. A value less than 0.5 indicates that the questions were not answered in a consistent way. For example, a student who performed well in question 1.1, did not necessarily performed well in the other questions testing this construct. Statistically the implication is that one cannot use the average percentage of 69.9% as a measure of this construct to compare it with the other two constructs.

Table 1. Questions and scores based on memorised reasoning (MR)

MR questions in the test		**Average % per question**	**Average (%) for MR**
1.1	Complete the following: $\lvert x \rvert = \begin{cases} \ldots \text{ if } x \geq 0 \\ \ldots \text{ if } x < 0 \end{cases}$	87.6	**69.9**
3.1	Define the concept function.	61.8	
4.1.1	Define the following concepts: $(f \circ g)(x) = \ldots$	83.5	
4.1.2	$(f \circ f^{-1})(x) = \ldots$		
6.1	Sketch the inverse of $\sin x,\ x \in [-1, 1]$	46.8	

Considering the questions separately, the percentages indicate that the students performed well in questions requiring memorised reasoning. The exception was question 6.1, which yielded 46.8%. A possible explanation is that this question was at the end of the test and that some students could not complete the test in the allocated time. According to the marking sheets, 37% of students did not attempt this question, which confirms that students ran out of time. This trend is confirmed in the analysis of question 6.2 (see next paragraph). One could argue that question 6.1 could be deleted from the Cronbach alpha analysis, but on subsequent investigation we did not find an improvement in the Cronbach alpha value after deletion of question 6.1.

Familiar algorithmic reasoning (FAR)

Questions requiring familiar algorithmic reasoning are listed in Table 2. These questions required students to solve an inequality (1.2) and to determine values of functions (2.1, 3.2, 4.2 and 6.2). For these questions there were at least three step-by-step explanations of how to get to the solution of similar problems in the study material. The questions in the test were not exact replicas, but the algorithms given in the study material could be used for answering these questions, hence the classification as FAR and not as MR. The Cronbach alpha value for this construct was 0.6, which means that these questions could be considered as a single construct and the average percentage of 65.85% could be used in comparison with other constructs with an appropriate Cronbach alpha coefficient. This average percentage indicates that the students performed well in questions requiring familiar algorithmic reasoning.

The average percentage of 39.4% for Question 6.2 was less than those of other questions in this construct (see, Table 2). Upon investigation we found that 43% of students did not attempt this question, leading to the deduction that these students did not have time to attempt Question 6.

Table 2. Questions based on familiar algorithmic reasoning (FAR)

FAR questions in the test	Average % per question	Average % for FAR
1.2 Solve the following inequality and give your answer in interval notation: $\lvert 5x-2 \rvert < 6$	67.6	
2.1 Complete:		
2.1.1 $-70^{o} = \ldots \text{rad}$	70.7	
2.1.2 $\frac{2\pi}{3} = \ldots .^{o}$		
3.2 Given: $f(x) = \begin{cases} x+1 & \text{if } x < -2 \\ x & \text{if } x \geq -2 \end{cases}$		
3.2.1 $f(-2) = \ldots$	77.5	**65.8**
3.2.2 $f(-4) = \ldots$		
4.2 Given: $f(x) = \sqrt{x+1}$ and $g(x) = \frac{x}{x+1}$		
4.2.1 D_f = ...	67.2	
4.2.2 D_g = ...	57.9	
4.2.3 $g \circ f$ = ...	81.8	
6.2 Find the value of the following expression: $\cos^{-1}(-\frac{1}{\sqrt{2}})$	39.4	

Local creative reasoning (LCR)

The remaining questions were classified as local creative reasoning and are listed in Table 3. These questions required students to graph an absolute value function (1.3), prove an identity (2.2), solve an equation containing reciprocal trigonometric functions (2.3), solve an equation containing natural logarithmic functions (5.1) and determine the inverse of a rational function (5.2). It must be noted that less than three or even no similar examples of these questions

appeared in the study material. The conceptual knowledge needed to solve these questions was contained in the study material. Another factor that contributed to this specific classification was that the reciprocal trigonometric functions, the absolute value functions and the natural logarithms were not part of these students' school curriculum.[6]

The Cronbach alpha value for this construct was 0.7, which means that these questions could be compared as a single construct. The average percentage of 33.2% indicates that the students performed poorly in questions requiring local creative reasoning.

Table 3. Questions based on local creative reasoning (LCR)

	LCR questions in the test	**Average % per question**	**Average % for LCR**
1.3	Graph the following function: $p(x) = \dfrac{2x + \lvert x \rvert}{x}$	40.2	
2.2	Prove the following identity: $\cot^2\theta + \sec^2\theta = \tan^2\theta + \csc^2\theta$	26.7	
2.3	Find all the values of θ which satisfy the following equation: $\csc\theta = -2,\ \theta \in [0, 2\pi]$	30.9	**33.2**
5.1	Solve x: $\ln(x+1) + \ln(x-1) = 1$	22.2	
5.2	Determine the inverse of $f(x)$: $f(x) = \dfrac{x+1}{2x+1}$	46.6	

Comparison of Constructs Based on Imitative and Creative Reasoning

The statistical results for the comparison of the constructs of familiar algorithmic reasoning and local creative reasoning appear in Table 4.

Table 4. Comparison of average scores for FAR and LCR

Mean for FAR (%)	Mean for LCR (%)	*p*-value	Effect size
65.8	33.2	0.000	1.43

The average percentage of 65.8% for familiar algorithmic reasoning in Table 4 indicates a far better achievement than the 33.2% of the local creative reasoning. This difference is validated by the statistical analysis. The p-value of 0.000 is less than the 0.05 required for a statistically significant difference. The d-value of 1.43 is greater than 0.8, which indicates that the difference is also practically significant. The statistical analysis of the first mathematics test written confirmed that the average mark for questions requiring creative (or more specifically local creative) reasoning were significantly lower than the questions requiring familiar algorithmic reasoning and that the difference was significant in practice.

Summary

Mathematical reasoning plays a crucial role in the transition from secondary to tertiary mathematics. In the context of the transition from secondary to tertiary mathematics the interpretation of mathematical reasoning in terms of imitative versus creative reasoning provides a useful framework to evaluate students' reasoning skills. In the current study the framework of Lithner [2], as implemented by Palm *et al*. [16] was used to classify the first mathematics test written by first year students at the institution in which this study was conducted. Furthermore, we classified the students' written answers in terms of memorised reasoning (MR), familiar algorithmic reasoning (FAR) and local creative reasoning (LCR). Students appeared to be confident and performed well (average of 66% for FAR) if the focus was on procedures that have been encountered before, but they struggled to answer problems requiring the transfer of knowledge from a familiar to an unfamiliar context (average 33% for LCR). The average for the memorized reasoning was 70%, but the distribution among the

students was not consistent, indicating that students did not exhibit a consistent trend when learning definitions and theorems.

Conclusions and Recommendations

We conclude that first year students at the institution where the current study was conducted performed well on questions requiring imitative reasoning, but struggled with questions that required creative reasoning. Creative reasoning skills are dependent on deep learning or learning with understanding. In the learning process students require more time to reflect on the processes of mathematics. There are many opportunities at school level to develop reasoning skills, however, deep learning is seemingly impeded by the time allocated to additional topics included in the new school curriculum and the drive to achieve good pass rates. Lecturers in mathematics want their students to be able to reason creatively, but for the majority of students this do not happen automatically. In order to develop creative thinking skills at school and tertiary level, students must be encouraged to use trial and error methods and to investigate different solution paths for the same problem. This will only come about if educators/lecturers teach consciously to assist and motivate students in the process of gaining skills to reason creatively. Assessment usually drives change; therefore assessment should include opportunities to develop creative reasoning skills. Problems that require students to develop their own strategies should be included in assessment, especially formative assessment. One proposal is to use the classification of Lithner as a taxonomy for tests and examination papers in an attempt to formalise the process of developing creative reasoning skills.

As a result of this study the first-year mathematics lecturers at the institution where the research took place adapted their teaching strategies to specifically include tasks that require creative reasoning elements. They also substituted Bloom's Taxonomy of cognitive

skills with Lithner's classification as a taxonomy to set their tests in order to make sure that questions from all the reasoning categories are included in the papers. We plan to do a follow-up study to investigate the impact of this move on the reasoning skills of the students.

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Simple rule, hidden meaning: the scalar product in engineering mathematics

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Abstract

Engineering is a highly mathematical field of study with different university courses requiring proficiency at different types of mathematics. Engineering dynamics requires the skilful use of vectors in various ways and proficiency at vector arithmetic, algebra and geometry is of vital importance to incoming students. This paper reports on findings from the administering of a vector proficiency assessment instrument across two semesters of a dynamics course. Findings suggest that problems requiring use of the scalar product embedded within a context are of the highest difficulty level. We argue that the geometric role of the scalar product is weakly understood by the majority of students, leading to poor performance at any problem requiring more than a basic calculation. We suggest that lecturers of engineering mathematics foreground the geometric role and that lecturers of engineering courses be aware of the level of challenge manifest in these problems.

Keywords: engineering dynamics; vectors; Rasch analysis; scalar product

Introduction

In modern engineering practice, and especially in computational mechanics, vector algebra is an indispensable tool for the solution of challenging problems, and, hence, proficiency in the algebraic manipulation of vectors is an absolute necessity for all engineering students. For this reason it is given great emphasis in first year mathematics courses taken by engineering students.

By contrast, vector geometry, that is the graphical representation of vectors using arrow headed line segments, while far from ignored, receives less emphasis. In particular, vector geometry is typically used for the description of problems, rather than the solution of problems, where algebraic manipulation is the preferred method. It could be argued that, within the context of typical first-year mathematics problems, vector diagrams, if used at all, are a halfway point between the ubiquitous vector arithmetic/algebra and the accurate and detailed vector diagrams used elsewhere, such as in dynamics. It is therefore imperative that the students grasp what vector geometry they encounter in first year in order to prepare them for the more advanced uses of vector geometry later. In this paper we discuss how even that minimal geometric understanding is worryingly absent, with serious implications for the teaching and learning of advanced vector use.

The Dynamics Education Research Group (DERG) was established at the University of Cape Town (UCT) to investigate a range of educational issues related to the teaching and learning of dynamics, of which several are mathematical in nature. One of the avenues under investigation is the degree to which students entering a second-year dynamics course retain the vector mechanics proficiency which they acquire in first-year mathematics. In particular, we are interested in the difference between the contexts in which students find a given vector manipulation to be evident or obscured.

In preparing for writing this paper, we searched the education literature using several databases and search engines and could find nothing other than textbooks and teaching guides looking at the teaching and learning of the scalar product. While we stumbled across quite a few scholarly treatments of the teaching and learning of the vector (cross) product, the literature has no work (or nothing easily found) on the scalar (dot) product [1,2,3]. We contend that this gap in the literature exists because the academic teacher's view of the topic is that it is so easily understood and so devoid of complexity that there is little to say. In this

paper we show empirically that a view of the scalar product as unproblematic is not correct and that students do struggle with using the scalar product in processes considered straightforward by the teacher or lecturer. We conclude with some ideas about why the students might find the scalar product unexpectedly challenging and offer suggestions for teachers of vector algebra and geometry.

Research methodology

Instrument and cohort

A test instrument was designed to assess proficiency at the vector algebra and geometry topics covered in the first-year mathematics course. This test was written in two consecutive semesters by the students registered for the Mechanical Engineering dynamics course for students in their second academic year of study at the University of Cape Town. Ethics approval was sought and obtained for the running of this study. The first and second semester cohorts consisted of 71 students and 129 students respectively, of whom 63 and 107 respectively gave consent for their data to be used in the analysis reported in this paper (Tables 1 and 2).

Table 1. Student numbers by Engineering programme

	Mech Eng	Mechatronics	Elec-Mech	Electrical	Totals
Semester 1	65	6	0	0	71
Semester 2	98	5	25	1	129
Totals	163	11	25	1	200

Table 2. Students stating consent for data use

	Gave consent	Withheld consent	
Semester 1	63	8	71
Semester 2	107	22	129
	170	30	200

Two of the students (both Semester 2 students) who gave consent for their data to be included in the analysis performed so well on the test that their results were excluded by the statistical software as the test was a poor fit for students of their level of proficiency. Therefore the data set finally used in the analysis discussed in this paper consisted of 168 students.

The first semester cohort saw an assessment instrument of 29 items. Item 13 in that instrument was not well constructed and has been removed from the data analysis. In the second semester, the students saw an instrument of 31 items, where Items 1-12 and 14-29 were identical to the first instrument, Item 13 was a better constructed version of the original Item 13 and Items 30 and 31 were new.

The items were chosen to represent a variety of typical vector topics encountered in first-year mathematics, including using the scalar product and the vector product, doing basic vector arithmetic, reading vector information off diagrams, calculating moduli, working with unit vectors, working with lines and planes and other geometric objects, such as spheres, and solving for parameters in vector parametric expressions. In this paper we discuss in detail the student responses to two of the scalar product items. We begin by briefly defining and contextualising the scalar product as encountered in a typical first-year mathematics course.

The scalar product (or dot product)

Given two vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, the projection of $\vec{a}$ onto $\vec{b}$ (or, alternately, the component of $\vec{a}$ in the direction of $\vec{b}$) can be calculated using trigonometry and vector scaling to give:

$$\frac{|\vec{a}|\cos\theta}{|\vec{b}|}\vec{b} = \frac{|\vec{a}||\vec{b}|\cos\theta}{|\vec{b}|^2}\vec{b}$$

where θ is the angle between the two vectors [4,5]. It is geometrically valuable to understand the factor $|\vec{a}|\cos\theta$as "the amount" of $\vec{a}$ in the direction of $\vec{b}$ and hence the product $|\vec{a}|\cos\theta \cdot \frac{\vec{b}}{|\vec{b}|}$turns that "amount" of $\vec{a}$ into a vector quantity. By defining the scalar product as

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

the projection expression above becomes

$$\frac{|\vec{a}|\cos\theta}{|\vec{b}|}\vec{b} = \frac{|\vec{a}||\vec{b}|\cos\theta}{|\vec{b}|^2}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\vec{b}.$$

Using the law of cosines we can extrapolate from the definition that

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3,$$

a calculation which is easy to remember and to perform. In fact, this form is so convenient that it is often remembered as the definition of the scalar product. Indeed, it is provided as the definition of the scalar product in many textbooks [4].

Use of the scalar product as part of a larger process, or embedded in a context, is usually related to angles, either determining an angle between two vectors, or imposing a

condition of orthogonality on a system of vectors since the scalar product of two orthogonal vectors is zero.

There are eight items in the assessment requiring use of the scalar product (see Appendix A). Two of those items simply provide two vectors and ask for their product (Items 1 and 14) and the other six require use of the scalar product as part of a larger process (Items 6, 7, 13, 21, 22 and 23). Items 6 and 23 ask for the component of a vector in the direction of another, both given. Item 21 asks for the distance between a point and a plane, requiring a self-chosen vector to be broken into components. Items 13 and 22 ask for the angle between two vectors, in both cases vectors which are not given directly and have to be determined from provided information. Item 7 requires use of the scalar triple product. We employed the Rasch measurement model to engage statistically with the items, to determine which items were found to be the most difficult and how students at different levels of proficiency responded to the items.

The Rasch measurement method

The Rasch measurement model is based on the requirement that measurement in the social sciences, including education, should aspire to the rigour of measurement instruments in the sciences, such as thermometers or rulers. The model originates with Georg Rasch [6] and has been discussed in detail elsewhere. [7, 8]

The first step in our process was to construct an instrument measuring a single construct of interest, in our case vector mechanics. The second was to administer the instrument to the study group, in our case second-year engineering students. The data were analysed using RUMM2030 software. [9] The items on the test were all multiple choice and were all on the single topic of vector mechanics. An important feature of this measurement

model is that the "construct of interest" be unidimensional, that is that the instrument is asking questions all centred around a single topic.

Results and discussion

The Rasch analysis software (RUMM2030) allows the data analyst to understand an assessment instrument in many different ways. Four outputs of the software will be included in this paper, namely (1) the fit statistics, (2) the item map, (3) the item characteristic curves for two of the items and (4) the multiple choice distractor curves for the same two items. To augment the statistical analysis, we include data from the students' rough working of the multiple choice items under scrutiny in a bid to understand where and how things are going wrong.

The fit statistics provide measures of the unidimensionality of the instrument under scrutiny and the fit of the instrument to the group responding to the instrument. The requirement behind the model is that a unidimensional property is being measured and that the difficulty level of the items will remain invariant across different cohorts and sub-cohorts. Our study is young, having been run in only two cohorts to date, one of them small, so the data are still being collected to measure robust invariance; however the fit statistics at this point in our journey suggest that the instrument is measuring a unidimensional property and that the instrument is an adequate fit to the cohorts. For those accustomed to reading such statistics we include some in Table 3. For those not familiar with such fit statistics, simply note the "acceptable" rating of unidimensionality, meaning that this test instrument was measuring a single construct of interest (vector mechanics) at an "acceptable" level.

Table 3. Fit statistics for the current analysis

Item fit residual		Person fit residual		Chi Square interaction		PSI (w/o extr.)	Unidimensionality t-tests (95%CI)
Mean	SD	Mean	SD	Value (df)	p		
-0.167	1.184	-0.188	0.765	145.07 (96)	0.00092	0.68736	2.7% (acceptable)

One output of the Rasch analysis process is an axis (the item map) which locates on the left individuals along a continuum of proficiency with the construct of interest and also on the right the items along a continuum of difficulty (see Figure 1). An individual (marked on Figure 1 with the × symbol) at the same point on the axis as an item indicates that a student of that level of proficiency has a 50% chance of answering an item of that level of difficulty correctly. If the item is located lower than the student on the axis, then it is easier for the student, and if the item is located higher than the student then the item is more challenging. For illustration, Figure 1 indicates that items 14 and 1 (marked as I0001.1 and I0014.1) were very easy for all the students. Items 10 and 15 were of moderate difficulty (50% chance of getting them correct) for the student with the lowest level of proficiency, that student identified as the lowest × on the left hand side of the axis.

Of particular interest in this paper is the clustering of the items requiring use of the dot, or scalar, product. The two items simply asking for the calculation of a scalar product are the two easiest items on the test. The other six items requiring use of the scalar product are six of the eight most difficult items of the test, as determined by Rasch analysis. We can contrast this bimodal behaviour to that of the items involving the vector product. The items involving the vector product are spread throughout the central portion of the continuum, more challenging than the "basic" scalar product and less challenging than the process-oriented

scalar product items. What's more, the "basic" vector product items (2 and 10) are themselves scattered among the spread of more process-oriented vector product items.

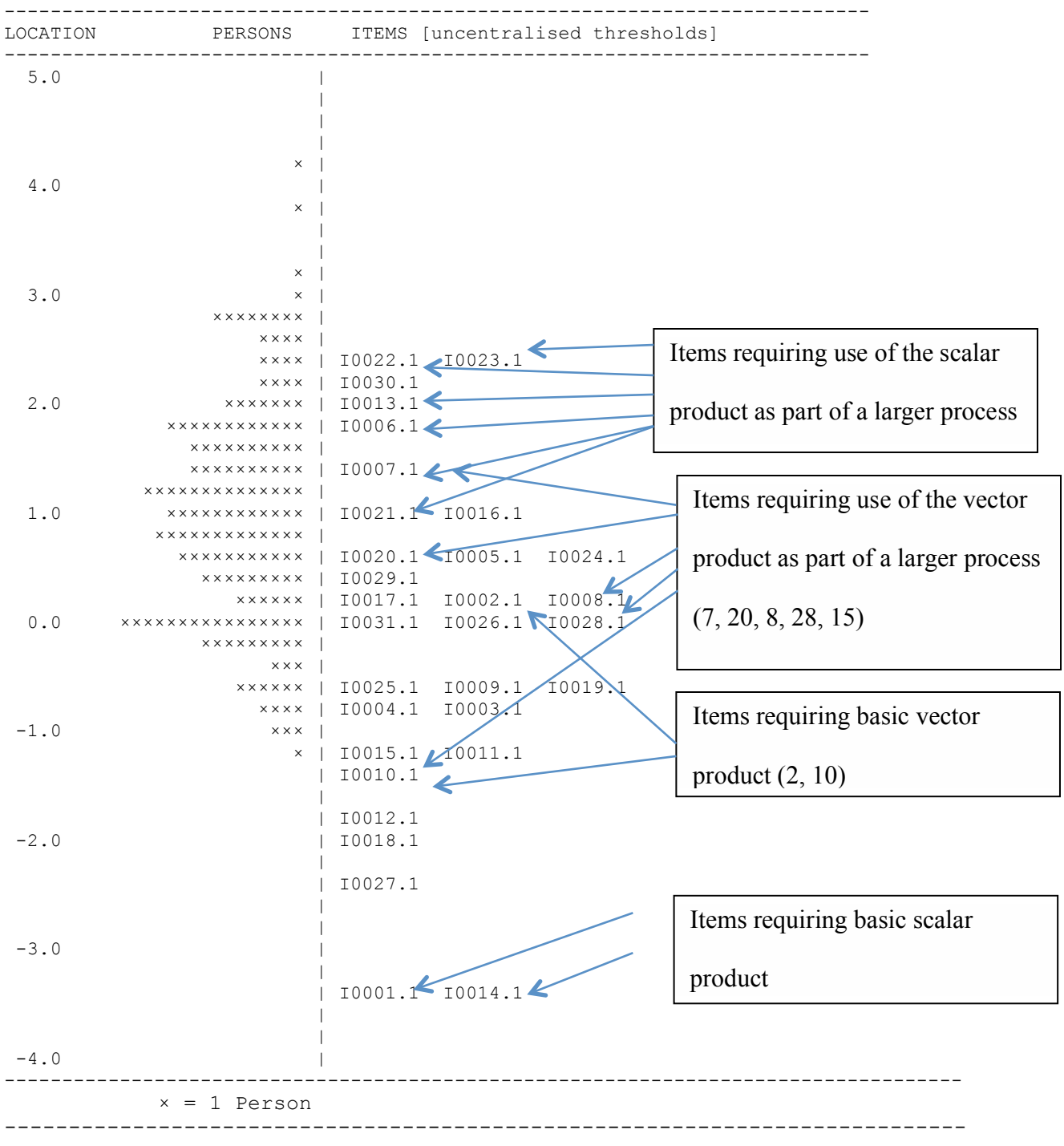


Figure 1. The item map

The fact that all of the items requiring use of the scalar product (other than simple calculations) are among the most difficult items on the assessment focusses attention on those items. Item 23, in particular, was found to be the most difficult item on the test by the students being assessed. Since Items 6 and 23 asked for the same process to be carried out, we looked more closely at these two items. In this paper we shall focus on items 6 and 23 (below, correct answers underlined). See Appendix B for solutions of these two items.

6. Resolve $\vec{a} =< 3,-5,-10 >$ into components parallel and orthogonal to $\vec{b} =< 1,3,4 >$. The parallel component is

(A) $< \frac{1}{2}, \frac{3}{2}, 2 >$ (B) $< 3, 9, 12 >$ <u>(C) $< -2,-6,-8 >$</u>

(D) $< -1,-3,-4 >$ (E) $< 1, 3, 4 >$

23. Resolve $\vec{v} = 2\hat{i} - 7\hat{j} + 15\hat{k}$ into components parallel and orthogonal to $\vec{u} = 2\hat{i} - \hat{j} + 3\hat{k}$. The parallel component is:

<u>(A) $8\hat{i} - 4\hat{j} + 12\hat{k}$</u> (B) $2\hat{i} - \hat{j} + 3\hat{k}$ (C) $-2\hat{i} + 1\hat{j} - 3\hat{k}$

(D) $4\hat{i} - 2\hat{j} + 6\hat{k}$ (E) $6\hat{i} - 3\hat{j} + 9\hat{k}$

For each of the item analyses below, we have included the item characteristic curve (the ICC) for each item as well as the MCQ distractor curves. The ICC positions a dot for each student proficiency quartile (located and indicated on the horizontal proficiency axis) at the probability level (vertical probability axis) at the collective probability (or expectation value) of students in that quartile answering the question correctly. In an ideal world where every student responds perfectly according to his or her proficiency level, those dots would lie along the indicated logistic curve. In an utterly random world where every student guessed

the answer, the dots would lie horizontally at the probability level of 0.2 (if, as in our case, there are 5 distractors per question). The MCQ distractor curves show what expectation value each quartile indicated of answering each specific distractor (where A=1, B=2, and so on) to the questions.

ICC and MCQ distractor graphs for Item 23

"Resolve $\vec{v} = 2\hat{i} - 7\hat{j} + 15\hat{k}$ into components parallel and orthogonal to $\vec{u} = 2\hat{i} - \hat{j} + 3\hat{k}$. The parallel component is:" [Item 23]

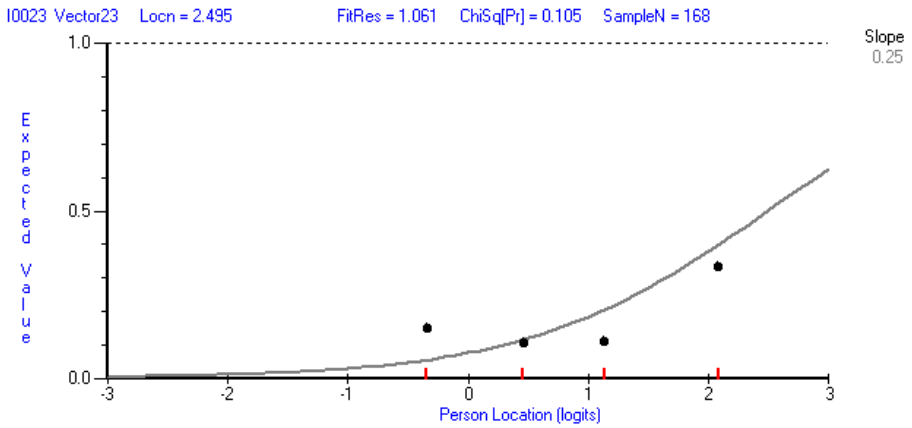


Figure 2. The ICC for Item 23.

The spread of the quartile responses (the quartiles' proficiencies are indicated by the small red bars on the horizontal axis) indicates some guessing, with the lowest quartile getting the answer correct more often than expected and highest quartiles less often (Figure 2).

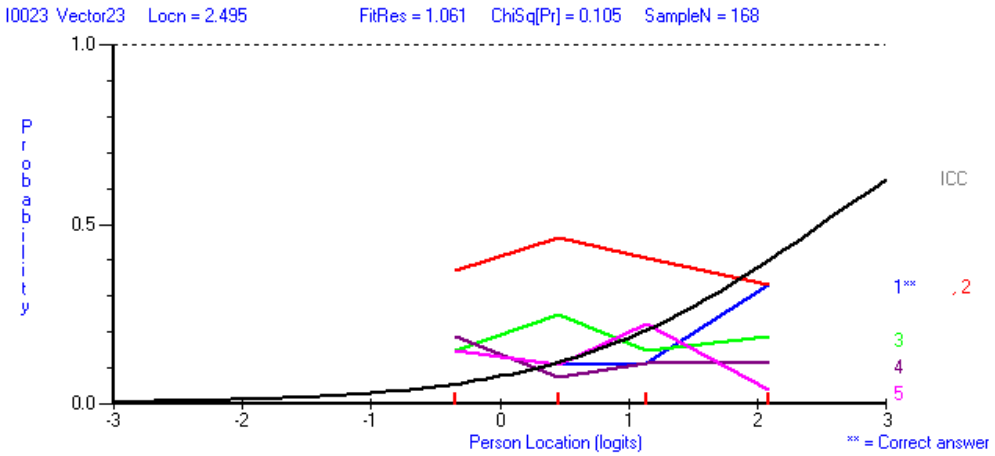


Figure 3. The MCQ distractor graph for Item 23

While A is the correct answer (shown as "1" and the blue line on the diagram above; also note the two little stars indicating 1 as the correct answer on the right hand side of the graph), option 2 (the red line) is more likely to be chosen by every quartile, with equal likelihood indicated for the top quartile (Figure 3). Option 2 (B) is the second vector ($\vec{u} = 2\hat{i} - \hat{j} + 3\hat{k}$) given in the question, suggesting either that students chose it as a likely looking guess, or that they believe that the component of vector $\vec{v}$ in the direction of vector $\vec{u}$ is the entirety of vector $\vec{u}$.

ICC and MCQ distractor graphs for Item 6

"Resolve $\vec{a} =< 3,-5,-10 >$ into components parallel and orthogonal to $\vec{b} =< 1,3,4 >$. The parallel component is:" [Item 6]

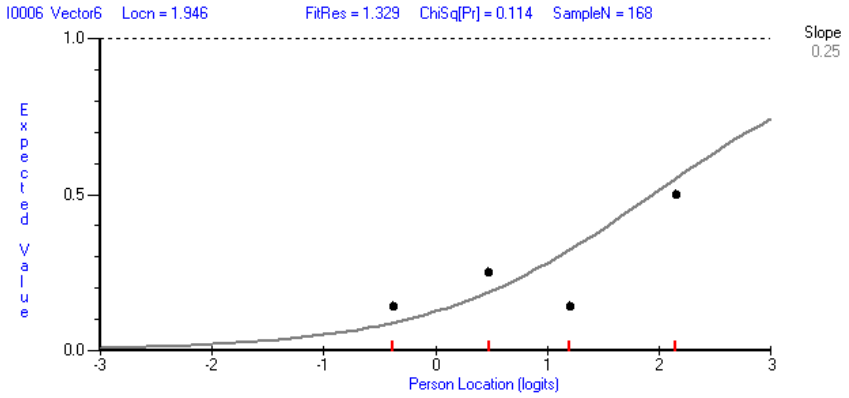


Figure 4. The ICC for Item 6

This item's ICC (Figure 4) shows a roughly similar guessing look to that of Item 23's ICC, with the lower two quartiles answering it correctly more often than expected and the top two quartiles less often. Nevertheless, this item was answered slightly better than Item 23.

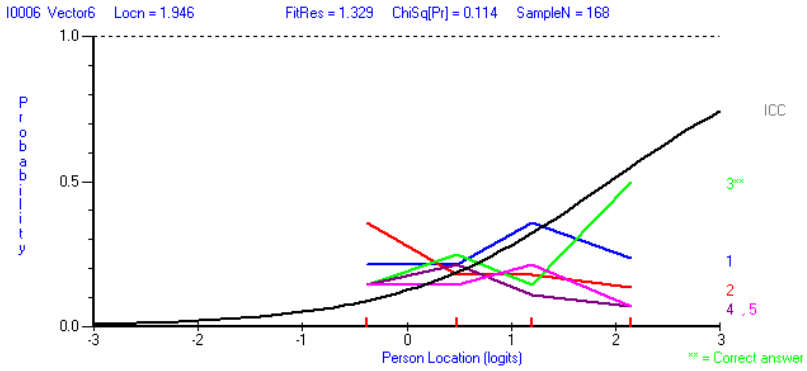


Figure 5 .The MCQ distractor curves for Item 6

In Figure 5, the correct answer of C (labelled 3 on the graph, the green line) is the most likely answer for the top quartile, but all of the distractors are contenders throughout, particularly A (1, blue) and B (2, red). The phenomenon noticed for Item 23, that of the second vector given in the question being the answer most popular with the students, is not observed here in Item 6; that option would be E (5, pink, on the graph above).

Types of errors

In order to understand what the students were doing when working on these items, we turned to looking through their rough work. Some students did their rough work in the margins of the question paper, others did it in a booklet provided and collected by the tutors. Since there was no stipulated requirement to make working apparent, in only a few cases (see Table 4) could working be identified for these two items.

Table 4. Number of identifiable instances of rough work

	On question paper	In booklet	Totals
Item 6	31	3	34
Item 23	2	8	10
Totals	33	11	44

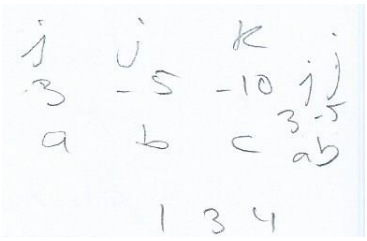
One interesting feature is that far less working was done for Item 23 than Item 6. One hypothesis is that the students recognised Item 23 as being very similar to Item 6, encountered earlier in the paper. Having found Item 6 to be challenging possibly led them to not attempt Item 23 with as much diligence. To test this hypothesis we shall swap the order of these two items in the next iteration of this instrument.

Some of the rough work was so minimal that it is impossible to see where the student made an error, for example simply writing a relevant formula and writing nothing further. In other cases, however, sufficient working was available to see patterns emerge across the cohort. Four types of error were identified, namely (1) using the vector product, (2) drawing the vectors in 3-dimensions and failing to find that helpful, (3) being confused as to how to use the scalar product and (4) weak skill at doing basic arithmetic.

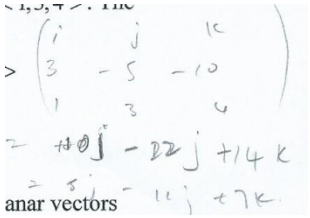
Two constraints prohibit us from inferring frequency of error from availability of rough work evidence. Firstly, rough work is not available for the majority of the students, for

a variety of reasons, and secondly, in several of the cases where rough work does provide interesting information the student has not given consent for us to publicise the data. Consequently, the examples given below are merely illustrative, not exhaustive and do not allow us to discern which errors have greater impact.

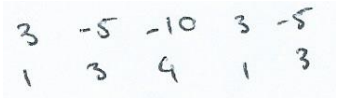
The students whose working is shown in Figure 6 are carrying out a vector product on the two given vectors, which indicates a lack of understanding of how to interpret the result of a vector product as well as a lack of knowledge of what process to use for the items in question.



Student 20152043



Student 20152026

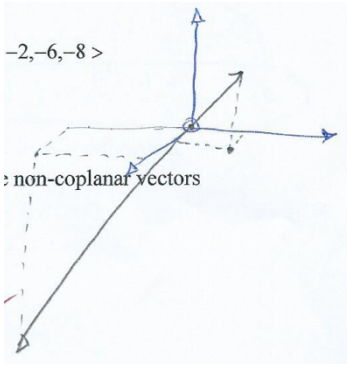


Student 20152023

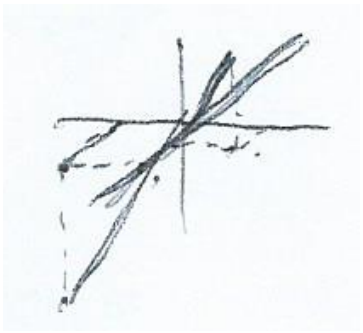
Figure 6. Vector product error

For exercises such as represented in Items 6 and 23, a two dimensional diagram serving as an aid to structuring the calculation correctly is as much as is needed. Students proficient with this type of problem might no longer need to draw any diagram as they can already "see" in their mind's eye what the geometric implications of the problem are. Some students in their rough work drew 3-dimensional diagrams. Certain students did so and went on to correctly answer the questions (falling back on the necessary algebra after drawing the diagram). Three dimensional vector sketches are not inherently bad; however they are not directly useful either in this context. Any diagram, 3-d or 2-d, might have helped some students conceptualise the problem and gain clarity in what algebra to use. Figure 7 shows the

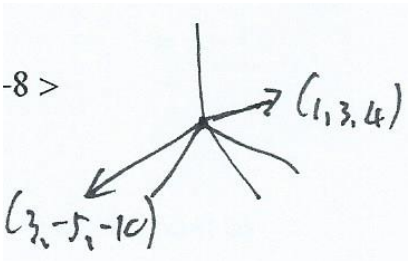
diagrams of students who got no further in their working than a 3-d diagram and ultimately answered the question incorrectly.



Student 20152089



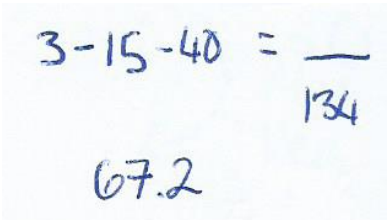
Student 20152030



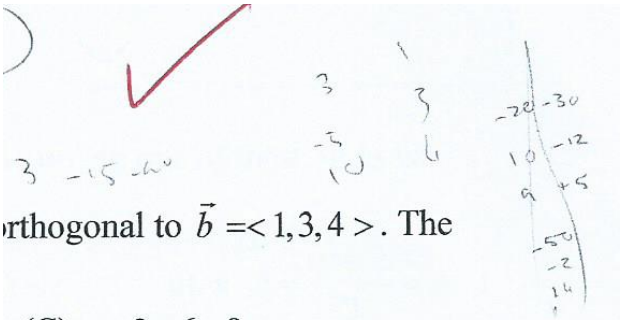
Student 20152111

Figure 7. 3-d diagrams with no further progress

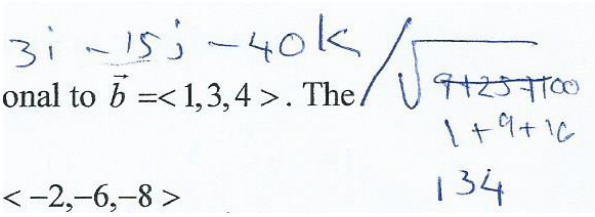
A third pattern in the types of error is failure to use the scalar product correctly. In Figure 8 below, Student 20152012 has carried out a scalar product and is using the modulus of one of the vectors, but has not put these otherwise useful pieces of information together in a useful way. (S/he seems to have calculated an angle in degrees as well, although the relevance of the 67.2 is unclear). Student 20152100 has carried out a scalar product and thereafter there is no apparent progress although the student seems to be trying out other calculations as well. Student 20152045 has rather worryingly carried out a version of the scalar product, yet has written the result in vector form, that is s/he has written $(3)(1)+(-5)(3)+(-10)(4)$ as $(3)(1)\hat{i}+(-5)(3)\hat{j}+(-10)(4)\hat{k}$. That student has then calculated the individual vector moduli but does not know what to do with them.



Student 20152012



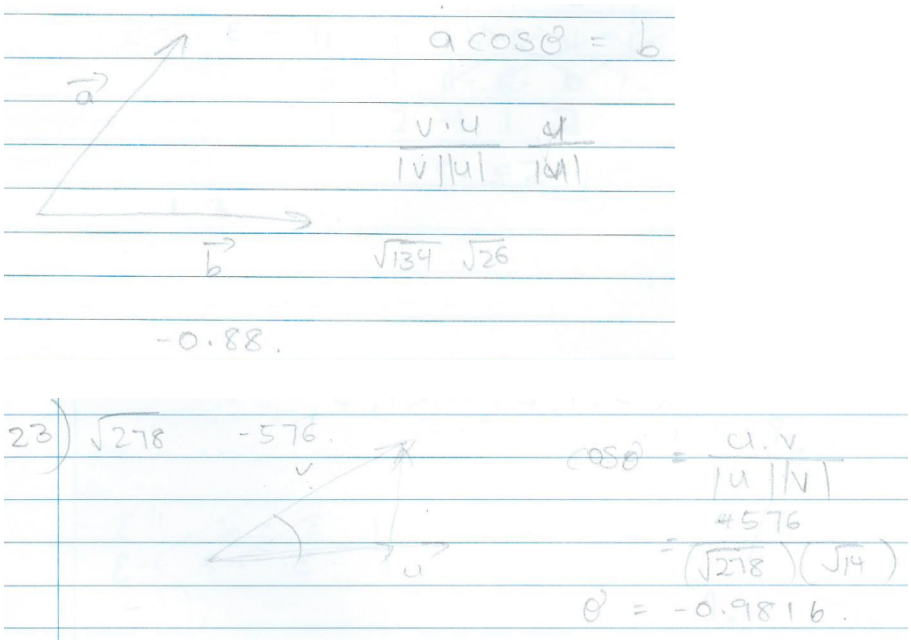
Student 20152100



Student 20152045

Figure 8. Problematic use of the scalar product

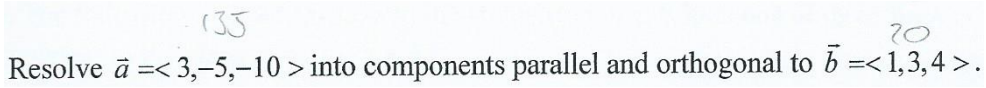
Student 20152046 (Figure 9) was one of the few students for whom hand-written working was found for both of the items 6 and 23. S/he has used the scalar product correctly, but in each case has used it to calculate an angle, not the vector required by the problem. It is as if the student recognises the problem format as requiring the scalar product, but then uses the scalar product in the only form with which s/he is comfortable, that of solving for an angle.



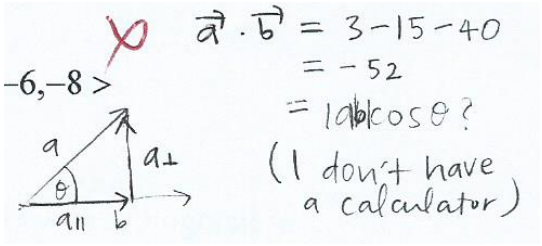
Student 20152046

Figure 9. Problematic use of the scalar product, continued

The final (and sadly commonplace) source of error is an inability to do error free mental arithmetic. In Figure 10 this finding is illustrated with one case of a student calculating moduli incorrectly (written over the vectors given) and another giving up since s/he did not have a calculator handy (for a calculation which did not strictly speaking demand one).



Student 20152016



Student 20152037

Figure 10. Inability to do arithmetic

Conclusion

We ran an assessment based on first-year mathematics vector mechanics. The students being assessed were students registered for the second-year engineering course in dynamics. We analysed the data using the Rasch measurement method. Analysis of the results drew our attention to a number of interesting phenomena, one of which was the surprising difficulty of several items involving the use of the scalar (dot) product. Of those six challenging items, two were identical in the process demanded, which was the finding of the component of a vector in the direction of a second vector, or alternatively, finding the projection of a vector onto a second vector.

Recognising the importance of this process in engineering contexts, such as dynamics, we looked more closely at the data available and discerned several facts about these two items.

The item encountered first (6) was analysed as being easier than the second (23). One possibility is that Item 23 was avoided (and the answer was either omitted or guessed) as a result of the difficulty of Item 6 and hence Item 23's inherent difficulty was skewed, a hypothesis backed up by less rough working being found for the later item. A second possibility resides in the fact that the notation used in Item 23 (the unit vector notation – and the preferred notation in the dynamics course) was less familiar to the students than the parentheses notation in Item 6, backed up by a similar pattern being observed throughout the test on other item pairs. Future uses of versions of this assessment instrument will swap the order and we shall observe any consequences of that change. Our ongoing research will investigate the apparent lack of familiarity with the

The errors we observed in student rough work included several which suggest that the geometric interpretation of the scalar product is poorly grasped, for example using the vector

product instead of the scalar product, drawing 3-dimensional diagrams when none are needed and using the scalar product to determine an angle when this result was not required.

As a result of the concerted efforts of the first-year mathematics lecturers, students in second-year engineering courses generally show great algebraic proficiency. However, when students encounter situations where physical problems are required to be represented using vectors they often encounter unexpected difficulties. Furthermore, in certain courses, such as second-year dynamics, students generally do not struggle to follow vector based solutions presented in class, but subsequently struggle to solve similar problem by themselves. It is a common refrain that students prefer to follow an algebraic approach because they struggle to 'see' the geometric interpretation.

We argue that the geometric role of the scalar product is understood weakly, if at all, by the majority of the students. This weak understanding is in spite of the geometric role of the scalar product being taught and demonstrated in the first-year mathematics course. The item map (Figure 1) suggests that the computational challenge of the vector product is greater than that of the scalar product, while using the vector product in a context is less challenging than using the scalar product in a context. Perhaps the very simplicity of the final form scalar product used for calculations results in an under appreciation of the geometric significance, whereas the greater complexity of the vector product provides a cognitive spur for engaging with the vector product's geometric interpretation.

We suggest to lecturers of first-year engineering and science mathematics that they strongly emphasise the geometric role of the scalar product, including exercises whose solution require geometric interpretation, and do not allow the simplicity of the arithmetic to upstage it. We suggest to lecturers of engineering or science courses which utilise the scalar product to be aware of the challenge which scalar product contexts might embody to students.

Acknowledgements

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Appendix A Scalar product items in increasing order of difficulty

The items are ordered here from easiest to most difficult, as determined through the analysis.

The correct answers are underlined.

1. Evaluate the dot product $< 5,1,1 > \cdot < 3,7,-1 >$.

(A) 32 (B) 16 (C) 63 <u>(D) 21</u> (E) 50

14. Evaluate the dot product of $8\hat{i} - \hat{j} + 4\hat{k}$ and $3\hat{i} + 6\hat{j} - 2\hat{k}$.

<u>(A) 10</u> (B) 24 (C) 11 (D) 25 (E) 38

21. Find the distance from the point $P(3,1,2)$ to the plane $x + 3y - 2z = 4$.

(A) $\frac{1}{7}$ (B) $\sqrt{14}$ (C) $\frac{1}{7}\sqrt{14}$ (D) $\sqrt{7}$ (E) $\frac{1}{7}\sqrt{2}$

7. Determine the volume of the parallelepiped defined by the three non-coplanar vectors

$\vec{m} =< 2,1,1 >$, $\vec{n} =< 4,1,2 >$, $\vec{p} =< -1,3,1 >$.

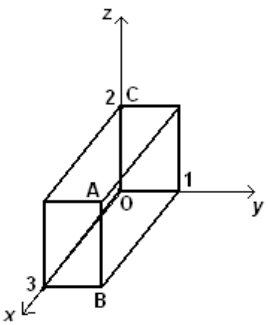
(A) -3 <u>(B) 3</u> (C) 8 (D) 2 (E) 5

6. Resolve $\vec{a} =< 3,-5,-10 >$ into components parallel and orthogonal to

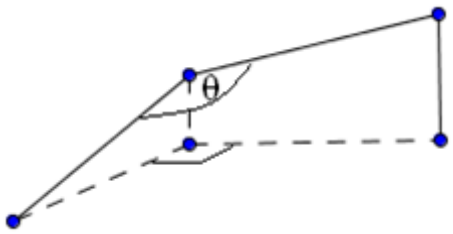
$\vec{b} =< 1,3,4 >$. The parallel component is

(A) $< \frac{1}{2}, \frac{3}{2}, 2 >$ (B) $< 3, 9, 12 >$ <u>(C) $< -2,-6,-8 >$</u>

(D) $< -1,-3,-4 >$ (E) $< 1, 3, 4 >$



13. Two water pipes are connected as shown in the diagram. The first pipe runs from south to north and rises up with a 20% grade. The second pipe runs from west to east and rises with a 10% grade. At the connection, determine the angle θ between the two pipes.



(A) 92.1° (B) 91.1° (C) 90° (D) 88.9° (E) 87.5°

22. What is the acute angle between the diagonals linking the corners A, B, C and O of the rectangular prism shown? (O is the origin.)

(A) 45° (B) 90° (C) 65° (D) 35° (E) 15°

23. Resolve $\vec{v} = 2\hat{i} - 7\hat{j} + 15\hat{k}$ into components parallel and orthogonal to $\vec{u} = 2\hat{i} - \hat{j} + 3\hat{k}$.

The parallel component is:

(A) $8\hat{i} - 4\hat{j} + 12\hat{k}$ (B) $2\hat{i} - \hat{j} + 3\hat{k}$ (C) $-2\hat{i} + 1\hat{j} - 3\hat{k}$

(D) $4\hat{i} - 2\hat{j} + 6\hat{k}$ (E) $6\hat{i} - 3\hat{j} + 9\hat{k}$

Appendix B Solutions to Items 6 and 23

6. Resolve $\vec{a} =< 3,-5,-10 >$ into components parallel and orthogonal to $\vec{b} =< 1,3,4 >$.

The parallel component is

(A) $< \frac{1}{2}, \frac{3}{2}, 2 >$ (B) $< 3, 9, 12 >$ (C) $< -2,-6,-8 >$

(D) $< -1,-3,-4 >$ (E) $< 1, 3, 4 >$

Let $\lambda\vec{b}$ be the parallel component and let $\vec{c}$ be the perpendicular component.

$$a = \lambda\vec{b} + \vec{c}$$
$$\vec{a}\cdot\vec{b} = \lambda\vec{b}\cdot\vec{b} = \lambda\left|\vec{b}\right|^2$$

$$\lambda\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\left|\vec{b}\right|^2}\vec{b}$$
$$= \frac{3-15-40}{1+9+16} < 1, 3, 4 >$$
$$= -2 < 1, 3, 4 >$$

23. Resolve $\vec{v} = 2\hat{i} - 7\hat{j} + 15\hat{k}$ into components parallel and orthogonal to $\vec{u} = 2\hat{i} - \hat{j} + 3\hat{k}$. The parallel component is:

(A) $8\hat{i} - 4\hat{j} + 12\hat{k}$ (B) $2\hat{i} - \hat{j} + 3\hat{k}$ (C) $-2\hat{i} + 1\hat{j} - 3\hat{k}$

(D) $4\hat{i} - 2\hat{j} + 6\hat{k}$ (E) $6\hat{i} - 3\hat{j} + 9\hat{k}$

Let $\lambda\vec{u}$ be the parallel component and let $\vec{c}$ be the perpendicular component.

$$\vec{v} = \lambda\vec{u} + \vec{c}$$
$$\vec{v} \cdot \vec{u} = \lambda\vec{u} \cdot \vec{u} = \lambda|\vec{u}|^2$$

$$\lambda\vec{u} = \frac{\vec{v} \cdot \vec{u}}{|\vec{u}|^2}\vec{u}$$
$$= \frac{56}{14}\vec{u}$$
$$= 8\hat{i} - 4\hat{j} + 12\hat{k}$$

Looking into the Private World of the Private Tuition Industry

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Abstract

Education is a top priority for many parents, initially as their child approaches school age and again as high stake examinations are close on the horizon. As the child progresses through the educational system, parents can supplement formal education in different ways, such as requesting extra homework from the current education institution; purchasing publications that target examination style questions; internet search for learning resources or seek private tuition. Private tuition (also known as coaching, shadow education, or parallel education) may be conducted in person as one-to-one, in small groups, or in larger groups in a classroom style setting. There are online options available such as self-guided learning programs and real-time interaction with tutors. As the uptake of employing private tutors is increasing globally, the industry has become one of the fastest-growing sectors of the education field. It is surprising that in many countries, there is very little regulation and monitoring of this industry given the potential income that can be generated. This scoping paper will use qualitative data to highlight the drivers behind the demand for private tuition and explore regulations and government interventions in the private tuition industry. This paper will define private tuition as any academic assistance rendered outside the education institution for a monetary fee.

Keywords: private tutor; mathematics tutoring; statistics tutoring; regulation of industry

Introduction

Education is often a top priority for parents initially, as their child approaches school age and again, as high stake examinations are close on the horizon. The learning of numbers and letters with the help of parents and siblings may happen in the informal

setting of the home before attending kindergarten. As the child progresses through the education system, parents can supplement formal education in different ways, such as requesting extra homework from the current education institution; purchasing publications that target examination style questions; internet search for learning resources or seek private tuition. Private tuition may be conducted in person as one-to-one, in small groups, or in larger groups in a classroom style setting. There are online options available such as self-guided learning programs and real-time interaction with tutors. As the uptake of employing private tutors is increasing globally, the industry has become one of the fastest-growing sectors of the education field. It is surprising that in many countries, there is very little regulation and monitoring of this industry given the potential income involved. There are many hurdles obstructing academic research into private tuition: there is not one universal name for the practice; there are different motives and expectations for seeking private tuition; the tutoring assistance can be organised through an education institution, private tutoring agency or a sole tutor. For the latter two, unlike formal education, there is no obligation to keep records of private tuition for educational purposes only for accounting purposes and the taxman. This paper will highlight the drivers behind the demand for private tuition and explore regulations and government interventions in the private tuition industry.

Background

The practice of hiring private tutors is not a new concept. Throughout history there have been many famous people including mathematicians who have tutored students. It is interesting to note that problems facing parents about the choice and price of tutors today have been present since 380 BC. The choice of Aristotle as tutor for Alexander the Great according to Merlan [1] was not solely based on qualifications there was an element of 'a friend of a friend' (although historically there may have been ulterior

motives):

> When Philip invited Aristotle to become the tutor of Alexander, primarily he was probably motivated not by Aristotle's fame as a philosopher or even by his membership in the Academy (although the latter was obviously also a center of political opinion). As Philip invited Alexander before the death of Hermias of Atarneus, he, as Jaeger proved, in all likelihood invited him as a confidential friend of Hermias, whose city-state was to serve as a bridgehead in the invasion of Persia planned by Philip.[1]

The cost of tuition is a major obstacle for many families. There may be rewards to gain by paying higher rates for tuition in order to procure the service of the better tutors. It appears that De l'Hôpital had this train of thought when he asked Bernoulli to tutor him:

> De l'Hôpital was delighted to discover that Johann Bernoulli understood the new calculus methods that Leibniz had just published and he asked Johann to teach him these methods. This Johann agreed to do and the lessons were taught both in Paris and also at de l'Hôpital's country house at Oucques. Bernoulli received generous payment from de l'Hôpital for these lessons, and indeed they were worth a lot for few other people would have been able to have given them. After Bernoulli returned to Basel he still continued his calculus lessons by correspondence, and this did not come cheap for de l'Hôpital who paid Bernoulli half a professor's salary for the instruction. However it did assure de l'Hôpital of a place in the history of mathematics since he published the first calculus book Analyse des infiniment petits pour l'intelligence des lignes courbes (1696) which was based on the lessons that Johann Bernoulli sent to him.[2]

Private tuition has morphed from humble beginnings as a little extra help from a neighbour or friend into a lucrative worldwide industry, Crotty commented:

> Market research firm, Global Industry Analysts, Inc. (G.I.A.) has released a study this past week stating that the global private tutoring market is projected to surpass $102.8 billion by 2018. According to GIA, the burgeoning private tutoring market is being driven by the failure of standard education systems to cater to the unique

> needs of students, combined with growing parental desire to secure the best possible education for their children in a highly competitive global economy.[3]

In Australia, The Australian noted a similar pattern in 2011:

> The tutoring industry has grown by almost 40 per cent in the past five years, adding around $6 billion to the economy. Nationally 36,100 people now say tutoring is their primary job, earning an average $1400 each a week. But the real number could be as high as 80,000 because the data does not include teachers moonlighting as tutors, or those who are in the tertiary sector. Some education critics claim the growth of tutoring is an indictment on the performance of schools, but the industry says the main reason is aspirational parents seeking extra advantage for their kids. Chief executive of the Australian Tutoring Association Mohan Dhall said families were spending billions each year on private coaching. "The annual spend on tutor wages alone is over $2.6 billion so the total annual value would be in the order of $6 billion," he said. "The Commonwealth Government's Jobs Outlook data indicates this market has grown by over 38 per cent in the past five years. [4]

It is, therefore, quite surprising that there is very little regulation of this industry, so anyone of any age, with limited to no teaching experience, can offer private tuition by utilising innovative marketing techniques. In Hong Kong, for instance, the status of some tutors [5] have been elevated to rock star level with television appearances, dedicated fan base and their images on billboards, so is it any wonder that high-achieving students feel tutoring would be an easy way to supplement their income?

> If you want to be a top tutor, it definitely helps if you are young and attractive. Students look at your appearance," said Kelly Mok, 26, a "tutor queen" at King's Glory, one of Hong Kong's largest tutorial establishments. Her designer clothes and accessories are not just for the billboards; it's how she likes to dress outside classes. But she is also careful to add that she wouldn't be in such high demand if she could not deliver top grades in her subject, English. [5]

Different descriptors and practices of private tuition

Private tuition has many synonyms leading to confusion when collecting and analysing data and comparing studies. Previous literature has described private tuition with terms such as shadow education, parallel education, coaching, private lessons, and supplementary education. The difficulty lies not only with the many labels given to private tuition, but that some descriptors of private tuition share certain qualities but relate to a variety of tuition practices. Although not the main theme of this paper, a brief overview of the two main labels given to private tuition, shadow and supplementary education and the range of practices involved. These descriptions will illustrate further, the complexity of this education research area.

Shadow Education

Shadow education is known as; parallel education in Greece; private lessons, after school support and coaching in France; and Juku and Kwa-woi in Japan and Korea respectively. Bray [6] describes shadow education as follows:

> First, private supplementary tutoring only exists because the mainstream education system exists; second, as the size and shape of the mainstream system change, so do the size and shape of supplementary tutoring; third, in almost all societies much more attention focuses on the mainstream than on its shadow; and fourth, the features of the shadow system are much less distinct than those of the mainstream system. [6]

Under the umbrella of shadow education, the tuition may be free or fee-paying and involve one-to-one tuition, small to large group size, or online programs following the curriculum. Extensive research by Bray [7] has noted the challenge created by the use of multiple names for private tuition.

Supplementary Education

Supplementary education, the term mainly used in the United States of America, appears similar to shadow education. The difference lies that supplementary education includes material beyond the curriculum whereas shadow education keeps to the school curriculum. Bridglall [8] describes:

> We define supplementary education as the formal and informal learning and development enrichment opportunities that are provided for students outside of school and beyond the regular school day or year. Some of these activities may occur inside the school building but are over and above those included in the formal curriculum of the school. [8]

Supplementary, as the name implies, enhances knowledge learnt in the formal education by the introduction of material outside the curriculum to challenge students. This includes activities such as learning a language not offered at school or help for pupils where English is a second language, singing lessons or playing musical instruments and enrichment classes usually for the more academic students. The added element of family and other social support systems is an extra element not included in the definition of shadow education.

Whatever term is used for the private tuition, this does not immediately indicate the purpose or delivery for tuition. For instance, in Japan, there are different types of Juku specialising from remedial, preparatory for tests, accelerated learning for the more academic student and drills.[9] The mode of delivery may be face to face, through televised live or recorded classes, online via Skype, or via structured online programs and so the class size can vary from one student to potentially thousands of students. Technology may remove the barriers of distance, location, and time, making private tuition more accessible and flexible.

Drivers behind the demand for private tuition

There is not one definitive list of drivers behind the growing demand for private tuition as each country has a different education system with assessment procedures allied with assorted economic and cultural factors. The intrinsic drivers are closely related to the family circumstances, such as household income, student's situation, and school environment.

Diskin [10] noted that:

> The juncture of trends, that of increased focus on attending school, parents' work commitments, school demands for parent involvement, and more transparency for individual school's testing results have all contributed to the sense of importance attached to what a school education can do for the child's eventual adult life.[10]

The household income may determine the amount of tuition, though it does not deter the pursuit of private tuition for those in a lower income bracket. Education, for many people, is seen as an investment for future employment in the same manner as banking where every small amount adds to the overall financial pot. Private tuition provides an extra boost to future career paths and financial security.

Depending on the student's situation, tuition will vary in nature. Absence from school due to illness, unable to understand the material in class and homework assistance are some of the main reasons put forward for seeking private tuition by parents and students. Lewis [11], also, noticed several motives behind seeking tuition, amongst them remedial and accelerated tuition. Remedial tuition is where the material introduced in the classroom during that week is studied and explained in a manner tailored to suit the student's needs. This type of tuition lasts for several terms or even the whole school year. Parents are usually concerned about the grounding work of mathematical foundations and will pay for a tutor so that their child will catch up or

keep up with class work. Studies such as Trends in International Mathematics and Science Study (also called TIMSS), the Southern and Eastern Africa Consortium for Monitoring Educational Quality (SACMEQ) and Programme for International Student Assessment (PISA) of the Organisation for Economic Co-operation and Development (OECD) have sought information on extra tuition received by students. Bray [7] noted the difficulty in ascertaining information as stated before with the cultural differences, when asked about private tuition there was no clear indication about reasons, location, payment, and mode of delivery.

Just as acceleration in a car is to progress to somewhere sooner, so accelerated tuition aims either to prepare the students for material yet to be taught in school or to enhance the material learnt beyond the formal curriculum. This is not the solitary domain for the advanced bright student. Parents can be keen for children to be challenged and often feel that being ahead of the curriculum is an advantage. Every parent boasts about their child's achievements, both physical and mental, though now it appears that kindergarten has lost the realm of play for the role of grade. The Boston Globe [12] quoted from a report by on the Boston-based non-profit Defending the Early Years organisation:

> What does earlier reading in kindergarten predict for reading proficiency and academic success in later grades? Not much, according to the report, which cites study findings that by fourth grade, children who were reading at age 4 were not significantly better at reading than their classmates who'd learned to read at age 7. The report also points out that in Finland and Sweden, kids don't even start formal schooling until they are 7 years old. Yet, Finnish and Swedish teenagers regularly trounce their American counterparts in international tests of reading, math, and science.[12]

Competition to obtain places at schools where parents feel children will receive a better education is fierce. Private tuition may be started before primary education, so

the first rung of the education ladder is reached though further tuition may be required to continue upwards. It is confusing that private tuition is believed to be needed to gain access to perceived 'better teaching' and further tuition is still needed to secure future education places. A National Assessment Program-Literacy and Numeracy (NAPLAN) was introduced in Australia in 2008 which was administered in Years 3, 5, 7 and 9 in all schools. The results would show students' position and ability in relation to national benchmarks. The original purpose of NAPLAN was to give a snapshot of students' abilities, identifying weakness and strengths, though now it may be a reason why private tuition is sought. These results are published so parents are more aware of where students are against benchmark and how schools are performing against each other. High-stakes examinations are exactly what the name indicates—parents and students realise the pressure and importance of the years leading up to and including these examinations. Hence, private tuition is seen as essential to securing a tertiary education place.[13] The demand for tuition is greater in years of high-stakes examinations.[14]

Private tuition is not limited to thriving economies and occurs in countries as diverse as Bangladesh, Cambodia, France, Canada, and the United Kingdom. All these countries have different education systems, both culturally and pedagogically, yet private tuition flourishes and ineffective teaching is one reason cited.[15] It is unclear whether the reason of ineffective teaching quoted by parents and students relates to lack of academic results or misunderstanding of material.

Extrinsic factors that affect the supply and demand of private tuition are market economy, labour market, public education system and cultural issues. Dang and Rogers [14] noted the following trends:

- Countries transitioning to a market economy like China, Vietnam and other nations have seen a rise in the rate of private tuition.

- Qualifications needed to enter the job market appear to focus more on tertiary rather than secondary marks. Preliminary screening processes use university degrees as a tool for choosing potential employees hence pressure to obtain tertiary places increases. Private tuition is deemed required to achieve this education level.
- A weak education system will make parents look for extra tuition to gain a competitive edge for the all-important tertiary education places.

The last point seems to suggest that private tuition should not be popular in countries with a strong public education system. There are many issues and motives for seeking extra help in the form of private tuition and so many choices. Commercial tutoring agencies regularly advertise fantastic results gained by attending their tutorial sessions, though little or any statistical evidence is shown to substantiate these claims. The old adage of 'how much is just too much?' can so easily be applied to tutoring.[16]

Government interventions concerning private tuition

The history of private tuition and efforts to regulate this industry depends on the country, so this section will discuss some of the actions taken by governments or local communities. The focus is concerned with the role of the government rather than the mode or reason for tuition, though a brief description will be given when required.

Governments mirror the practices of private tuition establishments

Governments may feel 'if you cannot beat them, join them' attitude may be the avenue to explore in the regulation of this industry. The introduction of government subsidised programmes, similar to the existing available private tuition, may ease the financial burden appealing to all socio-economic groups.

In France, three types of tuition are available, private lessons, coaching and after-school support. The after-school support is provided by non-profit associations and concentrates mainly on homework and is funded by either local or national grants or assistance with free use of facilities. This type of support is usually situated in working class areas with a small or no fee and can be one-to-one through to large groups organised by volunteers with the help of teachers. There may be different grade levels within the groups. Private lessons and coaching are associated with one-to-one and small groups and a substantial fee is charged. The main focus is examination preparation, though remedial tuition is offered on a short term basis.

The government introduced programmes such as 'educational tutoring' and 'tailor-made support'. Both were supposed to create an environment to assist with homework though the latter became more involved with students who had learning difficulties. Both had free lessons and a fee paying (much cheaper than that for private lessons) programme to take place during the summer holidays. Retired teachers, current teachers and students would deliver these lessons either during or outside school hours. A reduction in the workforce of all teachers, including some specifically employed in this area of tuition, sadly has not helped this initiative to grow.

In Cyprus, the Ministry of Education and Culture (MoEC) regulates both private and state owned 'frontistiria' (the name for private tuition establishments) as well as private and public schools. These establishments need to comply with certain regulations and are inspected by MoEC. It should be noted that the education system in Cyprus is controlled by the state so it appears somewhat of a paradox that the state creates a state-owned frontistiria. There are many illegal frontistiria operating in Cyprus which are not registered and operate as small frontistiria under a cloak of relative secrecy. The government has unsuccessfully tried to close these establishments. [17]

There are several strands of frontistiria from development of academic skills and knowledge; high-stakes examination preparation; overseas high-stakes examinations and professional bodies through to languages, arts and sports.[18]

Family financial incentives from governments

Voucher schemes have been utilised in Australia, South Africa, United States and England for students who do not reach some particular educational benchmark.[19] This appears to be a good way of helping students who are falling behind in literacy and numeracy. The implementation of this scheme in Australia had some administrative problems due to legislation reasons, locating eligible students, and privacy issues surrounding the identification of students to non-education departments which led to a delay in families being contacted and ultimately a lower than expected uptake. Watson [20] found similar problems in other countries:

> Although private tutoring programs have been introduced by governments in several countries, there is scant research demonstrating their effectiveness. Studies of the implementation of the Supplemental Education Services program in various states of the USA have concluded that the program is compromised by inadequate funding (ie. insufficient funds for eligible students), low take-up rates (less than 15-20 %) among eligible students, high attrition rates, and lack of accountability for outcomes by the private agencies providing the services (Burch 2007, Rickles and Barhart 2007, Sunderman 2007). In Australia, the pilot Tutorial Voucher Initiative experienced similar limitations, such as variable take-up rates and lack of evidence about sustainable gains in student achievement.[20]

In France, the government provided state aid for private tuition for tax-exempt families and tax relief for others on private tuition thus reducing the costs considerably for families.[18]

Current regulations concerning private tuition

This is the most confusing part of private tuition as there are so many discrepancies. Many countries have at some stage attempted to ban or reduce private tuition- South Korea in 1980; Mauritius in 1991; Cambodia in 1994; and Hong Kong in 1997 [15]. Total bans have been hard to enforce as it would take many resources to patrol all possible venues where private tuition may take place. The ban was lifted in South Korea and the private tuition industry is flourishing and has become part of the culture. Many governments will espouse the evil of private tuition, though often are indirectly aiding by providing venues for free. In many developing countries where teachers' salaries are low and added income is generated by private tuition, teachers are accused of deliberately not covering the curriculum fully in regular classes. This accusation could be directed at any teacher that works in both mainstream and private tuition regardless of pay-rate.

In Australia, there is little regulation of the industry or any monitoring of qualifications. Anyone who works with children in any capacity i.e. basketball coach, piano teacher requires a Working with Children Check and a background check by the police. It is unclear how many parents take this into consideration when seeking private tuition or ask to see these credentials or profession qualifications of private tutors. In 2005, the Australian Tutoring Association (ATA) was founded as a representative body for the private tuition industry. It has no actual powers, membership is voluntary for an annual fee and the ATA provides extra help with resources and advertising. The code of conduct that it promotes is no more than any tutor would abide by to gain more clients. The association is hoping to raise the profile of the industry and gain government support.

Conclusion

Education systems vary throughout the world just as the private tuition industry operates on different levels. The author has found this a fascinating journey into relatively unchartered waters of the private tuition. This paper has only touched on a few aspects of the private tuition industry, but has opened more doors of potential research. Most research in this paper dealt with primary and secondary education, what is the situation of private tuition at tertiary level? Are educational qualifications essential to be a good tutor? What regulations, if any, should be applied to private tuition and would they vary between primary, secondary and tertiary education levels.

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Engineering students' use of intuition and visualization in the mathematical problem solving

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Although deciding on the truth value of mathematical statements is an important part of the proving process, students are rarely engaged in making such decisions. Thus, little is known about the ways in which students' use mathematical reasoning to evaluate mathematical statements. In this study, task-based interviews were conducted with engineering students in which they were asked to determine the truth value of mathematical statements. Students' reasoning on the tasks will be classified and then further categorized according to the findings of current research, with new categories added as needed. This study should contribute to our understanding of the ways in which students' reason when dealing with uncertainty in the problem solving process. The findings suggest that the factors the participant failed to solve problems include: mathematical intuition hindered the constructing of counterexamples, pre-visual and post-visual errors mislead students' visualization. Additionally, this study may suggest ways in which educators can assist students in navigating the often difficult process of refuting mathematical statements.

Keywords: intuition; mathematical problem solving; visualization

Introduction

Determining the truth value of mathematical statements is an important component of the problem solving process. When dealing with uncertainty, mathematicians often try to decide on a statement's truth value with some degree of confidence before investing time in a proof or refutation attempt. [1, 2] The proving process is complex and encompasses a multitude of reasoning activities including intuitive, informal, and formal reasoning. *Formal* reasoning is based on logic and deduction, and *informal*

reasoning includes reasoning strategies such as visual, example-based, or pattern-based reasoning.

This study hereby attempts to explore the use of intuitive and visual reasoning in advanced mathematical problem solving by engineering students in an interview setting. Through an analysis of engineering students' problem-solving protocols and responses, I examined the relationship between intuition and visualization in justifying the truth value of a mathematical statement. This study explores (a) the ways intuition and visualization interact in the decision-making process, and (b) the ways this decision-making process influences students' constructions of associated counterexamples for the statements.

Intuitive reasoning

Intuition is especially important for deciding on the truth value of a mathematical statement because it can suggest what is plausible in the absence of a proof [3, 4] and "provides a justification for, but is prior to, the search for convincing argument and, ultimately, proof". [5, p.32] In the limited research on intuition in mathematics education, researchers have found a variety of types of intuitive reasoning used by students and mathematicians to evaluate mathematical conjectures. Inglis et al. [2] found that mathematicians' intuitive support for the truth or falsity of a mathematical statement was based on either suspected properties about mathematical objects or known relationships between mathematical concepts. Intuition constructs an automatic mental representation of a task, taking into consideration task cues, prior knowledge, and experience, and operates independently of working memory. [6, 7, 8]

This study is based on the theoretical framework on intuitions exposed by Fischbein. [7, 9] In this work, an intuition is "a representation, an explanation or an interpretation directly accepted by us as something natural, self-evident, intrinsically

meaningful, like a simple, given fact″ .[9,p.10] Fischbein [7] offered two approaches for classifying intuitions, one based on roles or origins. In this classification system, intuitions can be affirmatory, conjectural, anticipatory, or conclusive. In the case of an affirmatory intuition, one affirms or makes a claim. A conjectural intuition is one in which an assumption about future events is expressed. Anticipatory and conclusive intuitions represent phases in the process of solving a problem. Anticipatory intuitions express a preliminary, global view that precedes an analytical solution to a problem. Conclusive intuitions summarize in a global, structured vision the solution to a problem that had previously been elaborated. Anticipatory intuitions are the cognition that implicitly emerges during an attempt at problem solving, immediately after a serious search for a problem-solving strategy. Anticipatory intuitions are holistic and associated with the feeling of conviction derived from comprehensive reasoning or proving.

Visual reasoning

Increasing attention has being paid to the centrality of visualization in learning and doing mathematics, not just for illustrative purposes but also as a key component of reasoning. [10] When considering the role of visual images in structuring intuitions, ‘ it is worth keeping in mind that visual representations are *not* by themselves intuitive knowledge’.[7,p.103] Visualization is a critical aspect of mathematical thinking, understanding, and reasoning. Researchers argue that visual thinking is an alternative and powerful resource for students to do mathematics, [11,12] it is different from linguistic, logic-propositional thinking and manipulation of symbols. According to Duval, [13] visualization can be produced in any register of representation as it refers to processes linked to the visual perception and then to vision. Zimmerman and Cunningham [14] contended that the use of the term “visualization” concerned a

concept or problem involving visualizing. Nemirovsky and Noble [15] defined visualization as a tool that penetrated or travelled back and forth between external representations and learners' mental perceptions. Dreyfus contended that what students "see" in a representation would be linked to their conceptual structure, and further proposed that visualization should be regarded as a learning tool.[16]

Visualization involves both external and internal representations, and thus following Zazkis et al. [17, p.441], that is, as an "act in which an individual establishes a strong connection between an internal construct and something to which access is gained through the senses. Such a connection can be made in either of two directions. An act of visualization may consist of any mental construction of objects or processes that an individual associates with objects or events perceived by her or him as external. Alternatively, an act of visualization may consist of the construction, on some external medium such as paper of objects or events that the individual identifies with objects or processes in her or his mind". This definition emphasizes that the act of visualization is a *translation* from external to mental (or vice versa) and, particularly, the connection made by the individual between the image and the mental. Therefore, visualization has a powerful role in promoting understanding both as a support and illustration of symbolic results and as a means to solve conflicts between wrong intuitions and right solutions. Visualization helps to grasp the hidden meaning of formal definitions.

Methods and general procedures

The participants in this study were 21 first-year engineering students at a university of technology in Taiwan, who had previously completed courses in derivatives and definite integrals. They were selected for convenience: participants were contacted by colleagues of the researcher and were recruited on the basis of their willingness to participate in the study. The questionnaire contained three mathematical statements, and

was designed to assess the students' abilities to generate counterexamples related to basic differentiation and integration concepts.

Statement 1: If $\int_a^b f(x)dx \geq \int_a^b g(x)dx$, then$f(x) \geq g(x), \forall x \in [a,b]$.

Statement 2: If f(x) and $g(x)$ are both differentiable and $(x) > g(x), \forall x \in (a,b)$, then $f'x) > g'(x), \forall x \in (a,b)$.

Statement 3: If f(x) and $g(x)$ are all differentiable and f$'(x) > g'(x), \forall x \in (a,b)$, then $f(x) > g(x), \forall x \in [a,b], \forall x \in (a,b)$.

The students were asked to determine the accuracy of the mathematical statements and justify their answers. Data were gathered concerning the mathematical reasoning that were used by the participants.

The data generated from (a) transcripts from the participants' task-based interviews using the think-aloud method, and (b) participants' written work on the tasks in the interviews were analysed using the grounded theory approach.[18] The procedure of data analysis involved open, axial, and selective coding processes for qualitative data, following Strauss and Corbin, to produce descriptive categories [19]. The data were analysed according to uses of intuitive and visual reasoning during the participants' processes of deciding whether a mathematical statement was true or false and constructing counterexamples. Additionally, students' decision-making and construction processes were analysed to determine students' intuitive and visual reasoning. I will classify reasoning as intuitive if the student (a) stated that it was a(n) intuition, instinct, gut feeling, or first thought; (b) used similarity to make an assessment of the task; or (c) was unable to justify the reasoning. Reasoning will be classified as

visual if the student (a) introduce diagrams; (b) exhibited an ability to explicitly think of pictures or diagrams rather than algebraic representations.

Findings

Statement 1

Four students proposed the converse of the statement 1 and they did not need to give any argumentation for this decision. The following episode is one example of many similar protocols:

> S14: If $f(x)$ is greater than $g(x)$, then the integral of $f(x)$ is greater than the integral of $g(x)$.
> Interviewer: How are you going to prove it?
> S14: No. It has been proved in the textbook.
> S14 is convinced, it is so evident for her/him that s/he does not need to propose any further argumentation.

Five participants asserted that statement 1 was correct and generated examples for verification; however, most students generated converse statement examples: Two students, including S4, made logical errors in generating supportive examples.

> S4: The greater the function is, the larger the integral is. For example, $f(x) = x^2$+1 and $g(x) = x^2$, $f(x)$ is larger than $g(x)$. The integral of f from 0 to 1 is 4/3, which is also greater than the integral of g, 1/3, over the same interval.

Six students used graphical representations to generate examples to support their assertions. Although they connected integrals to areas, they did not understand the true relationship between the two. A typical response is as follows.

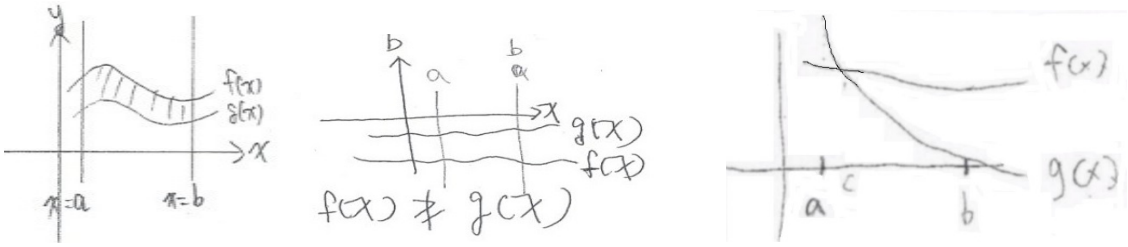
> S6: The integral represents the area under the curve. A greater integral corresponds to a greater area [Fig. 1(a)]. Here, the greater integral is represented by the higher graph, so the function is greater.

Five students asserted that the statement was false, but provided a counterexample that failed to refute the statement; they also did not understand the relation between integrals and areas.

> S9: Integral is area, so a larger integral means a greater area. The area bounded by $f(x)$, x = a, x = b and the x-axis is larger than that of $g(x)$ [Fig. 1(b)]. However, $f(x)$ is smaller than $g(x)$.
> Interviewer: The integral value may be negative, but area is positive.
> S9: (10 seconds of silence) If the area above the x-axis, then the integral value equals to area. If the area below the x-axis, then the absolute value of integral vale equals to area. (10 seconds of silence) I think I make a mistake, the statement is correct.

Only one students correctly determined the truth value of statement 1. S/he took a trial-and-error strategy to generate examples to evaluate this statement, until he constructed a counterexample:

S13: In the graph [Fig. 1 (c)] that I drew, the area that is bounded by $f(x)$, x= a, x= b and the x-axis is larger than that bounded by $g(x)$, x = a, x = b and the x-axis; therefore, the integral of f in [a, b] is greater than the integral of g in [a, b]. However, the function value of f in the interval [a, c] is smaller than the function value of g in the interval [a, c]. Therefore, the statement is incorrect.



(a) (b) (c)

Figure 1. Supporting examples and counterexamples to statement 1

Statement 2

Eight students decided that the statement was true, they did not define the range of variable and so failed to generate supporting examples. A typical response is from S5.

S5: If $f(x) = x^2+1$ and $g(x) = x$, $f(x)$ is greater than $g(x)$, and $f'(x) = 2x$, $g'(x) = 1$, and $f'(x)$ is greater than $g'(x)$.

Interviewer: why $f'(x)$ is not always greater than $g'(x)$?

S5: It is true obviously, for example, if x equals to 1, then 2 is greater than 1.

Six students including S7 provided similar examples but to refute this statement.

S7: $f(x) = x^2$ and $g(x) = x$, $f(x)$ is greater than $g(x)$, $f'(x) = 2x$, $g'(x) = 1$, when x is less than 1/2, then $g'(x)$ is greater than $f'(x)$.

Interviewer: When x is less than 1/2, is $f(x)$ always greater than $g(x)$?

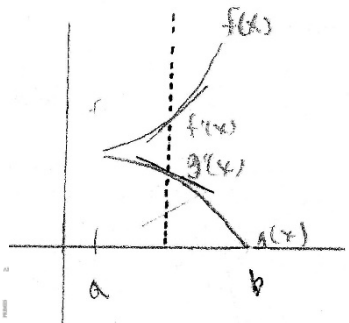
S7: No. $f(x)$ is not greater than $g(x)$, x must be less than 1/2, and x^2 must be greater than x. Oh! I see, for this example, x must be less than zero.

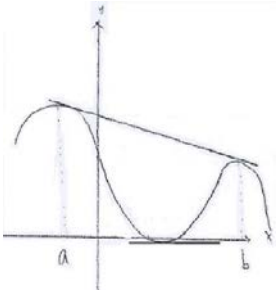
These students focused on algebraic manipulation, but did not appreciate the importance of the domain. Five students used graphical representations and slopes of tangents to argue that the mathematical statement was true. The following response is just one example of many similar protocols (Fig. 2(a)):

S8: Since $f(x)$ is greater than g(x), so the graph of $f(x)$ is higher than $g(x)$, and $f'(x)$ must be larger than $g'(x)$.So…, if $f(x)$ is concave up and $g(x)$ is concave down, like this figure. It is obvious that this statement is true.

Two students used graphical representations and slopes of tangents to argue that the mathematical statement was false (Fig. 2(b)). Interestingly, these students all marked the domain of the functions on the graphs. For example,

S12: I think a greater function does not imply a greater derivative. This statement is false. Thus, I would like to find a counterexample to prove its falsity. The graph of $f(x)$ is above that of $g(x)$. The graph of $g(x)$ is a straight line with a constant slope. However, the slope of $f(x)$ in the interval of (a, b) is not always greater than that of $g(x)$. Therefore, I proved that this statement is false.





(a) (b)

Figure 2. Supporting examples and counterexamples to statement 2

Statement 3

This statement is the converse of Statement 2. However, the students demonstrated a distinct performance in response to these two statements, indicating that Statement 3 was more challenging to assess than was Statement 2. However, most of the students neither noticed the interval nor generated correct examples or counterexamples to verify or refute the statement. For example, S10 considered $f'(x) = 4x$ and $g'(x) = 2x$, and claimed that $f'(x)$ was greater than$g'(x)$, and then provided $f(x) = 2x^2$ and $g(x) = x^2$ and claimed that $f(x)$ is greater than $g(x)$. S/he did not realize that this example did not satisfy the conditions of the statement, ignoring the range of x and the constant. Unlike S10, S11 also said that the statement was true, and used algebraic representation to generate an example. However, s/he neglected the arbitrary constant c. S/he considered $f'(x) = 2x + 1$ and $g'(x) = 1$, stating $f'(x)$is greater than$g'(x)$, for all > 0; $f(x)$is $x^2 + x + c$; $g(x)$ is $x + c$, and $f(x)$ is greater than $g(x)$, for all $x > 0$.

Interviewer: Are these two c the same?

S11: The same, because all are c.

Interviewer: Is c a fixed constant?

S11: No. C can be any constant.

Interviewer: So these two constant c can be any constant, right?

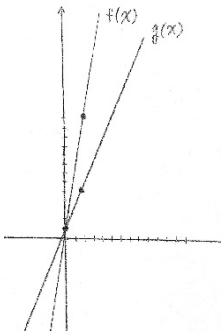
S11: Yes, but (10 seconds of silence) I make a mistake, a big mistake, c can be any constant, so these two c can be different. It didn't occur to me. (7 seconds of silence) Now I claim this statement is not true, the constant c of $f(x)$ is -100, and the constant c of $g(x)$ is 100, and x should between zero and 1.

There were five students that used graphical representations and slopes of tangents to argue that the mathematical statement was true. The following response is just one example of many similar protocols:

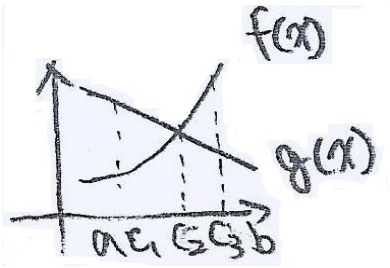
S2: $f'(x)$ is larger than $g'(x)$, so the slope of $f(x)$ is greater than the slope of $g(x)$, like this figure (Fig. 3(a)), and $f(x)$ is greater than $g(x)$.

These students did not notice that $f(x)$ is not always greater than $g(x)$. Three students successfully generated counterexamples. Unlike S11, S15 noted the effect of the constant of integration and the range of the variable; s/he considered "$f'(x) = 3, g'(x) = 2,$ and stated that $f'(x)$is greater than$g'(x)$, claiming that for $f(x)$ is $3x + a, g(x)$ is $2x + b$, $f(x)$ isgreater than $g(x)$ if $0 \leq x \leq 10$, and b $> 10 +$ a." Three students used graphical representations to generate counterexamples. For instance, S1 connected the derivatives to the slope of the tangent [Fig. 5(b)]: "the slopes of the

tangents of $f(x)$ is greater than 0; the slopes of the tangent of $g(x)$ is less than 0, so $f'(x)$is greater than $g'(x)$, but $g(x)$ is greater than $f(x)$ in the interval $[a, c_1]$." As in their responses to Statement 3, these students all marked the domain of the functions on their graphs.



(a)



(b)

Figure 3. Supporting examples and Counterexamples to statement 3

Discussion and conclusion

Students' use of intuition and systematic intuitive errors

From students' problem solving behaviour of this study, and from judging the mathematical statements, the kind of intuition that can be used for students classified in this type of exercise is affirmatory intuition. In planning the solution, students used intuition to try to use symbolic and graphical representations. This intuition is classified as anticipatory intuition. In implementing the solution successfully, the intuition is used is by way of trial and error, the kind of intuition that is used in this type of student learning is classified as anticipatory intuition.

Systematic intuitive errors are errors of intuitive reasoning that cause misrepresentations of situations and persist across situations and people. Many students made logical errors in deciding the truth value of mathematical statement (e.g., S14). According to Fischbein, [7] I can affirm that the (false) equivalence between a statement and the converse is an intuition. Moreover, it is important to guide the students to the awareness of the structure of their argumentations, so that the knowledge

of the non - equivalence between the statement and the converse becomes intuitive knowledge. As Fischbein [7, p.81] wrote:

> The training of logical capacities is a basic condition for success in mathematics and science education. We refer not only to a formal-algorithmic training. The main concern has to be the conversion of these mental schemas into intuitive efficient tools that is to say in mechanisms organically incorporated in the mental behavioral abilities of the individual.

The participant of this study also hold the false intuition ' More A-More B ' [20] in the statement 1, for example, ' The greater the function is, the larger the integral is ' (S4) and ' With a greater integral, the area is greater ' (S6). What is interesting is, students do not hold similar false intuition regarding the statement 2 and statement 3 about the concept of derivative, but about the concept of function. Such as S10, because 2 is greater than 1, so ' $2x$ is greater than x'; because 4 is greater than 2, so ' $4x$ is greater than $2x$ '. Differentiation is usually easier than integration, and the representation of area is easier than the representation of slope of tangent, according to the problem solving processes of the participants. This finding shows the calculation complexity of the mathematical statement involved and the intensity of its connection with the graphical representation. These observations seem to relate to the inclination of students in using intuitive laws. Many systematic intuitive errors can be classified as accessibility errors.[8, 21] Accessibility is the ease with which certain knowledge is evoked or certain task features are perceived and is a crucial component of intuitive reasoning and decision-making.[21] These intuitive errors involve attribute substitution [21], when a more readily accessible attribute is substituted in a task for a less readily accessible attribute. For example, similarity is an attribute that is always accessible because it is processed intuitively.[22] Participants may intuitively notice similarities

between a given concept (integral, $2x$, or $4x$) and familiar concepts (area, 2, or 4) and substitute more accessible attributes for less accessible ones based on these similarities.

Students' visual reasoning

One of the main heuristic strategies in many calculus tasks is to draw a graph of the function involved. However, most of the students had the strong inclination to use symbol representation. What is interesting is, even if students used graphical representations to generate examples, only a few students could generate correct counterexamples. A possible reason is that there is no visual component in their concept image of the derivative and definite integral, this makes it difficult for them to "see" the statements. This is particularly true for statement 1. It is difficult to find an appropriate counterexample, if one can't expand one's limited concept image. The most significant expansion in the evoked concept image of function, in terms of being associated with learning events, is the use of visualization in the sense of Zimmermann and Cunningham [14, P.3]: "Mathematical visualization is the process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for mathematical discovery and understanding". Why is visualization important? The examples generated by students show that those using the symbolic representation were unable to meet the condition"$\forall x \in (a, b)$". On the contrary, the use of graphical representation allowed students to control more assumed conditions at the same time while generating an example. The global not the local idea that the graph had could be associated with the statement, allowing the graph to act as a kind of generic example. In other words, visualization allowed students to control larger number of conditions simultaneously, while in the symbolic representation students may only control one requirement at a time. This finding provided some support to corroborate

Fischbein's [7, p.104] claim that visualization 'not only organizes data at hand in meaningful structures, but it is also an important factor guiding the analytical development of a solution.' I suggest that visualization can be more than that: it can be the analytical process itself which concludes with a generic solution.

There are two dangers in visualizing. The first danger in visualizing is that *figures can induce false conclusions*. In fact, in this case (e.g., S8 and S9), it is not the figure that is incorrect and that brings us to the false conclusion according to which all triangles are isosceles. Rather, what is misleading is the reasoning 'behind' the figure. These incorrect—propositionally expressed—hypotheses activated an inaccurate figure, and this is what brings one to a false conclusion. Nevertheless, this does not mean that the figure is incorrect as a figure. Rather it is the role of this figure as the activation of some incorrect hypotheses. Therefore, the error is in the informal reasoning which is behind the construction of these figures, and not in the figures themselves, or in the possibility of putting them to the test. This kind of error in using figures is pre-visual, since it depends on wrong hypotheses that are made before the figures are drawn. The second danger in visualizing is that f*igures can mislead our reasoning*. This can happen when the reasoning is performed on the particular image that represents the mathematical statement without considering the consequences implied by it. Concerning problem solving performance of S2 and S6, though students know the mathematical concepts of derivative and definite integral, they are not capable of solving the mathematical statements. These kinds of errors in using figures are post-visual, since they depend on wrong hypotheses that are made on the drawn figures.

If we consider examples taken from mathematical problem solving, we see that the appeal to visualization is not direct, because it strongly depends on expertise. Moreover, discovery by visualization is mediated by the intuition of the generality of

the conclusions obtained by means of it. On the contrary it can be fallacious, at least in two senses. First, there can be pre-visual errors, if an erroneous hypothesis is made on how to draw the figure. Secondly, there can be post-visual errors, if an erroneous hypothesis is made on properties of the figure which are not relevant to the mathematical problem. Nevertheless, intuition and visualization are interconnected parts of a vast web of knowledge that results in the learning and in the application of a mathematical problem solving. It is the preservation of these interconnections that allows for the intuition of the generality of some conclusion and the consequent stabilization of certain beliefs.

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Multicultural lecturing: some challenges

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Abstract

In this paper I discuss the cultural influences on mathematics education from my perspective as a teacher, and from recent migrant students' perspectives on learning in a new country. I reflect on the assumptions I have made in my teaching and learning context that spans three decades, two countries (South Africa and New Zealand), one medium of instruction (English) and a shift from mono-cultural students to diverse multi-cultural, multi-national and multi-lingual students. Today, in New Zealand, there is no 'elephant' in classrooms, instead it seems to be a whole jungle. This is because my typical class includes: indigenous New Zealanders (Maori); immigrants from Europe (mainly from the United Kingdom); Pacific Islanders (Polynesians from Samoa, Tonga, and the Cook Islands); and more recently, immigrants (from South Africa, Asia, and occasionally South America); and I hear at least ten languages. What does this mean for me? Does it change my teaching? Have these students come to New Zealand to be indoctrinated into New Zealand styles of teaching and learning? These questions need to be considered from two perspectives, mine as lecturer, and from students as learners.

Keywords: mathematics education; culture and mathematics; languages and mathematics; experiences and mathematics; experiences of immigrant students

Introduction

I find that while institutions are keen to get international students they do little to help lecturers cope with the resulting diversity; and the students are often reluctant to talk in ways that may be critical of the institution. This paper draws from three sources of data; from my personal experiences as a student and teacher educated in the South African

apartheid era and now teaching in a university in New Zealand, from a research project I completed with younger students who were new immigrants, and from literature.

With the range of nationalities in my multicultural classrooms I have come to realize that it is more than language to be considered; language, beliefs, and experiences are fundamental factors embedded in one's cultural construct. These factors will form the structure of the discussion of the paper.

Culture in the mathematics context was unheard of until three decades ago, and was first proposed in the 1980s by D'Ambrosio as ethnomathematics.[1] In spite of an awareness of ethnomathematics in the sphere of mathematics education research, it seems to me that mathematics education practices and attitudes continue to be projected as culturally neutral ideologies.

Methodological approach

This personal narrative provides a critical reflection on my experiences as a student, and on my practice as a teacher and teacher educator. According to Connelly and Clandinin,[2] narrative inquiry is the study of humans experience the world.

As a student in the South African context, I experienced mathematics education as if it was culture-free, factual and based on theorems. I manipulated formulae or geometric theorems to arrive at a predetermined answer, and measured my success when the teacher affirmed my reasoning. I did not consider how indoctrinated I had been by the seemingly culturally neutral ideology of mathematics education. It was only after the apartheid era in the mid-1990s when students of diverse language and cultural backgrounds entered my classroom that I begin to see the importance of culture in mathematics education. I subsequently found myself having to reconsider mathematics concepts and learning in a context where some students spoke a different language at home to the dominant classroom language, and potentially held different beliefs and

experiences of mathematics understandings to those presented in the classroom, often from a western perspective.

The New Zealand context presented a similar western approach to mathematics, in culturally and linguistically diverse classrooms. New Zealand has an increasing ethnic and culturally diverse society with increasing numbers of bilingual and multilingual students in the classrooms.[3] My curiosity initiated a small hermeneutic phenomenological study of the lived experiences of ten immigrant students in their New Zealand mathematics classroom. According to Laverty [4] this approach sets out to understand the lived experiences. The ten students had come from eight different countries and from non-English backgrounds. The students were observed between three to five times in their classrooms and were interviewed after each observation to gain an understanding of their perceptions of their lived experiences. The study was not confined to a specific topic in mathematics but varied and included a range of topics that were being studied at different times. Although the study was based in a secondary school mathematics education context, I believe the experiences of the students are transferable to the tertiary level.

In the tertiary context, while preserving knowledge of the past is important [5] the days of the sage on the stage as the ultimate transmitter of knowledge and the gate-keeping practices of knowledge for academic success may be over. At the heart of our teaching and learning classroom should be "enquiry and intellectual debate" [6,p.35] where the teacher and students form a partnership in steering multiple learning directions by exploring different ideas of mathematics engagement. This notion of including diverse ideas initiated my study of literature for alternative teaching and learning ideas. All classroom participants bring their own 'history, social construction, knowledge and life experiences' to the classroom dialogue therefore, teachers are urged

not to impose their ideas on the learners [7,p.95] but rather to explore multiple ideas that students may have of learning, which are often informed by their own cultural background and beliefs.[8]

Discussion

The section that follows offers a discussion of language and mathematics education, beliefs and mathematics education, and experiences and mathematics education with regards to the challenges of multicultural mathematics education. The discussion emerges from my gaze on my practice, and from the voices of immigrant students in the mathematics education context.

Language and mathematics education

As a student I had been taught mathematics as a body of proofs transmitted to me by the teacher and mathematics textbooks. This occurred within timetabled blocks allocated for algebra and geometry, inclusive of trigonometry and measurement. The fragmented series of topics and subtopics were based on the assessment structure of the matriculation examinations, not on understanding. The teaching comprised teacher telling and teacher demonstration of step-by step methods for students to follow in an environment where the language of instruction was taken for granted. This was based on the view that mathematics was culture neutral and that all in the classroom shared a universal language. 'A negative number times a negative number gives a positive number' [9,p.145] in any geographical region, was the argument. According to Bishop this may be attributed to the decontextualising of ideas for applicability in any context.

The examples that follow show the fundamental importance of language and how it affects mathematical understanding, particularly in an environment where it is assumed that once a mathematical term is defined by the teacher, students gain

understanding. This ignores the idea that for students for whom the medium of instruction is different to their home language, understanding mathematics involves thinking between two languages. The following excerpt shows the language related challenges that students sometimes face.[10] Ian (pseudonym), a recent immigrant student to New Zealand at the time of the study, shared his experiences of learning mathematics:

> If I learnt the maths in Nigeria, I think about it in Nigerian [Igbo], if I learn the maths in New Zealand then I think about it in English... the maths we did in Nigeria is different than this one here [in New Zealand]… I'm listening but I don't know how to do it because I haven't done it like that before.

Language switching also seemed to be a popular strategy employed to cope with mathematics content in South African schools as a means for teachers and students to communicate their ideas and thinking. According to Setati,[11] language switching helped improve student-student and student-teacher interactions in the mathematics classroom. These interactions between participants are often framed by their individual language structure and culture.[12] A student's cultural frame may be dissimilar to the classroom culture and this difference has the potential to affect the student's learning success.[13] As suggested by Parvanehnezhad and Clarkson,[14] teachers have an important role in facilitating the learning of immigrant students by encouraging them to switch between their first language and the dominant classroom language to promote understanding. However, a study by Ohia [15] describes the New Zealand classroom as being mono-cultural and representative of English attitudes in which non-English speaking students are expected to only communicate in English. Perhaps the statement, *I know how to add them, I didn't know I had to add them*, made by Ian about completing the totals of the rows and columns on a table could have been avoided had the instruction been more accessible to him.

In addition, Ian's response to a probability task reiterates Boaler's,[16] assertion that an inability to solve problems may not be due to a lack of mathematical knowledge but rather to the student's interpretations of the demands of the problem. In response to a task requiring students to draw a probability tree of all possible gender outcomes of children if there were three children in the family, Ian drew a picture including a tree, a duck, and three houses. Besides the multilevel language specific difficulties that may be indicative in Ian's response, his drawing and his explanation for the various items in his picture may suggest that he was trying to draw from the context of his own worldview and experiences, "*for the kids… a tree and duck for the kids … to play with … and house to live in*." While a statistically flawed response, perhaps it is suggestive of a non-fragmented or holistic approach within a specific cultural context. As asserted by Begg,[17,p.4] western concepts of probability may contradict Asian beliefs of probability such as 'karma', or religious beliefs associated with probability of the gender of a child. From a linguistic perspective the context of the task embedded in words that may have been unfamiliar, such as the use of the word 'gender', and the use of logical connective 'if', required a higher level of language skills for understanding to have occurred, thus presenting particular difficulty;[10] and perhaps a confusion of the context because of the unfamiliar words.[17]

While electronic dictionaries may be seen by some to be the answer to students' understanding of mathematics in a language that is dissimilar to their spoken language. From my experience in the classrooms that involved students engaging in tasks involving a locus of points, a student from China looked bemused at his electronic dictionary which displayed a picture of a locust. This indicated that direct translations are sometimes problematic and confusing in the context of mathematics education and may be inconsistent from culture to culture.[17] In addition parents may discourage

their children from using electronic dictionaries because they believe that relying on dictionaries will inhibit their child's ability to become proficient in the dominant classroom language [10] and this is essential in accessing a 'better life'.[17,p.3]

The role of language cannot be overemphasised; it gives students the tools to make sense of their learning and become active participants in the mathematics classroom. Being unable to communicate with others or to understand people speaking is not only frustrating but can be a lonely environment for immigrant students. They may feel isolated, invisible and discredited, and according to Davidson and Kramer,[18,p.139] for the student it would be like looking "into a mirror" and not seeing themselves. Such an environment is bound to stunt academic growth, because experiences are manifested from the students' social, cultural and historical contexts and learning occurs from active involvement in the environment.[19]

While teachers may invite all students to contribute to discussion, international students for whom English is not their spoken language, may find this difficult. My experience with students in the secondary school context and the university context is that initially they tend to prefer to be observers. For Ian, contributing to class discussions was a daunting experience, as he pointed out, "*I don't like that… I don't like doing that. Too shy and I don't like it"*. This silence may sometimes be misinterpreted as non-participatory attitudes, however, it is important to understand the silent behaviour in the context of language and culture. According to Endo [19] students' silence may be attributed to their lack of confidence in conversing in the dominant classroom language, and to teachers' avoidance to engage in conversation with the immigrant students because of the fear of not being able to understand the student. Other studies found that for some students silence may be a necessary incubatory phase to develop language proficiency.[21] Contrary to the study by Endo [21] an earlier

study found that teachers may not direct questions at immigrant students because they are respectful of this incubatory phase for second language students [21] in the mathematics context.

I have found peer learning to be beneficial for student engagement, and particularly effective for second language learners to cope with the language demands of the classroom. The benefit of peer learning had previously been noted in the findings of Bose and Choudhury,[23] and Parvanehnezhad and Clarkson,[14] who asserted that code mixing and code switching play a significant role in students negotiating meanings, especially within groups of students that share a common language. According to Bose and Choudhury,[23] unlike language switching or code switching, code mixing involves conversations in one language with certain words being substituted in a second language without interfering with the structure and meaning of the original language. Further, it is beneficial for students to have the opportunity to work in groups as it promotes their mathematics learning and, for second language learners, it develops their first and second language skills.[10] Encouraging students to communicate in their first language is powerful in terms of developing mutual respect and inclusiveness,[10] especially in our increasingly multilingual classrooms.[24]

While some educators advocate for the use of first language by students in class, many students come from backgrounds that emphasise oral language learning over written language. However, while pedagogical practices place students at the centre of learning,[25] assessments practice seem enshrined in traditional written forms and do not promote oral language communication. Thus, students from predominantly oral language cultures may find difficulty in expressing themselves mathematically in written forms and this could result in their underachievement.[1] While language in the mathematics education context plays an important part in students learning, and their

ability and motivation to participate, I believe that as a teacher I can influence the learning environment, through the language I use in the classroom and the extent to which I invite other language/languages into my classroom.

Beliefs and mathematics education

As a student, I did not realise how being exposed to mathematics education from a western perspective had impacted on my views and beliefs. My mathematics education at school comprised being taught by a teacher who was knowledgeable in the subject, reinforcing my understandings that by practising from particular mathematics textbooks, would lead to a career in mathematics education and a secure employment future.

My focus is on the underlying cultural beliefs of mathematics education that may be apparent in resources that are used in the classroom context. From my experience, textbooks seem to have formed an integral part of mathematics education especially at secondary school and tertiary levels. Not only is the textbook seen as providing the method in solving mathematics problems, it also seemed to affirm the teachers approach, thus portraying a singular perspective perhaps of the dominant culture. Shan and Bailey,[1] have associated the lack of representation of people of colour in mathematics textbooks with the invisibility of non-white students and a hidden curriculum.

The invisibility of students in the context of learning according to Davidson and Kramer would be like looking "into a mirror" and not seeing their own reflection.[18,p.139] In an earlier study Begg [17] too spoke of Polynesian students who saw themselves as ‘outside this knowledge’ when exposed to European schooling. Furthermore a study by Baker [26] found that when students do not see themselves in the textbook, they view the textbook as an artefact of a foreign culture to which they do

not belong. It may also reinforce offensive stereotypical ideas of groups of people [1] in statistical data. A New Zealand study by Ohia,[15] attributed a similar lack of representation of Maori students in mathematics context to their underachievement because of their lack of inclusion and indoctrination towards the dominant English worldview. While mathematics texts are now attempting to be more inclusive, they still need to present the various cultural contributions that have been made to mathematics education. A multiplicative strategy, for example, that may be traced to India and mentioned in early Chinese, Arab, and Persian documents [27] has been attributed to John Napier who introduced it to the western world. It became popularly known as Napier's bones or multiplication rods. Shan and Bailey [1] advocate for the acknowledgement of those mathematicians responsible for the creation of the original ideas of mathematics, rather than those who took the ideas to the west.

My study of immigrant students' experiences [10] revealed numerous differences. In some cultures the teacher is viewed as the fountain of knowledge, "*listen*[ed] *to the teacher and … know what to do"* and as the authority that should not be questioned. Questioning the teacher in certain cultures is seen to be disrespectful and a display of the student's deficiency "*When in class I listen to teachers … I can go home and ask my mum for help,"* because of the high esteem in which teachers are held as the 'givers of knowledge.'[26,p.139] This statement suggests that parents of immigrant students play a key role. Furthermore some parents of immigrant students believe that their children will benefit from learning mathematics in their host country and at the same time being exposed to assessments from their home country, "*my mum ask her Chinese friend to post Chinese* [mathematics] *tests to New Zealand then I will write them."* Such beliefs may be indicative of the parents assessing the education of their

child in the new context and may have implications in the context of international students whose tertiary education abroad may be funded by their parents.

Not only are parental beliefs fundamental in the learning experience of international and immigrant students, the social background beliefs that students come from may also impact their learning. A study by Li,[28] established that the beliefs of students of eastern origins is premised on self-perfection morally, and on learning virtues for the purpose of contributing to society, while the beliefs of students from the west centred on the mind, the processes of learning and personal excellence. The eastern belief of failure is viewed as the student's deficiency and shameful while from a western perspective failure is seen as a stepping stone to success.[10] Understanding these worldviews is important to the success of students in the context of mathematics education at all levels.

This idea of student deficiency can extend to students' views of working in peer groups. In spite of social interdependence theory which advocates for peer learning,[29] my experience with international and immigrant students in the university context, students seem to be that they are hesitant to work in groups with their peers. This notion is echoed in other studies that speak of the discomfort that international and immigrant students experience associated with their feelings of powerlessness, intimidation and isolation in peer group situations.[20]

The influences of the different beliefs that students bring into the learning context has implications for mathematics educators in all sectors of education. The challenge is to create a multicultural learning environment which is not about separatist and tokenistic practices of occasionally including an origami activity, a Maori kowhaiwhai pattern, or basket weaving in a geometry lesson, but rather it is about teachers acknowledging various cultural contributions to mathematics education in an

environment where students can access their own cultural background to enhance their mathematics understandings.

Experiences and mathematics education

The habits and experiences that we are exposed to seem to mould the lenses through which we view and engage all education, including mathematics. My pedagogical practice and assumptions as a mathematics teacher in South Africa often resembled the way I had been taught as a student, and not much had changed in the school context including the mathematics education context. The changing South African context and the different New Zealand context challenged me to rethink my practice, from the resources I used to my interactions with the diverse students in my classroom.

Student diversity included the different backgrounds that they came from and their experiences such as language spoken, gender, interests, hobbies, religion, and socio-economic status. My first reaction to a student who did not look at me when I spoke to them was that they were being disrespectful. This teacher assumption is what Begg [17] has referred to as being misconstrued for bad behaviour by teachers. I have learnt that for students of Pasifika background, lowering their eyes and not looking directly at an adult was a sign of respect. This was also found to be consistent with the study by Rosenblum, Goldblatt and Moin [30] who asserted that Ethiopian culture promoted silent, non-confrontational and non-assertive behaviours towards adults. I began to realise just how blurred my own view was of the numerous cultures, habits and experiences that students brought into the learning context.

Mathematics education in eastern cultures, specifically Chinese culture, encourages a silent learning environment.[31] Silence seems to be the prerequisite for thinking in this context. According to Han and Scull,[32] the experience of students that have been exposed to Confucian heritage cultures is that they should only speak when

they are spoken to by the teacher, otherwise student utterances are seen to disrupt the flow of teaching. For students that come from such mathematics education environments, the New Zealand classroom of inquiry must seem strange. Ian, an immigrant student from Nigeria in my research, seemed to have transferred his experiences of acceptable classroom behaviour to his New Zealand classroom context. "*I have to be quiet … I'm not allowed to speak to another person,"* and from this he attributed his inhibition to participate in class discussions and to interact with his peers and teacher in his mathematics classroom.[10]

Another student from my study Babeloo (pseudonym) who emigrated from South Africa, found the New Zealand classroom very distracting, "*I just don't concentrate that well when people are that loud around me."* While this statement may reflect her personal preference to work in a quiet environment, it may also be reflective of Kaufman's,[33] finding that immigrant students are used to working in silence and are not familiar with noisy classrooms. This dissimilar experience of international and immigrant students could well have implications for their learning at the tertiary level where classroom discussions may be perceived as noisy and present an overwhelming experience for them.

The dissimilar mathematics education experiences that students face may also extend to other contexts such as money which could result in a myriad of possible solutions. When I reflect on my experiences as an immigrant to New Zealand, I was in the habit of mentally converting all my purchases in the New Zealand dollar to the South African rand. Similarly one can only imagine the compounding problems that immigrant and international students may be faced with when solving contextualised problems involving currencies.

In addition to students experiences with mathematics learning, it is important to note that while many cultures may use the base ten number system, others may use base 20 or base 60.[34] As averred by Begg,[35,p.3] "Puzzles are useful because they do not usually assume a particular way of starting – instead they begin with a situation that may be approached in many ways." The use of certain ideas and concepts may be linked to a specific context of reference therefore when students are asked to solve non-contextualised mathematics questions, their responses may be multiple and dependent on their contextual reference at that time. Drawing from my experience in the university context that involved looking at multiplication from a Japanese perspective, I realised how indoctrinated we might be in the way we think about solving mathematical problems. In my diversity class I presented the lines method (Figure 1) to multiply numbers and the overwhelming response from my students was that they did not know that multiplication could be done in such a different way. This approach shifts one from thinking about number manipulation towards a visualising the process. The visual form also seemed to have transcended the language barrier.

Varied ways in which one might interpret non-contextualised sequential ideas is particularly relevant to the sports context. Non-contextualised tasks involving students to complete the next occurrence in a sequence such as:

i. 42.1, 42.2, 42.3, 42.4, 42.5, …

ii. 15, 30, 40, …

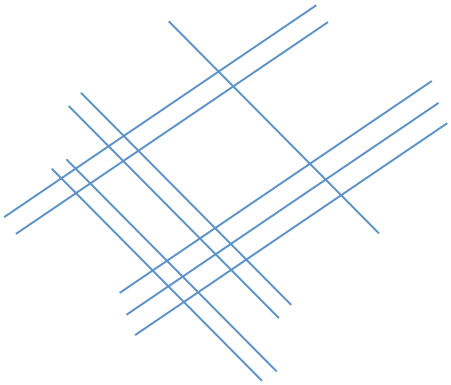
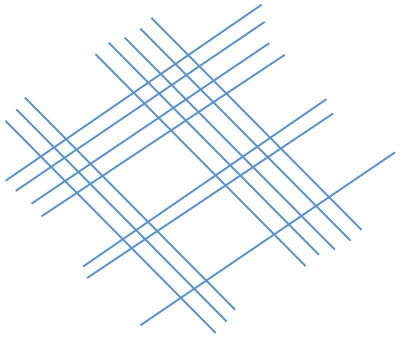
23 x 41	**421 x 35**
 For 23 - Draw 2 lines, space, then 3 lines from bottom left to top right. For 41 – Draw 4 lines, space, then 1 line from top left to bottom right. The intersections are counted vertically from left to right: 8 2+12 3 Answer 943	 For 421 – Draw 4 lines, space, 2 lines, space then, 1 line from bottom left to top right For 35 – Draw 3 lines, space then, 5 lines from to top left to bottom right. The intersections are counted vertically from left to right: 12 26 13 5 Answer: 14735

Figure 1: A Japanese multiplication strategy with lines and intersections

In example (i) the next expected occurrence in the sequence might have been 42.6 however in the context of cricket the next occurrence would be 43. The number 43.6 does not occur in the game and the end of the over is signalled by 43, meaning that 43 overs have been bowled. Furthermore, in light of the recent 2015 international Cricket World Cup semi-final match between South Africa and New Zealand, 43 could also mean the end of one team's innings or the end of the match owing to shortened

play. In example (ii) while it could result in a range of responses when considering the algebraic rule of the pattern, tennis avid fans or players may be quick to respond with any one of the following as the next occurrence: deuce; advantage server; advantage receiver; game; set; and match. While it may be argued that the game context is different to the mathematics education context, my point is that the game context offers mathematical thinking opportunities and students responses are dependent on their familiar and cultural experience, in this case a sporting lens. In my teaching and learning environment I found that when students are invited to interpret non-contextualised learning becomes more powerful, more meaningful, and more multi-faceted when students bring their own experiences into the context.

Concluding thoughts

Differences in international and immigrant students' experiences in the mathematics education classrooms may contribute to incongruities in their understandings a new learning context. Begg, Bakalevu, Edwards, Koloto, and Sharma [36] urge mathematics educators into the realisation of the different worldviews of teaching and learning, and unlike the western phenomenon of partitioning of knowledge other cultures experience mathematics education as transcending contexts for its use. Consequently, mathematics education is influenced by the cultural tenets of languages, beliefs and experiences of the teachers and the learners.

From my perspective as a teacher in the university context, and from immigrant students' experiences, I firmly believe that diversity ought to be embraced as an integral part of mathematics education. An environment that enculturates the learners within their own cultural beliefs, by not attempting to acculturate them as deficits into a foreign culture [9] may help alleviate some of the challenges faced by international students and provide the platform for multicultural education. Finally, students see their

lecturer/teacher as the "'elephant' on the delta" who will always win out, but the elephant has been influenced by a disruptive small creature such as a mouse!

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Zimbabwean Pre-service Teachers' responses to Matrix algebra Assessment items

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Abstract

Linear algebra is often regarded as a challenging course to learn and to teach and has received much attention in mathematics education research studies. However there are limited studies that have been carried out in Africa. This qualitative study was exploratory in nature with the purpose of identifying trends in the performance of the participant students in selected items based on the concepts of determinants, inverse matrices and solution of systems of equations. Twenty seven pre-service mathematics teachers from a university in Zimbabwe who were studying a course in linear algebra participated in the study. Data was collected from the students' written responses to an assessment instrument and analysed using document analysis techniques. The findings reveal that many students had low levels of engagement with the concepts and seem to have learnt off certain techniques which they applied to solve the problems. The study recommends that the university authorities should consider planning further support and interventions that would provide opportunities for deeper engagement with the concepts.

Key words: Matrix Algebra, determinants, inverse, systems of equations, APOS,

Introduction

Linear algebra, which includes the topic of matrix algebra is often the first mathematics course that first year mathematics undergraduate students encounter, since it requires limited mathematical prerequisites. Hillel and Sierpinska [1, p.65] argue that, "both the

teaching and learning of linear algebra at the university level is almost universally regarded as a frustrating experience". Some of the reasons for the difficulties faced by the students are not confined to the content of Linear Algebra, but are a result of the transition from elementary to advanced mathematics. The move from elementary to advanced mathematical thinking involves a significant transition: that from describing to defining, from convincing to proving in a logical manner based on those definitions, and from the coherence of elementary mathematics to the consequence of advanced mathematics.[2] Research about such difficulties has been documented in many countries, but there have been few that have focused on students in Africa. In this study we explore some of these difficulties experienced by a group of pre-service students in a university in Zimbabwe by focusing on their responses to an assessment in an introductory linear algebra course. The research questions guiding this study are:

1) How do the students perform on selected assessment items in matrix algebra?

2) What do the written responses of the students reveal about the engagement levels of students with the concepts? It is hoped that the identification of these trends can be used to inform the design and delivery of courses run in similar situations.

Literature Review

Bolgomolny [3] contends that understanding matrix algebra concepts is more than performing calculations. It is being aware of how procedures work, developing an intuitive expectation of the result without actually performing all the calculations, being able to work with variations of algorithms, being able to notice connections and to organise experiences.[3]

Formalism is among the reasons for students' struggles with linear algebra concepts.[4] Formalism entails a wide range of use of notations and symbols to the use of structures to represent ideas.[4] Parker [5] found that the quality of language use was

more closely aligned with students' definitional understanding than with their problem solving skills.

Some difficulties in understanding linear algebra are also related to underlying misconceptions that are held by students. Misconceptions are systematic conceptual errors caused by beliefs and principles in the cognitive structure. Aygor and Ozdag [6] in their study investigated misconceptions revealed by undergraduate students while solving problems on matrices and determinants. Their results revealed many misconceptions which are related to confusion between matrices and the determinant of matrices. For example, some students took the relationship det $A = -$ det B (that is the determinant of matrix A equal to minus one times the determinant of matrix B) to mean $A = -B$ (that is, matrix A equals minus one times matrix B). Some student also took the relationship det $A = k$ det B (that is determinant of matrix A equal to k times determinant of matrix B) to mean $A = k\,B$ (that is matrix A equals k times matrix B). Again some students took the relationship det $A +$ det B (determinant of matrix A plus determinant of matrix B) to mean $A + B$ (matrix A plus matrix B).

Theoretical framework

In researching students' conceptions in mathematics, a theory that has proved to be very useful is APOS (action, process, object, schema) theory which describes possible cognitive paths taken by students when developing an understanding of mathematics concepts. APOS theory is a constructivist theory focusing on individual's mental constructions of mathematical knowledge.[7-9]

The mental constructions of action, process, object and schema is hierarchical in the sense that an action conception develops before a process conception and a process understanding can be transformed to an object understanding. An action may be defined

as any physical or mental transformation of objects to obtain other objects.[7, 8] A process is viewed as a transformation of an object or objects the individual is in control of without the need of external stimuli. A constructed process can be transformed in several ways, that is, it may be reversed or coordinated with other processes.[7, 8] When the individual becomes aware of the process as a totality, and realises that transformations can act on that totality, and can actually construct such a transformation explicitly, the individual has encapsulated the process into a cognitive object. There have been many studies [3, 5, 7, 10–13] which have conducted research on students' understanding of linear algebra and development of linear algebra using APOS theory. Dubinsky [7] argued that research is needed to determine the specific mental constructions that a student might make in order to understand linear algebra concepts as well as the pedagogical strategies that need to be developed that can lead to students making these constructions. For example, Mathews [13] noted that solving a system consisting of a single linear equation is easier than solving a system with two or more linear equations and this has implications for developing pedagogical strategies.

APOS theory emphasises the important role that a person's existing schema of mathematics concepts takes on in the construction of new knowledge. Dubinsky [7] and Ndlovu [14] pointed that background of mathematical concepts that are not part of linear algebra are essential to learn it. Ndlovu [14] argues that a strong foundation in concepts such as functions, equations and algebraic reasoning will support the development of schema in systems of equations. On the other hand a lack of appropriate mental constructions in these prerequisite concepts hinders the development of an understanding of concepts such as solutions to systems of equations.

Methodology

This study used an interpretative approach which is embodied by an individual concern

to comprehend the personal world of human experiences.[15] The study is an exploration of the understanding of matrix algebra through interpretation of individual responses. The participants were 27 pre-service student teachers who were enrolled on a linear algebra course. Data were generated from their written responses to an assessment task consisting of seven items, four of which were selected for analysis. These items were based on determinants, solutions to systems of equations and inverse matrices. The other three items did not test matrix algebra concepts.

The participants' written responses were analysed and themes relating to the ways in which they worked with and reasoned about determinants, inverse and solutions to systems of equations were then identified. This method is an example of the application of content analysis which extends to assessment of all types of communicative material either structured or unstructured.[15] In addition inductive and deductive analyses were used through coding the written responses of the participants. After coding, the responses were grouped according to identified categories according to a preliminary genetic decomposition. These categories aided in identifying different components of the theoretical framework in various types of correct and not correct responses. Informed consent was obtained from the participants and further ethical procedures were followed according to the university's protocol. The four tasks appear below.

Results

We report the results of the various items under the headings of determinants, inverses and solutions to system of equations. References to responses by particular students are organised by using a number 1 to 27, so S16, for example refers to student number 16 from the list of students in the class (where the list was not arranged in any meaningful manner).

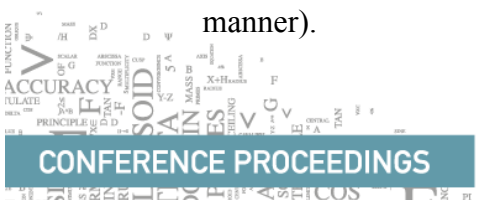

Table 1 The four items

1. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 4 \\ -1 & -1 & 1 \end{bmatrix}$. Find (a) the determinant of A; (b) A^{-1}(the inverse of A) (c) Hence, solve the system of equations $x - 2y + z = 1$ $2x - 3y + 4z = 2$ $-x - y + z = 0$	3. (a) Let A be a non singular matrix. Give a formula connecting $A^{-1}, \det(A)$ and $adj\ (A)$. (b) Find x such that the $\det(A) = 0$ if $A = \begin{bmatrix} 1 & x & x \\ -x & -2 & x \\ x & x & 3 \end{bmatrix}$
2. Find the determinant of $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 0 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 0 & 2 & 2 \end{bmatrix}$	4. For which values of a will the following system of equations have $x + 2y - 3z = 4$ $3x - y + 4z = 2$ $4x + y + (a^2 - 14)z = a + 2$ (a) no solution, (b) infinitely many solutions. (c) Solve the equation for which the system of equations has infinitely many solutions.

Items related to determinants

For Item 1(a), 26 out of the 27 students who wrote the test produced a correct response, indicating that these 26 students were at least at an action conception on finding

determinants of 3×3 matrices. The one student (S8) made an error when evaluating one of the minors, $\begin{vmatrix} 2 & -3 \\ -1 & -1 \end{vmatrix}$. The student wrote $-2-4$ instead of $-2-3$, adding -1 to -3 instead of multiplying.

For Item 2, there were six students who presented correct responses, while 20 presented incorrect solutions while one student did not respond. Besides getting an incorrect determinant some students used matrix brackets instead of bars when evaluating determinants, suggesting that they could not distinguish between the notation for a matrix A and that for the determinant of A, that is they viewed $\det(A) = A$. The 20 students who failed to get the correct determinant value were still at an action conception of finding determinants. Perhaps it was a bit burdensome trying to work with three 3×3 determinants although they managed to get the single determinant of a 3×3 matrix correct in Item 1(a).

For Item 3(b) 15 out of 27 students correctly evaluated the determinant of a 3×3 matrix containing algebraic terms, managing to find the two values of x correctly. These students have most likely interiorised the procedure of finding the determinant of a 3×3 matrix. Ten students gave an incorrect response and two had no response. It is of interest that 12 students had managed to evaluate the determinant of the 3×3 matrix in Item 1 yet they were unable to evaluate the determinant of a matrix of the same size but one whose entries included algebraic terms. Hence these 12 students are at an action conception of evaluating determinant of a 3×3 matrix. From 10 incorrect responses, four students struggled with simplifying the algebraic expression representing the determinant. Six students obtained the correct expression for the determinant but displayed misconceptions concerning the use of the square root function. Three students wrote $x^2 = \frac{3}{2} \rightarrow x = \sqrt{\frac{3}{2}}$, left the negative value, while the other 3 students wrote

$x = \pm\sqrt{\frac{6}{4}} = \pm\frac{\sqrt{6}}{4}$, effectively applying the square root function to the numerator of the fraction only. The results for the items on determinants are summarised in Table 2.

Table 2 Summary of responses on tasks on determinants

Item	Number who get correct determinant expression	Number who get correct response	Number who get incorrect response	Number with no response
1(a)	26	26	1	0
2	6	6	20	1
3(b)	21	15	4	2

Overall for the items on determinants, of the 26 students who presented correct responses for Item 1(a), only 6 produced the correct value of the determinant of the 4×4 matrix in item 2. For Item 3(b), there were 21 students who derived the correct algebraic expression for the determinant, but for six of them, their misconceptions of the square root function stopped them from arriving at a correct value of x. Four others attempted the item but did not have the requisite algebraic simplification skills to get correct algebraic expression for the determinant. One student did not get correct response to any of the Items 1(a), 2 and 3(b) suggesting a very limited conception, not even at an action level. The written responses of three students who got all the three items correct suggest that they have interiorised the concept of a determinant as a process being able to evaluate determinants of matrices of different sizes and solve problems involving determinants.

Items related to working with the inverse of a matrix

The students' responses to Items 1(b), 1(c) and 3(a) matrix are discussed in this section. The fact that seventeen students produced correct responses to Item 3(a) suggests that their understanding of the inverse of matrices was at process level. They could write the correct formula connecting A^{-1}, det (A) and det (A). Three students gave no response. The seven students who wrote incorrect response, three illustrated the formula using a general 2×2 or 3×3 matrix. Four students gave incorrect versions of the formula such as $A^{-1} = \det(A).\text{adj}(A)$ and $A^{-1} = \frac{1}{\det(A).\text{adj}(A)}$. S20 wrote $A^{-1} = \det(A).\text{adj}(A)$ in Item 1(c). S/he tried to use the row echelon method to find the inverse to find the solution of equation in Item 1(c) but was not successful. Her/his responses to Item 3(a) and Item 1(c) show that s/he has not interiorised the concept of finding an inverse by using the adjoint method.

For Item 1(b), 19 students who managed to get the correct inverse of matrix A were most likely at process conception. These students calculated all the minors correctly, worked out the adjoint of the matrix correctly by transposing the matrix of cofactors correctly and then multiplied the adjoint by the reciprocal of the determinant. Six out of the 19 students produced correct responses without using matrix brackets and just inserted the brackets in the final answer so they have challenges in matrix notation.

Of the 8 students who failed to get a correct response, one failed to calculate the correct determinant but managed to get the correct adjoint matrix of A. Seven students did not evaluate some of the minors correctly, leading to a wrong matrix of cofactors. These students' slips, [13, 14] suggest that they were responding to external prompts, and are operating within an action conception. In terms of Item 1(c) 18 students manage to get the correct solution to the system of equations using the inverse matrix method. From the nine students who wrote incorrect response, one of whom did not complete his

work. Eight students arrived at an incorrect solution because they used an incorrect matrix, say K (for the purpose of this discussion) instead of the correct A^{-1} (answer to Item 1(b)). It was interesting that 7 of these students wrote the first step as $KA = K.b$ where b = $\begin{bmatrix}1\\2\\0\end{bmatrix}$ and continued in the next step to $I_3.x = K.b$, where x=$\begin{bmatrix}x\\y\\z\end{bmatrix}$ and I_3 is the 3× 3 identity matrix. The students assumed that the product (KA) of their matrix K with matrix A was the identity matrix which is a necessary result required in the procedure. The response of S19 is as follows:

$$\frac{1}{8}\begin{bmatrix}1 & 1 & -5\\-6 & 0 & -2\\-5 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & -2 & 1\\2 & -3 & 4\\-1 & -1 & 1\end{bmatrix} = \frac{1}{8}\begin{bmatrix}1 & 1 & -5\\-6 & 0 & -2\\-5 & -1 & 1\end{bmatrix}\begin{bmatrix}1\\2\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \frac{1}{8}\begin{bmatrix}1+2\\-6+0\\-5+2\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}\frac{3}{8}\\\frac{-3}{4}\\\frac{-7}{8}\end{bmatrix}$$

$$x = \frac{3}{8} \quad y = \frac{-3}{4} \quad z = \frac{-7}{8}$$

The student used an incorrect matrix in the matrix product in the first step, instead of using ($A^{-1}.A$), and continued, assuming that the product would still yield the identity matrix. S/he might have not known the uniqueness property of inverses. This error was found in seven responses. They did not find it necessary to check that their matrix was actually A^{-1} since $K \neq A^{-1}$. Thus it is clear that S19 and the others just carried out the steps in the algorithm without having interiorised them into a process.

There was one student (S3) who arrived at the incorrect inverse matrix in Item 1(b) and did not continue in the same manner as the seven students described above. In her second step she multiplied the two matrices K and A and did not get the identity

matrix. The student then continued solving the system of equations that she was left with ($Kx = b$, where $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$). Hence s/he continued using the row reduction method to find the value of x, y and z in her system of equations, effectively mixing up the algorithms. The student S3 wrote:

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 \\ -6 & 2 & -3 \\ -5 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 4 \\ -1 & -1 & 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 \\ -6 & 2 & -3 \\ -5 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\frac{1}{8}\begin{bmatrix} 1+2-1 & -2-3-1 & 1+4+1 \\ -6+4+3 & 12-6+3 & -6+8-3 \\ -5+6-1 & 10-9-1 & -5+12+1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+2+1 \\ -6+4+0 \\ -5+15+0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4} & \frac{-3}{4} & \frac{3}{4} \\ \frac{1}{8} & \frac{9}{8} & \frac{-1}{8} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 10 \end{bmatrix}$$

The student continued with the solving the system.

$$z = 10 \qquad \frac{1}{4}x - \frac{3}{4}y + \frac{3}{4}z = 3$$

$$\frac{1}{4}x - \frac{3}{4}y = -\frac{9}{2}$$

$$x + 9y = 8$$

$$y = \frac{70}{21}, \quad x = \frac{-86}{7}$$

S3 did not consider going back to correct her answer in Item 1(b) when her identity matrix did not materialise in step 3, but continued with her own contrived algorithm. This indicates that the student did not understand the solution method very well. In summary the results for the three questions related to inverse matrices are in Table 3.

Table 3 Summary of results for items based on inverse of a matrix

Item	No correct	No incorrect	No Blank/incomplete
1(b)	19	8	0
1(c)	18	8	1
3(a)	15	8	3

Overall in the items on matrix inverses, 12 students provided correct responses to all three questions. From the 17 who managed to write the correct formula connecting A^{-1}, det (A) and adj (A), eight did not get the correct solution for the system of equations using the inverse matrix method and nine managed to get the correct solution for Item 1(b) and 1(c) and are at action conception. Thus a person who can reproduce a formula may not necessarily be able to carry out the procedure completely. The data also provides evidence that the reverse is true. There were seven students who failed to write the correct expression for Item 3(a), but six of them produced the correct inverse for Item 1(b) and the correct solution of the system of equations in Item 1(c) using the inverse matrix method. This shows that even though these students could not express the relationship in words, they were able to carry out the procedure, suggesting that their engagement with the concept was on an action level. The steps were done as a series of externally directed transformations and the concept of inverses had not been interiorised into a process. A process level of engagement with the notion of inverse would have entailed an understanding of how the procedure was linked to properties of the inverse. There were 12 students whose written responses suggest that they are working at a process level of the notion of inverse.

Results on solutions of system of equations

For item 4, there were 17 students who struggled with row reduction and algebraic

manipulation. Ten out of the 27 students were able to reduce the augmented matrix correctly to echelon form indicating that they at least exhibited action conceptions. Of these, seven students were unable to make the deduction from the last row, $(0\ 0\ \ a^2-16 : a-4)$ about the values a for which the system would have no solution or infinitely many solutions. These students had problems in distinguishing the necessary conditions for the system to have no solution, a unique solution or an infinitely many solutions. For Item 4(a), three students presented correct responses, six did not give any response and 18 gave incorrect responses. Eleven out of 18 students revealed difficulties with algebraic manipulation when reducing row three which involved unknowns as coefficient of a and the constant. For example S5 wrote:

$$\begin{pmatrix} 1 & 2 & -3 & \vdots & 4 \\ 3 & -1 & 5 & \vdots & 2 \\ 4 & 1 & a^2-14 & \vdots & a+2 \end{pmatrix} r_2 \to r_2 - 3r_1, r_4 \to r_4 - 4r_1$$

$$\begin{pmatrix} 1 & 2 & -3 & \vdots & 4 \\ 0 & -7 & 14 & \vdots & -10 \\ 0 & -3 & a^2-2 & \vdots & a-14 \end{pmatrix} r_4 \to r_4 - 3r_2,$$

$$\begin{pmatrix} 1 & 2 & -3 & \vdots & 4 \\ 0 & -7 & 14 & \vdots & -10 \\ 0 & 0 & 7a^2-56 & \vdots & 7a-68 \end{pmatrix}$$

$$7a^2 - 56 = 7a - 68$$

$$7a^2 - 7a + 12 = 0$$

$$a = 2\sqrt{2},\ -2\sqrt{2},\ \frac{68}{7}$$

The seven students who reduced the matrix correctly to row echelon form but had problems with deducing the necessary conditions for a system to have no solutions and linking that condition to the particular matrix in row echelon form are likely to be operating at an action level. For part (b) two students had correct responses, six did not present any response and 19 students produced incorrect responses. For part (c) seven students who gave correct responses are most likely on a process conception, six gave no response and 14 had incorrect responses. Five out of the seven students with correct

answers just wrote zeros in the last row and then found the solution of the system of the equation using parameters. For example S5 was one such student who had an incorrect response for Item 4(b) but managed get a correct response for Item 4(c). S5 wrote:

$$\begin{pmatrix} 1 & 2 & -3 & \vdots & 4 \\ 0 & -7 & 14 & \vdots & -10 \\ 0 & 0 & 0 & \vdots & 0 \end{pmatrix}$$

Let $-7x_2 + 10x_3 = -10$

$7x_2 = 10 + 14x_3$

$x_2 = \frac{10+14x_3}{7}$ Let $x_3 = t$

$x_2 = \frac{10+14t}{7}$

Hence even though these five students presented correct answers to Item 4(c) their engagement with the concept of systems of equations was at an action level. Three students who got a correct answer for Item 4(b) were able to solve the system of equations with infinitely many solutions for $a = 4$. Fourteen students who gave incorrect responses had difficulties in finding the values of x, y and z for a system with infinitely many solutions. These students did not know what procedures to follow and did not indicate even action conceptions. Table 4 is a summary of the results for systems of equations.

Table 4 Results for items based on system of equations

Item	Correct row reduction	Correct solution	Incorrect solution	No response
4(a)	10	3	18	6
4(b)	7	3	18	6

Only one out of the 27 students responded correctly to all the sub-questions in Question 4. It seems this student has mastered the algebraic manipulation involved in

row reduction and has also encapsulated the necessary and sufficient conditions for the system to have no solution and infinitely many solutions. The written response of this student suggests his engagement with the concept of systems of equation is at an object level of engagement

Discussion

The data revealed that the students' problems with basic numerical fluency and algebraic manipulation skills hampered their performance in the matrix algebra items. For example in Item 3(b), 60% of the students who produced incorrect responses did so because of their problems with basic concepts such as finding the square roots of fractions. 40% of the incorrect attempts in Item 3(b) were because of algebraic manipulation skills such as multiplication of terms within brackets. Algebraic manipulation skills also emerged as a major reason for 61% of the incorrect responses to Item 4 where students were unable to find the correct expressions or carry out a series of operations on algebraic expressions. Although some of the mistakes are related to what [16] calls slips, others are misconceptions arising from school level mathematics. The non-encapsulation of previously taught concepts was a serious impediment in constructing further mathematics concepts. Ndlovu [14] found similar results in her study where students' engagements with the concept of systems of equations were limited because of their poor numeracy and algebraic skills.

The data also revealed that students had problems with notations used to distinguish between a matrix and the determinant of matrices, similar to findings reported in [6]. Although this may seem like a minor notational error, it signals a deeper confusion between carrying out a procedure on the object (matrix) and the object itself. The determinant of a matrix is a single value (or expression) while a matrix is an $m \times n$ array of numbers. Hence students who have not developed this deeper understanding of

the differences between the two are more likely to interchange the notation without appreciating the different connotations that are conveyed. The problem with matrix brackets also emerged in Item 1(c) where students left out brackets and only inserted them in the last step. This tendency also signals a problem with distinguishing between the matrices whose entry values are obtained from finding the results of determinants and the procedure of actually finding the determinants.

There were many instances where students completed certain procedures without understanding what they were doing. Only four percent of students could identify the condition when the solution has infinitely many solutions and no solution and were able to understand algebraic manipulation. Many were able to reduce the augmented matrix to row echelon form but they stopped at that point because they were unable to apply their knowledge of systems of equations to explain what values of the variables could cause the system to have no solutions or an infinite number of solutions. In addition when asked to actually solve the system for an infinite number of solutions, some did not proceed even though they had the necessary information. On the other hand in Item 4, there were five students who solved the system even though they did not work out correct values of a in Item 4(b). It seems that they had learnt off the procedure for finding the infinite solutions and went ahead and applied it. This may have been so because students may have become accustomed to certain types of questions which appear regularly in tests and assessments in the course and they have learnt the steps without understanding. For these students the primary method of engaging with the material is by studying previous examination questions whose solutions are available. Developing a conceptual understanding is a secondary goal while the primary one is to pass the module. Dorier and Sierpinska's work [17], Ndlovu [14] and Siyepu [16] also found that students often cope with the procedural aspects of the course, solving linear

systems and manipulating matrices but struggle to understand the crucial conceptual ideas underpinning them.

APOS theory [7-9] asserts that engagement with a concept can take place at different levels. In this study it was clear that most students were stuck at the action level understanding of determinants which is characterised by procedural ways of working. A conceptual understanding develops as students move from seeing a concept as an action to having an interiorised conception at a process level. With determinants, there was evidence of three students having developed a process understanding; however most were stuck at action level conceptions. This finding concurs with [10, 14] who assert that students lack conceptual understanding. With respect to the notion of inverse, from the questions that were done, there are possibly 12 students who may be working at a process level or even at a higher level, but further evidence is needed before more definitive findings can be made. With respect to the understanding of systems of equations, only 1 student showed some evidence of possibly having an object conception, with most other students performing unevenly across the questions which indicate that they were working mostly at an action level which differs with [18] who had 29% of participants at process conception.

Conclusion

In this paper we analysed the written responses of 27 Zimbabwean pre-service students to assessment items based on the matrix algebra concepts of determinants, matrix inverse and solutions of systems of equations. It was found that students were at action conception since they were able to work out the items requiring procedural ways of working. Very few were operating on process and object conception for the four items. Students seemed to have been operating at action conception to certain types of problems which they then reproduced without showing evidence of deeper engagement

of object conception with the underlying concepts. It is a concern that some students revealed a lack of fluency algebraic manipulation skills as well as certain numerical operations and conceptual understanding which lead to schema development. This limited their engagement with the matrix algebra concepts. Hence for the participants to develop the appropriate mental structures they need structured opportunities to discover the underlying principles and relationships between objects in matrix algebra. It is therefore important for the university administrators to develop delivery plans around these needs of the students in a manner which allows developing of the necessary understandings which lead to object conception.

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Developing interactive applets with GeoGebra: processes, technologies

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Abstract

With continuing technological improvements, it is increasingly easy to produce high-quality interactive online resources to support learning of mathematics & statistics concepts. Software like GeoGebra (www.geogebra.org) allows users to produce dynamic, interactive constructions, with relatively low technical demands and moderate time and resource requirements. In this paper, we describe a project at the School of Mathematics and Statistics, The University of Melbourne, to produce a collection of interactive applets to enhance teaching in a range of undergraduate subjects offered by the school. The applets target specific teaching and learning needs of our school and are tailored precisely to our local teaching and learning context. We will give an overview of the project, and describe the approach taken by the project to identify teaching needs that could be serviced by new applets, and the design and feedback processes used to produce the applets. We give a preliminary survey of the outcomes of the project and discuss its impact on teaching and learning, both directly on the student experience as well as on academic staff through professional development. We offer a discussion of the technical merits and drawbacks of GeoGebra as a technological platform for the development of interactive applets for undergraduate mathematics and statistics, including data on the development time required to produce applets using GeoGebra.

Keywords: applets, GeoGebra, undergraduate, mathematics, statisticsIntroduction

Applets in mathematics and statistics education

Applets, small pieces of software which serve a specific task and run within a web browser, are used widely in undergraduate mathematics and statistics. Applets can offer interactive, dynamic visual representations of mathematical & statistical concepts. The

interactivity that applets offer can "extend and enhance" the communicative power of graphical representations of mathematical concepts [1]. Applets often have a specific conceptual focus (for example [2, 3]), so they can be used selectively by instructors to support understanding of key concepts or to enhance an instructor's 'story-telling' or educational narrative. They are flexible, allowing use in a classroom or by students outside of class, and are usually easy for users to master without training or previous experience with their use. Although many of these benefits are not unique to applets, the potential flexibility and ease of use that applets offer make them a popular choice of computer-based learning resource. Research shows that computer-based tools, such as applets, can be effective in mathematics and statistics education [4]. Many collections of applets exist on the web; for instance MERLOT (www.merlot.org) or GeoGebraTube (www.geogebratube.org), although the resources they provide can be of varying quality.

Applets may be built with a range of technological platforms. Java (www.java.com) was the most common platform for applets through the late 1990's and 2000's, and is still used widely; however, improvements in web browser technology mean that applets can now be built using HTML5, which run in a web browser without requiring any additional browser plugins or other software. An example of an applet is shown in Figure 1.

GeoGebra

GeoGebra (www.geogebra.org) is a dynamic geometry, algebra and calculus software system [5]. It is open-source software and freely available to download. Development of GeoGebra and related research is led by a non-profit foundation, the International Geogebra Institute [6]. Materials produced in GeoGebra can be uploaded to the web via the GeoGebraTube repository (www.geogebratube.org) or embedded within another website such as a learning management system (LMS). An outline of the technical

aspects of GeoGebra, and a discussion of the strengths and weaknesses of GeoGebra as a platform for applet development, is given in Section 0.

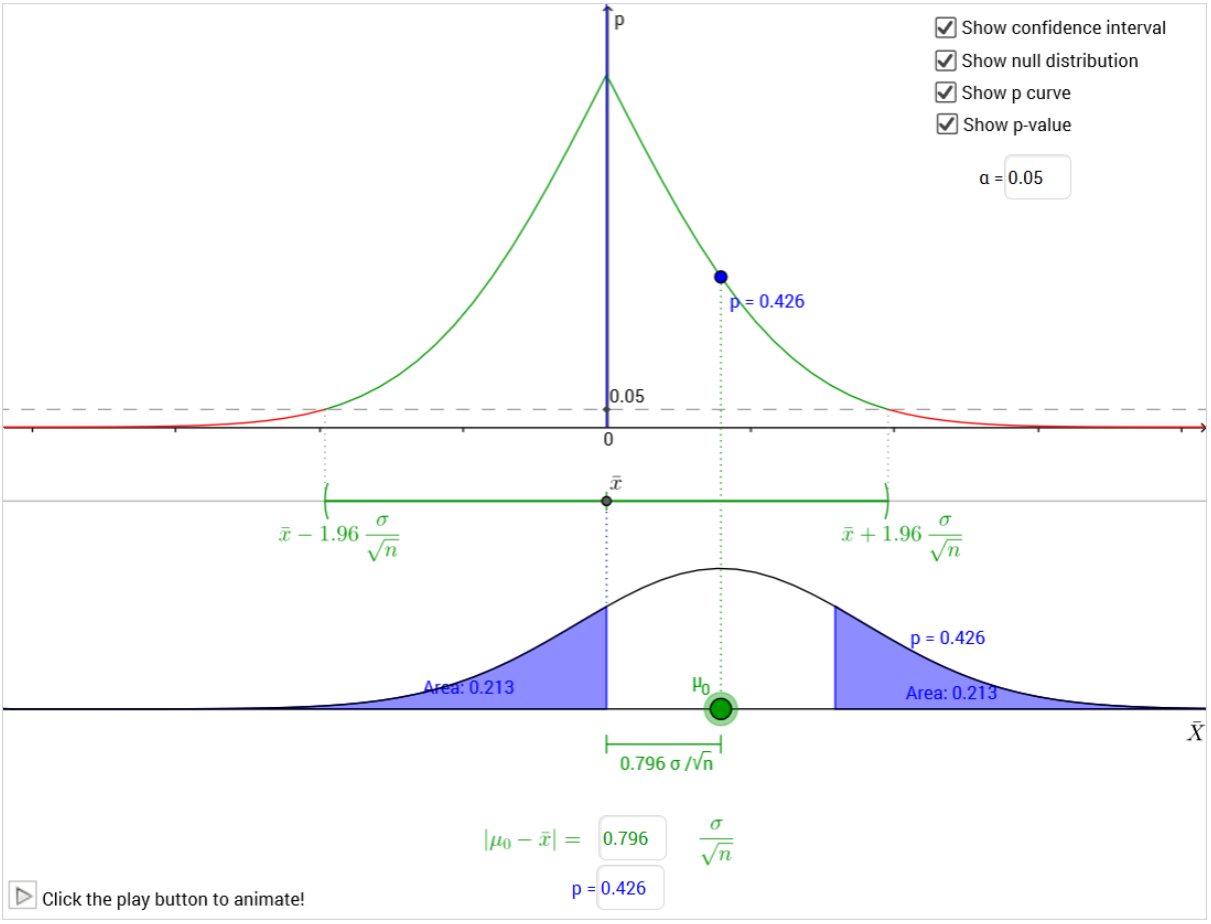

Figure 1: Screenshot showing an applet linking confidence intervals, hypothesis testing and p-values. Surrounding web browser window not shown. This applet is accessible at http://www.melbapplets.ms.unimelb.edu.au/?portfolio=confidence-intervals-hypothesis-testing-and-p-values

Project: Conceptual learning with interactive applets

Genesis of the project

Applets have been used in the School of Mathematics and Statistics at the University of Melbourne, in various ways and in several subjects, for many years. However, despite a wealth of applets publicly available on the web, many of the applets being used were

not perfectly suited to the local teaching context. For instance, the notation or terminology used in an applet may differ from that used in lecture notes, or between one applet and the next; some applets contained superfluous features, or lacked features which would have been of use; others have technological obstacles which make their use difficult. Despite the efforts of projects like MERLOT [7] to provide quality control for online teaching resources through peer review, it is often still difficult to find resources that meet a particular combination of teaching needs. During discussions between the authors, it was realised that, due to technological advancement over recent years, it is now feasible for academics to build custom-made interactive applets, tailored precisely to their needs, with relatively moderate requirements of time or technical expertise. As a demonstration of this, the authors produced two applets using GeoGebra, a software package which one of the authors had recently become acquainted with. The applets were designed around demonstrations used in undergraduate statistics lab classes, and anecdotally showed good potential for enhancing conceptual understanding in early trials with students and colleagues. The potential for such resources to make a notable impact on teaching and learning in the department for only a moderate investment of resources was recognised.

A small amount of seed funding was obtained from the School of Mathematics and Statistics for further applet development. This funding allowed us to produce two additional statistics applets, with the development done by a graduate student research assistant. A larger grant of $10,000 was obtained from the University to extend the project to several first and second-year mathematics and statistics subjects. The main aim was to enhance learning by producing new applets targeting specific teaching and learning needs in our school which were not met by existing resources. The applets would be tailored precisely to the local teaching and learning context. An additional

aim of the project was to develop a pool of expertise in development and use of such applets, which may be drawn upon in the future by other academics in the school seeking to produce or use similar resources for their own needs. A further aim, although more tacit, was professional development: to trigger colleagues to reflect on their teaching practice, by prompting them to make explicit their pedagogical goals and approaches during the applet design and feedback process.

Identifying teaching needs

Once the project was underway, areas in the school's core undergraduate programme were identified where new applets could have a positive impact on teaching and learning. This was done in discussion with interested teaching staff as well as drawing on the team members' own experiences. Areas were sought which were inadequately served by existing interactive resources, and for which the project could make a contribution within its resource and technical capabilities. Such areas were identified in a range of subject areas, including introductory and intermediate statistics and probability, first-year calculus, second-year real analysis, and first-year mathematics for biomedical sciences.

The areas identified were driven largely by three main motivators, which aligned with the motivations of the academics participating in the project. Each of the three motivators feeds in to the main aims of the project. This is discussed in Section 0.

Processes: Design and feedback

Once an area for an applet had been identified, team members would briefly survey existing resources to identify anything that might be pertinent. The team would produce draft designs for one or more applets, in consultation with the relevant academics when appropriate. These designs would be implemented in GeoGebra either by a research

assistant or by a member of the project team (the authors of this paper). The resulting applet(s) would then undergo a lengthy cycle of review, feedback and refinement. The applet(s) were provided to relevant academics who were encouraged to play with them, with particular attention to whether they met their teaching needs. In some cases, academics would be prompted with specific questions, such as what terminology or notation should be used in the applet, what key examples they would want the applet to support, or what additional functionality would be needed to make effective use of the applet in their teaching. Academics would provide feedback and suggestions to the project team, who would revise the applets accordingly. This cycle would be repeated several times. A further cycle would usually be performed after the applets were used by staff or students in the course of teaching. The duration of this process, from first discussions to finished product, was typically 2-3 months.

The project team also ran workshops with academics to refine and test the applets. Two workshops were held. One focussed on statistics applets and targeted practicing statisticians and staff involved in teaching statistics; the other focussed on mathematics applets and targeted lecturers of core first and second-year mathematics subjects. Each workshop had about 10 participants and ran for about 90 minutes. In the workshops, brief demos of selected applets were given, and then participants were invited to play with the applets, individually or collaboratively, and give constructive feedback. These workshops prompted discussion about subtle pedagogical and design issues, brought to light through collaborative discussion, which were not identified in the previous feedback cycles. Further revisions were made to the applets following the workshops.

Several design principles informed the design and development of the applets. These include the use of colour coding to re-inforce semantic relationships between

visual elements; the use of directly manipulable user interface elements, acknowledging that knowledge is embodied; and the use of representations, notation and terminology consistent with other course materials, such as lecture notes, to ensure coherence with existing resources. These principles are guided by cognitive load theory [8] and computer user interface research [9], and broadly align with those of similar projects such as [2, 3]. These will be discussed further in a future paper.

In addition, the project also produced supporting resources including instructors' notes, sample online tutorials and sample assessment items. These are discussed further in Section 0.

Project outcomes to date

The project has produced 29 applets at time of writing[1]. These are available on the project website at http://www.melbapplets.ms.unimelb.edu.au . The applets address concepts from a range of subjects and levels, including introductory calculus, statistics and probability, analysis, and discrete mathematics. A list of the concepts addressed by the applets is given in Table 5. These applets range in complexity from simple demonstrations with one or two visual components to rich applets with complex animations and multiple representations of key concepts. Two applets were adapted from existing freely available resources from Geogebratube; the remainder were designed and built by the project. A screenshot of a typical applet is given in Figure 1.

Table 5: Concepts addressed by applets produced by the project, grouped by subject area. One applet was produced for each concept, unless indicated otherwise.

[1] This excludes some applets which are minor variations on other applets, and 3 applets which, at time of writing, are incomplete and still under development.

Subject area	Concepts addressed by applets
Calculus (first-year level)	Sequences & series Inverse of a function Parametric curves (2 applets) Continuity ODEs: direction fields and Euler's method Mathematical models: population models without harvesting and with harvesting, springs (3 applets)
Introductory statistics	Discrete and continuous probability distributions (2 applets) Confidence intervals, hypothesis testing and p-values Power of a hypothesis test Partitioning of variability (2 applets) Normal probability plots
Intermediate probability & statistics (second-year level)	QQ plots Distribution of order statistics Maximum likelihood estimators Central Limit Theorem and Law of Large Numbers * Prisoners' Paradox *
Real analysis (second-year level)	Formal ϵ-M definition of convergence of a sequence Formal ϵ-δ definition of convergence and continuity Differentiability Riemann sums and integral
Discrete mathematics	Difference equations (2 applets) Discrete time population genetics model
Linear Algebra	Linear transformations of the plane and eigenvectors

* These applets have not yet been published publicly so links are not yet available.

In addition to the applets, the project also produced resources to support the use of applets in teaching. Instructors' notes were produced for several applets. The notes begin with a short description of the rationale for the applet, including key concepts for which it was designed to illustrate. This is followed by an outline of the user interface components of the applet. The notes typically also offer suggestions for ways that the applet may be used in teaching, and/or list specific examples (for instance, functions or combinations of parameter values) that can be used with the applet to highlight key

aspects of the concept(s) in question. The examples provided have been tested in the applet to ensure that they work smoothly and display clearly. For examples which require a function or equation to be input into the applet, the equation is provided in both standard mathematical notation and in GeoGebra's input notation, so that the input may easily be copied-and-pasted directly into the applet. These documents also note any technical or mathematical details which are relevant to the teaching of the concept but may not be outwardly obvious. Such details may include, for example, any assumptions being made in the mathematical model underlying the applet. The project team felt it important to document such details to enable instructors, when discussing or demonstrating the applet to students, to present a mathematically correct narrative that avoids conveying subtle misconceptions. Overall, the instructors' notes are intended assist teaching staff by reducing technical or cognitive barriers to effective use of the applet.

The project also produced sample online tutorials associated with two of the applets. These take the form of a web page with a sequence of questions guiding students through a learning activity using an applet. These were produced for two statistics applets; statistics was targeted for the tutorials because the statistics subjects offered by our school include computer lab classes in which the online tutorial exercises could potentially be embedded. Further improvement of the tutorials, informed by cognitive learning theory as in [10] and cognitive theory of multimedia learning [11], also remains for the future. The project team is in the process of producing sample assessment tasks, including assignment and online quiz questions, which make use of the applets. These will be published to the project website after they have been trialled and refined.

It was not our aim to produce supporting resources for every applet in the collection; rather, it was intended to produce a few exemplar resources as part of a showcase, which may act as models for further resources produced in the future.

Integration and evaluation

The design and development phase of the project, during which new applets are produced, is now nearing completion, with most applets complete or approaching the end of the feedback cycle. However, work is continuing to integrate the applets and other resources into subjects offered in the school. Some applets will be used in teaching for the first time during semester 2, 2015. During the next phase of the project we will explore how to further integrate applets into the existing teaching models used in the school, which include tutorials, computer labs, individual consultations, and various forms of assessment, as well as lectures. In some cases it may be desirable to adjust the teaching model to take full benefit of the learning opportunities that the applets offer.

The project also includes an evaluation phase. During the evaluation phase, data will be collected to investigate the project's impact on student learning, as well as on teaching staff in the school. Preliminary evaluation has already been conducted in 4 subjects which made use of the project's applets. This was done using data from online student surveys, as well as from analytics data and from assessment, and results have been positive. We briefly report here on the evaluation results from one subject. In semester 2, 2014, a collection of applets were used in a mathematics subject for first-year biomedicine students. The applets related to difference equations[2,3], discrete-time

[2] http://www.melbapplets.ms.unimelb.edu.au/?portfolio=iterating-a-difference-equation

[3] http://www.melbapplets.ms.unimelb.edu.au/?portfolio=cobwebbing

models of population genetics[4], and first-order ODEs[5]. The applets were used in lectures, and provided for students to access outside of class. Students were required to use one or more of the applets in assignments during the semester. Enrolment in the subject was 183 students, mostly first-year, all enrolled in a Bachelor of Biomedicine. An online survey was administered to the students over 5 days during weeks 7-8 of semester, after the applets had been used in lectures and assignments. The survey contained 6 questions, of which we will discuss only two. These are shown, along with results, in Table 6 below. Thirty-eight students responded to the survey, a return rate of 20.8%.

The first question asked students to rate their agreement with the statement *"I found that the applets improved my understanding of concepts from this subject"*. The response options were *Strongly agree*, *Agree*, *Unable to judge*, *Disagree*, *Strongly disagree*. Of the 38 respondents, 34 agreed or strongly agreed with the statement; the remaining 4 chose *Unable to judge*. No respondents disagreed or strongly disagreed with the statement. This question was intended to assess students' perceptions of the impact of the applets on their learning. This strong response indicates that students do indeed believe the applets to be beneficial for their learning.

The fourth question asked "*How did you use applets in this subject? Please select all that apply.*" The response options, and percentage of respondents that selected that option, are shown in Table 6 below. Thirty-seven percent of respondents answered that they used applets to help understand lecture content; 68% used applets to help answer exercises, and 97% (all but one) used applets as part of an assignment. These responses appear to show a preference towards task-oriented use of applets: students

[4] http://www.melbapplets.ms.unimelb.edu.au/?portfolio=population-genetics-by-fhw-model

[5] http://www.melbapplets.ms.unimelb.edu.au/?portfolio=exploring-an-ode

favour applet use to help complete specific learning tasks, rather than to assist with a general understanding of concepts. However, in this subject some assignment questions and non-assessed practice exercises explicitly direct students to use applets, which likely contributed to the high rates for the task-oriented responses. Whether the apparent preference for task-oriented use is genuine or a consequence of the explicit directions given in assignments and exercises may be answered by further evaluation. The remaining questions on the survey addressed technical aspects such as ease-of-use or were open-response. We will not discuss them here.

Table 6: Select survey questions for first-year biomedical mathematics class and summary of results.

Q1:	**I found that the applets improved my understanding of concepts from this subject.**				
	Strongly agree	Agree	Unable to judge	Disagree	Strongly disagree
Number	13	21	4	0	0
%	34%	55%	11%	0%	0%

Q4:	**How did you use applets in this subject? Please select all that apply.**	
Number	%	Response
14	37%	*I used applets to help me understand content from lectures*
26	68%	*I used applets to help me answer exercise sheet questions*
37	97%	*I used applets as part of an assignment*
17	45%	*When I used applets, I was working/studying on my own*
5	13%	*When I used applets, I was working/studying with friends*
0	0%	*I did not use applets in this subject*
0	0%	*Other*

Note: Respondents could choose multiple answers in question 4.

Total responses: 38

These results indicate that students perceive the applets as beneficial for their learning. Preliminary evaluation results from other subjects (not discussed here) paint a similar picture. More work is required before reaching definite conclusions however, in particular regarding student motivations for applet use. The evaluation will also include interviews with academic staff, which will explore the impact of the project on teaching practice in the school, including its role in professional development.

Early impacts

So far, the applets produced by the project have been used in 8 subjects over 3

semesters, potentially reaching an estimated 5000 enrolments and over 3000 distinct students. The authors have used the applets in lectures, tutorials, computer labs and one-on-one consultations and have observed first-hand the power of the applets to strengthen understanding of concepts, clear up misconceptions, and support students to see relationships between concepts and to reflect on their learning. An evaluation of the impact on student experience is ongoing. The impact of the project has also been felt more widely in the school. Over 20 academic staff were involved in the development and feedback processes during the project, including teaching specialist staff, research/teaching academics, consulting statisticians and casual tutors. By involving many academic staff from the school in the development and feedback processes, the project helped foster discussion and collaboration between staff, some of whom would not otherwise have been involved in the development of teaching resources. By actively seeking input and feedback from academics, the project prompted academics to reflect on their teaching practice. In this way, the project helped encourage collegiality and reflective practice in a non-confrontational and respectful way. We have only anecdotal evidence for these impacts so far, but this will be the focus of further evaluation in the future.

A framework for analysing applet impact

During the first phase of the project, the project team met with academics to identify areas in their undergraduate teaching where new applets might be of benefit. The involvement of individual academics in the project, and the development of ensuing applets, were driven largely by three (overlapping) motivators:

1. Pedagogical impetus. The original motivation for the project was to develop powerful interactive visuals for mathematical & statistical concepts. The visuals are tailored to a specific pedagogy (while providing enough flexibility to adapt

to other pedagogies if needed), with the aim of strengthening links between internal visual images and mathematical formalism.

2. Reduce technological obstacles. Some lecturers had used existing resources (typically Java-based applets or Excel spreadsheets) in their teaching for many years. These resources generally served the pedagogical requirements adequately, but had in recent years become increasingly difficult to use due to changing software configurations – in particular, stricter browser security settings, which require users to click through several security warnings or modify browser configuration before the software would load. In these cases, the lecturers wanted similar applets that did not require Java or other additional software.
3. Enhance subject delivery. In some subjects, opportunities were identified where interactive applets could enhance existing teaching practice, for instance by replacing a series of hand-drawn diagrams used by lecturers with a single applet. Lecturers typically wanted to enhance engagement and interactivity in lectures using media beyond the conventional lecture slides or document camera.

Two main aims of the project were, ultimately, to improve student learning, and to provide professional development by prompting reflection amongst teaching academics. We posit that each of the three motivators above feeds into these main aims. This is represented schematically in Figure 2. For example, constructing pedagogically tailored visual aids helps students link symbolic and visual representations of concepts, feeding in to improved conceptual understanding. Designing such applets requires teachers to reflect deeply about the concepts in question and the narratives they use to explain them, a form of (informal) professional development. Removing technological obstacles reduces barriers to student engagement, which feeds into improved learning.

Designing replacements for old applets necessitates an examination of the role that the applets were playing, leading to reflection on teaching practice. Enhancing subject delivery leads to improved engagement and learning; seeking feedback on a new interactive resource prompts staff to experiment with new examples or narrative sequences. Investigating these links will be a task for the ongoing evaluation phase of the project.

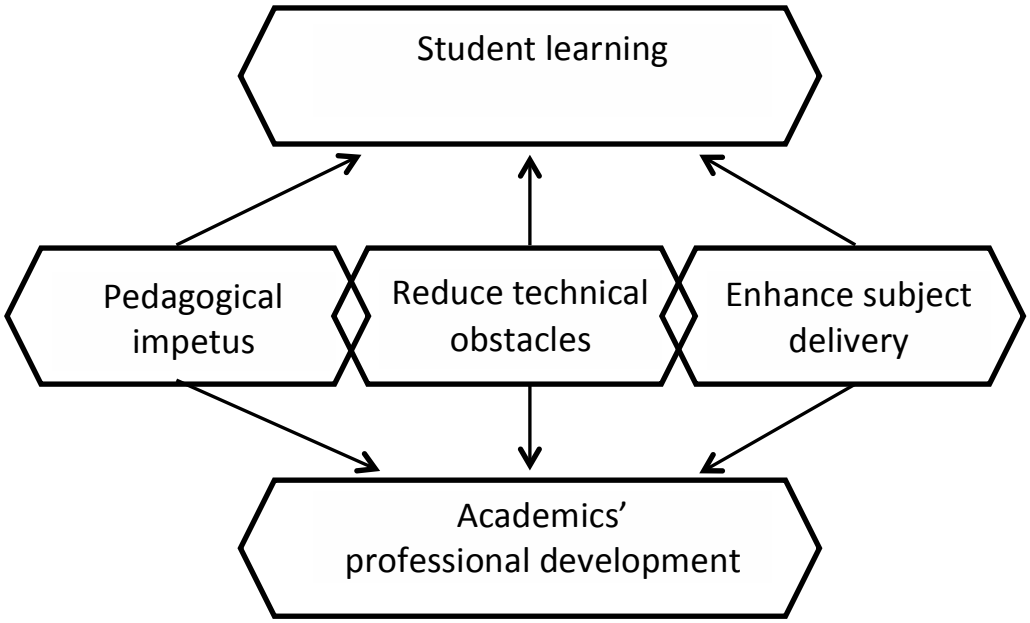

Figure 2: Emerging framework for evaluation of project impact

Technology: GeoGebra as a platform for interactive resource development

In this section we will discuss GeoGebra as a technological platform for applet development. We will outline some technical aspects which informed the choice of GeoGebra as a technological platform for our applet development, and discuss some strengths and weaknesses of GeoGebra that we have encountered for this kind of project. We give some data on the development time required to produce applets in GeoGebra, which may be of use to those planning similar projects or for comparison of the relative ease of development of different technological platforms.

GeoGebra is an open-source mathematical software package, freely available at www.geogebra.org . It includes dynamic geometry, calculus, symbolic algebra and statistics functionality. GeoGebra can be used in several ways, including as a standalone desktop application, as a tablet app, as a browser-based web app, or embedded as an applet into a webpage. The GeoGebra project also provide a repository, GeoGebraTube, to which users can upload and share GeoGebra worksheets. Development of our applets was done in the desktop version of GeoGebra, and then uploaded to GeoGebraTube for deployment to students.

GeoGebra applets embedded in a web site or on GeoGebraTube can be run in two modes: HTML5 mode, which requires no additional software or browser plugins, but will not work on very old web browsers, or Java mode, which requires that Java software be installed on the user's computer but can work a wider range of browsers, providing that the Java plugin is installed and appropriately configured. By default, the HTML5 version is used by GeoGebraTube but it is possible to specify that the Java version should be used instead.

Low technological barriers

A significant strength of GeoGebra, from our perspective, is its low technological barriers to use. Once uploaded to the web as HTML5, a GeoGebra applet usually runs entirely within a web browser, without requiring any additional software, plugins or configuration. This is in contrast to Java applets, which require the Java software package and browser plugin to be installed, and can require the user to click through several security warnings before the applet will load. To use a Java applet on their own PC, a student must first download and install Java, and keep it up-to-date with updates if it is to continue to work reliably. Due to the wide variety of devices and configurations, and time commitment, teaching staff typically cannot provide technical support with

this process so students are 'on their own' with installing and configuring the software. We have found that, even on University-maintained PCs in computer labs, students must complete additional configuration steps (such as clicking through a security warning) before a Java applet will load. Moreover, guidance about this process must be provided by teaching staff if applet use is a required part of the course. A major strength of GeoGebra is that it removes these potential technological barriers from students; to access a GeoGebra applet, a student typically needs only to follow a web link, after which the applet will load and execute automatically a few seconds later. Moreover, being based on HTML5 standards, the applets 'just work' on a wide range of devices (eg. desktop PC, mobile tablet) and browsers (eg, Firefox, Safari). This greatly reduces the technological issues experienced by students and staff, and the effort expended by teaching staff providing technical support.

Ease of development

From a developer's perspective, we found building applets in GeoGebra to be very fast. Similarly to other dynamic geometry systems, GeoGebra operates by construction: each mathematical object or visual element is constructed by specifying its relationship with previously defined objects. If an object is changed, for instance as a result of a user interaction, then all objects descended from it are automatically updated to reflect the change. Hence the work required to construct a mathematical applet using GeoGebra is close to the theoretical minimum: to fully specify an applet, the developer must, at some point, define each object in the applet and their relationship to other objects. Building 'by construction' in GeoGebra means that these definitions are, in many cases, all that is required; once an object is defined, GeoGebra takes care of rendering, updates and user interaction with the object automatically.

To quantify the development time required for such applets, we can look at the time spent by research assistants developing applets for the project. Of the applets produced by the project, 10 were developed primarily by casual research assistants employed by the project (8 of which are fully complete, 1 has minor modifications pending, and 1 has substantial development work remaining). The research assistants had no prior experience working with GeoGebra before taking on this work, and were required to learn the system as they went. The 10 applets produced by research assistants, and the time taken by the research assistants, are given in Table 7. The table also gives an indication of the technical complexity of the applet. The technical complexity is mainly determined by the number of elements in the construction and the presence or absence of multi-stage animations. The average development time required for these applets was 18.2 hours. The development work for each applet was typically spread over 2-4 weeks. The times given in Table 7 are the times spent implementing the applet in GeoGebra; they do not include time spent in the design or feedback phases (which in some cases drew on hands-on experience with students going back years), but they do include time spent implementing changes as a result of feedback.

Table 7: Development times for applets developed by research assistants.

Applet	Technical complexity	Research assistant development time (hours)	Team member development time estimate (hours)	Total development time (hours)
Partitioning of variability in ANOVA	High	25	4	29
Partitioning of variability in regression	High	18	4	22
Maximum likelihood estimators	Low	16	1	17
Distribution of order statistics	Low	10	1	11
Discrete and continuous distributions	Low	9	1	10
Prisoner's Paradox	Medium	15	2	17
QQ plots	Medium	17	2	19
Spring with forced vibrations	High	18	4	22
Central Limit Theorem and Law of Large Numbers	Low	3	1*	4
Random variables	Very high	25	6*	31
	Average	**15.6**	**2.6**	**18.2**

*At time of writing these applets are still under development so these times are forecasts.

For these applets, initial development was done by a research assistant under guidance from the project team. During the feedback cycle, minor changes or corrections were usually made directly by the team members. Team members did not formally record the hours spent on these tasks so the times given for team members are estimates. In some cases, technical issues arose during the development of low or medium complexity applets (as discussed in sections 0 and 0) which increased the development time required. In other cases, changes were made to the design while development was underway; the consequent modifications also increased development time.

Community support

The documentation provided by the GeoGebra project is generally reliable and comprehensive. Moreover, the large pool of resources that are available on GeoGebraTube or elsewhere on the web provide many sources of ideas or techniques to draw from. Finally, the forums are a good source of expertise or help for technical issues; typically, our posts would be answered within a day with useful suggestions for resolving problems.

Poor performance

The main obstacle that we encountered with GeoGebra as a platform for applet development was poor responsiveness of certain applets when running in HTML mode. In some cases, animations would be jerky or a noticeable lag would occur between a user interaction (for instance, clicking a button) and the response appearing on screen. This was a problem mainly for larger applets with many components, extensive computational requirements, or complex animations, and generally only when the applet was displayed in HTML mode. Performance varied with hardware and software configuration. Sometimes, performance could be improved significantly by redesigning aspects of the construction to be more efficient. In one case, where the performance in HTML mode was unacceptable, we elected to use the Java version of the applet by default, as the Java version generally has better performance, but at the cost of potentially higher technical barriers such as Java security warnings, update notifications or plugin problems.

Immaturity of software

Over the course of our project we ran into several bugs or other software issues with GeoGebra, for instance cases where a function does not work as expected, or

constructions which cause a crash. In a small number of cases, an applet had to be modified or redesigned to avoid bugs or technical limitations. However, the GeoGebra developers were quick to respond to reports of such issues on the forum, often providing a fix within a day or two. We also note that GeoGebra's library of built-in mathematical functions, although sufficient for most needs in introductory calculus and statistics, is not as extensive as that of more specialised mathematical software such as Mathematica or Maple, which may be a limitation for more advanced applications.

Further work

Integration of the new applets into the teaching activities of the school is continuing. Several of the applets are being used in teaching for the first time in semester 2, 2015, so a full picture of the impact of the project may not be seen until 2016. One particular challenge is to explore how to successfully integrate online applet use into interactive 'whiteboard' tutorial classes.[12] Applets have many attributes that potentially encourage interaction, groupwork and creativity, key features of the whiteboard tutorial style, but it is not clear to us how to leverage these aspects in a way that is not disruptive to the active work taking place at the whiteboards, and does not disadvantage students who did not bring a suitable mobile device to the tutorial.

There are several avenues to investigate further in the project evaluation. Additional evaluation is required of the supporting resources (instructors' notes and online tutorials). Do the resources provide the kind of support needed by teaching staff? Evaluation of the impact of the project on both student learning and staff professional development will also continue, guided by the framework from Section 0, which in turn may lead to further refinement of the framework.

Conclusion

We have described a project to develop interactive applets targeting specific teaching and learning needs in our school's undergraduate teaching. Over 20 applets were produced by the project, covering a range of subjects and concepts. An iterative, consultative approach was taken to design the applets, with repeated cycles of feedback between the project team and relevant academics. This approach led to applets which were highly tailored to our local teaching and learning context, as well as fostering collaboration and discussion between a range of staff, and prompting academics to reflect on their teaching practice. Hence the impact of the project extended beyond the immediate utility of the applets themselves for teaching and learning, by encouraging professional development and reflective practice amongst academics. Further assessment of this impact is a focus for future evaluation. Preliminary evaluation with students shows that students are generally favourable towards the applets and feel that the applets have a positive impact on learning.

The software used by the project, GeoGebra, was found to allow very rapid development of applets, without requiring programmers or other specialist skills. The resulting applets also present low technical barriers to use by students and staff. Despite occasional limitations or shortcomings with GeoGebra, overall we found it to be a very good technological platform for this kind of project.

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